

ON LINEAR SERIES
ON GENERAL k -GONAL PROJECTIVE CURVES

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ABSTRACT. Let X be a general k -gonal curve of genus g . Here we prove a strong upper bound for the dimension of linear series on X , i.e. we prove that $\dim(W_d^r(X)) \leq \rho(g, r, d) + (g - 2k + 2) := g - (r + 1)(r + g - d) + (g - 2k + 2)$.

In recent years several papers were devoted to the study of linear series on a general k -gonal curve. This note contains the proof of the following result.

Theorem 0.1. *Fix integers g, r, d, k with $k \geq 2$, $2k - 2 \leq g \leq 4k - 4$, $2 \leq r \leq d \leq g - 1$. Assume $\text{char}(\mathbf{K}) = 0$ or $\text{char}(\mathbf{K}) > 2g$. Let X be a general k -gonal curve of genus g . Then*

$$\dim(W_d^r(X)) \leq \rho(g, r, d) + (g - 2k + 2) := g - (r + 1)(r + g - d) + (g - 2k + 2).$$

Here $\rho(g, r, d) = g - (r + 1)(r + d - g) = (r + 1)(d + r) - rg$ is the Brill-Noether number. Note that the bound in the statement of 0.1 is just $\rho(2k - 2, r, d) - (r - 1)(g - 2k + 2) = (r + 1)(d + r) - r(2k - 2) - (r - 1)(g - 2k - 2)$ and this is a hint of the way in which we will prove the theorem in section 1 by induction on g starting from the case $g = 2k - 2$ (which is known to be true because a general curve of genus $2k - 2$ is a k -gonal curve). The tools are the theory of limit linear series of Eisenbud and Harris (see [EH2]) and the theory of admissible coverings of Harris-Mumford (see [HM]). The reader is assumed to have a working knowledge of these theories. In this paper we work always over an algebraically closed base field \mathbf{K} with either $\text{char}(\mathbf{K}) = 0$ or $\text{char}(\mathbf{K}) > 2g$.

We believe that Theorem 0.1 is quite good for high r , but not sharp. For instance, it says nothing if $r = 1$, while much is known about the pencils on X . The upper bound for $\dim(W_d^r(X))$ in its statement just comes from part (d) of its proof.

Proof of 0.1. The proof of Theorem 0.1 is divided into 4 steps.

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(a) The assumption on $\text{char}(\mathbf{K})$ allows us to use freely in the range of integers that we will consider the results on admissible coverings in [HM] and limit linear series in [EH1], [EH2], [EH3] and [EH4].

(b) Set $m := 2k - 2$, $s := g - m$, and let D be a general curve of genus m . By the generality of D , D satisfies the Brill-Noether conditions, has the expected number of g_k^1 and each of them has $6k - 6$ simple ramification points. Fix one g_k^1 on D and s of its ramification points, P_1, \dots, P_s . Let A be the stable genus g curve of compact type union of D and s disjoint elliptic curves E_1, \dots, E_s , with each E_i intersecting D at P_i .

(c) First we will check that A is a limit in \overline{M}_g of smooth k -gonal curves of genus g . Indeed this curve is the stable reduction of an admissible degree k covering $f: X' \rightarrow U'$ (in the sense of Harris-Mumford [HM]) (or see [EH2], sect. 5) with the following description. U' is a connected nodal curve with $p_a(U') = 0$. X' is the union of D and $s(k - 2)$ \mathbf{P}^1 's. The morphism f sends D to one of the \mathbf{P}^1 's, say J , of U' as the chosen g_k^1 ; each of the s families of $k - 2$ \mathbf{P}^1 's of X' are linked to D at the other points of the g_k^1 over the image of $D \cap E_i$ and mapped by f to a different component, say J_i , of U' with $u(\text{Sing}(X')) = \text{Sing}(U')$.

(d) To prove Theorem 0.1 it is sufficient to prove that the dimension of limit linear series (in the sense of Eisenbud-Harris) is bounded by $\rho(g, r, d) + s$ (and that the same is true for any curve which is obtained from A by inserting chains of \mathbf{P}^1 's at each point of $\text{Sing}(A)$). This (and all the Brill-Noether-Gieseker package) would hold if we had chosen general points of D as $\text{Sing}(A)$ instead of the points P_i ; this was remarked (more or less explicitly) several times (see [EH3], Theorem 1.1, or [EH4], Proposition 5.2 and Corollary 5.3, or [W], Remarks 1.12 and 2.9). But of course, since D has only finitely many g_k^1 , the set $\{P_i\}_{1 \leq i \leq s}$ is not a general set of s points of D . It is a standard calculation in the theory of limit linear series that each general point of D to which an elliptic tail is linked “drops” the dimension of the set of all limits g_d^r on D (which have dimension $\rho(m, r, d)$) by r . We want to check that each E_i imposes at least $r - 1$ independent conditions to the set of all limits g_d^r on D , proving 0.1. Indeed we claim that this holds for any choice of the points P_i . To prove the claim we degenerate D to the union of a general curve Z of genus $m - s$ (which by assumption is ≥ 0) and s elliptic curves F_i with E_i linked to F_i . By the theory of admissible coverings of Harris-Mumford, to preserve the condition that the arithmetic genus g curve W obtained in this way is still a limit of smooth k -gonal curve, it is sufficient that the singular points of each F_i , say R_i and S_i (with R_i on Z and S_i on E_i), are such that $2R_i$ is linearly equivalent to $2S_i$ on F_i and that the points R_i are ramification points of a g_k^1 on Z . It is possible to satisfy the last condition by Riemann-Hurwitz because $4k - 4 \geq g$. It is the torsion condition on $R_i - S_i$ which assures us that there is a 2:1 morphism $F_i \in \mathbf{P}^1$ ramified over R_i and S_i . It is this torsion condition which makes (in general) false the Brill-Noether bound for W (see [W], Remarks 1.12 and 2.9, [EH1], Lemma 7.3, [EH3], Theorem 1.1, [EH4], Proposition 5.2 and Corollary 5.3). To prove our claim (and hence 0.1), look at the proofs in [EH1] in which the case of a curve with general moduli is proved. To get the upper bound $\rho(g, r, d)$ for $\dim(W_d^r)$ for the genus g starting from the bound $\rho(2k - 2, r, d)$ for the genus $2k - 2$, it would be sufficient to perform the numerical calculations in the first half of page 134 of [EH1]. To make those calculations the only missing step in our situation is part (iii) of Lemma 7 of [EH1]. To have our weaker upper bound it is sufficient to have, for each of the s elliptic tails, instead of part (iii) of Lemma 7 of [EH1] which says

“... for more than one value of the index called i in [EH1] only if there are two or more independent sections of V_{F_i} vanishing only at the two points R_i and S_i ”, the weaker statement “... for more than two values of i only if there are three or more independent sections of V_{F_i} vanishing only at the two points R_i and S_i ”. To prove this weaker statement it is sufficient to note that on each F_i there is no g_d^2 with a positive divisor supported on $\{R_i, S_i\}$ because no degree 0 line bundle on F_i has two linearly independent sections. \square

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