

RADIAL IMAGES BY HOLOMORPHIC MAPPINGS

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ABSTRACT. Let \mathcal{R} be a nonexceptional Riemann surface, other than the punctured disk. We prove that if f is a holomorphic mapping from the unit disk Δ of the complex plane into \mathcal{R} , then the set of radial images that remain bounded in the Poincaré metric of \mathcal{R} has Hausdorff dimension at least $\delta(\mathcal{R})$, the exponent of convergence of \mathcal{R} . The result is best possible. This is a hyperbolic analog of the result of N. G. Makarov that Bloch functions are bounded on a set of radii of dimension one.

1. INTRODUCTION

In recent years, a number of results has been obtained in Function Theory showing that some sets describing a certain “exceptional” boundary behavior are always there and although they are usually of zero length they have large dimensions. See, for instance, Makarov [M1], Rohde [R], Anderson and Pitt [AP], or Bourgain [Bo].

In this note we consider holomorphic mappings f from the unit disc Δ of the complex plane and we shall see that, given the range domain Ω of f , there is a sharp estimate of the dimension of the set of radii where f remains bounded away from the boundary of Ω .

Our result is best illustrated by the following elementary observation: *if f is a holomorphic mapping from Δ into a ring $\{z \in \mathbb{C} : 1/R < |z| < R\}$, $R > 1$, then there is at least one radius where f remains bounded away from $\partial\Omega$.* Of course, we may assume that almost every radial limit of f belongs to $\partial\Omega$. Then, we consider $r \in (1/R, R)$ such that f has no critical values in $\{|z| = r\}$. Suppose, for simplicity, that $r = 1$ and $|f(0)| = 1$, and find a curve $\gamma : [0, 1) \rightarrow \Delta$ such that $|f(\gamma(t))| = 1$, $\gamma(0) = 0$, $\lim_{t \rightarrow 1} |\gamma(t)| = 1$. The above assumption on radial limits implies that $\lim_{t \rightarrow 1} \gamma(t)$ exists, say, $\lim_{t \rightarrow 1} \gamma(t) = e^{i\theta}$. But then harmonic majorization implies that

$$\frac{1}{\sqrt{R}} \leq \liminf_{t \rightarrow 1} |f(te^{i\theta})| \leq \limsup_{t \rightarrow 1} |f(te^{i\theta})| \leq \sqrt{R}.$$

By the way, covering maps from the disc into rings have the stated property only at two radii and for a degenerate annulus, $\Delta \setminus \{0\}$, there could exist none of those radii.

It turns out that these two examples, rings and punctured disk, are exceptional. To state our result we need the concept of exponent of convergence of a Riemann

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surface, which we now recall. Let \mathcal{R} be a nonexceptional Riemann surface, i.e., a Riemann surface other than the Riemann sphere, the complex plane \mathbb{C} , the punctured plane, $\mathbb{C} \setminus \{0\}$, or the tori. Then \mathcal{R} can be represented as the quotient of Δ by a Fuchsian group, Γ . The exponent of convergence, $\delta(\mathcal{R})$, is defined as

$$\delta(\mathcal{R}) = \inf\{\alpha > 0 : \sum_{T \in \Gamma} (1 - |T(0)|)^\alpha < +\infty\}.$$

The exponent $\delta(\mathcal{R})$ measures in a subtle way the size of \mathcal{R} . For basic background see, for example, [Ni].

For planar domains, our result reads as follows:

Theorem 1. *Let Ω be a planar domain, other than \mathbb{C} , $\mathbb{C} \setminus \{a\}$, and which is not conformal to $\Delta \setminus \{0\}$. Then, if $f : \Delta \rightarrow \Omega$ is holomorphic,*

$$(1) \quad \text{Dim}\{\theta \in [0, 2\pi] : \inf_{0 \leq r < 1} \text{dist}(f(re^{i\theta}), \partial\Omega) > 0\} \geq \delta(\Omega).$$

The result is sharp in the sense that for each such Ω and for f a covering map one has equality in (1). Moreover for each $\eta \in [\delta(\Omega), 1]$, there is f as above with the dimension in (1) equal to η .

Here dist means spherical distance, and Dim denotes Hausdorff dimension.

We remark that for any domain Ω which is not simply or doubly connected, $\delta(\Omega) > 0$. Also, for $\Omega = \Delta \setminus \{0\}$, $\delta(\Omega) = 1/2$, so that, as we have remarked above, (1) does not hold in this case. In [FR] one can find conditions in terms of the euclidean geometry of Ω to decide whether $\delta(\Omega) = 1$.

Notice that \mathbb{C} , $\mathbb{C} \setminus \{a\}$ are exceptional Riemann surfaces. For $\Omega = \mathbb{C}$ there is a parallel result due to Makarov [M1] (see also [R]), but in that case one needs the assumption that the functions are Lipschitz from Δ endowed with its Poincaré metric into \mathbb{C} endowed with the euclidean metric, i.e. that f is in the Bloch class; for such an f , f is radially bounded in a set of Hausdorff dimension one. The same result holds for the cylinder, $\Omega = \mathbb{C} \setminus \{0\}$, with the metric whose density is $1/|z|$.

In our case the proper metric in Ω is its Poincaré metric. On the one hand, since this metric is complete, Theorem 1 claims that f is radially bounded (in the Poincaré metric) in a set of radii of Hausdorff dimension at least $\delta(\Omega)$, and, on the other hand, Schwarz' lemma implies that such f 's are Lipschitz in the Poincaré metric.

Theorem 1 is valid for general Riemann surfaces and actually this is its natural context because of the geometry involved. Here and hereafter by a Riemann surface we shall mean a nonexceptional Riemann surface, and its intrinsic Poincaré distance will be denoted by $\rho = \rho_{\mathcal{R}}$.

Theorem 2. *Let \mathcal{R} be a Riemann surface other than a punctured disk, and let f be a holomorphic mapping from Δ into \mathcal{R} . Then*

$$(2) \quad \text{Dim}\{\theta \in [0, 2\pi] : d_f(e^{i\theta}) < +\infty\} \geq \delta(\mathcal{R}),$$

where

$$d_f(e^{i\theta}) = \sup_{0 \leq r < 1} \rho(f(re^{i\theta}), f(0)).$$

For a covering map the images of radii are the geodesics in \mathcal{R} from $f(0)$. In that case one has equality in (2), and this is due to Fernández and Melián [FM] (see also Stratmann [St], but only for the case when \mathcal{R} has finite type; recently, Bishop and Jones [BJ] have also obtained a proof of this result). This result can be stated as follows.

Theorem FM. *Let \mathcal{R} be a Riemann surface, other than $\Delta \setminus \{0\}$, and let $p \in \mathcal{R}$. If $B(\mathcal{R}, p)$ denotes the set of directions v such that the geodesic of \mathcal{R} emanating from p in the direction v remains at all times $t > 0$ at a bounded distance from p , then*

$$\text{Dim}(B(\mathcal{R}, p)) = \delta(\mathcal{R}).$$

Theorem FM shows, in particular, that Theorem 2 is sharp. And, of course, Theorem 1 is a direct corollary of Theorem 2. Notice that the last statement of Theorem 1 follows directly from Theorem FM and the following construction: If $\eta \in [\delta(\Omega), 1]$, let \mathcal{C}_η be a regular Cantor set with Hausdorff dimension η , and let μ_η be the standard Cantor measure associated to \mathcal{C}_η . Consider the function $f = F \circ b$, where F is a universal covering map of Ω and b is the singular inner function whose associated singular measure is μ_η . One can check easily for f that the dimension in (1) is exactly η .

We remark here that Theorem FM is also a basic ingredient in the proof of Theorem 2. We shall give the proof of Theorem 2 in Section 3; some preliminary material is collected in Section 2.

2. SOME LEMMAS

In order to prove Theorem 2 we will need two results (Lemmas 1 and 2 below) about distortion of boundary sets under conformal mappings and inner functions. Both results are extensions to fractional dimensions of Löwner's Lemma (see, e.g., [A1, p. 12], [T, p. 322]).

Lemma 1 (Makarov [M2], Hamilton [H]). *Let f be a univalent function in Δ with $f(\Delta) \subset \Delta$. If E is a Borel subset of $\partial\Delta$ such that $f(E) \subset \partial\Delta$, then*

$$\text{Dim}(f(E)) \geq \text{Dim}(E).$$

Recall that an *inner function* is a holomorphic function from Δ into Δ such that the radial boundary values have modulus 1 almost everywhere.

Lemma 2 ([FP]). *Let $f : \Delta \rightarrow \Delta$ be an inner function. If E is a Borel subset of $\partial\Delta$, and if we denote by $f^{-1}(E)$ the set*

$$f^{-1}(E) = \{e^{i\theta} : \lim_{r \rightarrow 1} f(re^{i\theta}) \text{ exists and belongs to } E\},$$

then

$$\text{Dim}(f^{-1}(E)) \geq \text{Dim}(E).$$

If I is an arc in $\partial\Delta$, we denote by $\omega(z, I)$ the harmonic measure in Δ of I from $z \in \Delta$ and by B_I the hyperbolic halfplane given by

$$B_I = \left\{ z \in \Delta : \omega(z, I) < \frac{1}{2} \right\}.$$

In the next two lemmas we shall consider a closed set $E \subset \partial\Delta$ with at least two points, and the collection $\{I_j\}$ of complementary intervals of E in $\partial\Delta$. If $\varepsilon \in (0, 1/2]$, then we define a generalized polygon $P = P(E, \varepsilon)$ as

$$\begin{aligned} P &= \{z \in \Delta : \omega(z, I_k) < 1 - \varepsilon \text{ for each } k\} \\ &= \left\{ z \in \Delta : \rho_\Delta(z, B_{I_k}) < \log \cotan\left(\frac{\pi}{2}\varepsilon\right) \text{ for each } k \right\}. \end{aligned}$$

Lemma 3. *Let $f : \Delta \rightarrow \Delta$ be a holomorphic function with $f(0) = 0$. Let $\varepsilon \in (0, 1/2]$ be such that $\omega(0, I_k) < 1 - \varepsilon$ for each k .*

The connected component Q of $f^{-1}(P)$ which contains 0 is simply connected. And, if $\gamma : [0, 1) \rightarrow \Delta$ is a curve contained in Q beginning at 0 and ending at a point $e^{i\theta} \in \partial Q \cap \partial\Delta$, then

$$f(re^{i\theta}) \in P(E, \varepsilon'), \quad 0 \leq r < 1,$$

for every ε' , $0 < \varepsilon' < \varepsilon/2$.

Proof. If $\omega_k = \omega(\cdot, I_k)$, then

$$f^{-1}(P) = \bigcap_{k=1}^{\infty} \{z \in \Delta : (\omega_k \circ f)(z) < 1 - \varepsilon\}.$$

From the maximum principle it follows that Q is simply connected.

Also, notice that for each k , $\omega_k \circ f < 1 - \varepsilon$ on γ , and $\omega_k \circ f \leq 1$ in the whole disk, and so by harmonic majorization $\omega_k \circ f \leq 1 - \varepsilon/2$ on the radius ending at $e^{i\theta}$. \square

We denote by $\text{rad}(E)$ the union of the set of radii ending at points of E , i.e.

$$\text{rad}(E) = \{re^{i\theta} \in \Delta : e^{i\theta} \in E\}.$$

The following lemma can be proved by a simple estimate of hyperbolic geometry (see e.g. [Be, p. 162]).

Lemma 4. *Let $\varepsilon \in (0, 1/2]$ such that $\omega(0, I_k) < 1 - \varepsilon$ for all I_k . Then, if $z \in P(E, \varepsilon)$,*

$$\rho_\Delta(z, \text{rad}(E)) \leq h(\varepsilon),$$

where $h(\varepsilon)$ is given by $\sinh h(\varepsilon) = \cotan(\pi\varepsilon/2)$.

3. PROOF OF THEOREM 2

Let $F : \Delta \rightarrow \mathcal{R}$ be a covering map such that $F(0) = f(0)$, and let us factorize f as $f = F \circ b$ where b is a holomorphic mapping from Δ into Δ with $b(0) = 0$. We will assume that b is an inner function since otherwise the result is trivial.

Fix a positive number η , and use Theorem FM to obtain a closed set $E \subset \partial\Delta$ such that

$$(3) \quad \text{Dim}(E) \geq \delta(\mathcal{R}) - \eta,$$

and such that for some constant k we have that

$$(4) \quad \sup_{0 \leq r < 1} \rho(F(re^{i\theta}, F(0))) \leq k, \quad \text{for each } e^{i\theta} \in E.$$

Let $\{I_j\}$ be the complementary intervals of E in $\partial\Delta$. Choose ε so close to 0 that $\omega(0, I_k) < 1 - \varepsilon$ for each k and denote by P the set $P = P(E, \varepsilon)$.

Now let Q be the connected component of $b^{-1}(P)$ containing 0. By Lemma 3, Q is simply connected, and, trivially, the same is true for P . Notice that if $z \in \partial Q \cap \Delta$, then $b(z) \in \partial P \cap \Delta$. Let $\Phi : \Delta \rightarrow Q$ and $\Psi : \Delta \rightarrow P$ be Riemann mappings chosen so that $\Phi(0) = 0$ and $\Psi(0) = 0$. We collect now some information about these two conformal mappings.

Φ has radial boundary values, and they belong to the boundary of Q , at all $e^{i\theta} \in \partial\Delta$ except at most for a set N of zero logarithmic capacity. Partition $\partial\Delta$ as $\partial\Delta = G \cup B \cup N$, where if $e^{i\theta} \in G$, $\Phi(e^{i\theta}) (= \lim_{r \rightarrow 1} \Phi(re^{i\theta})) \in \partial Q \cap \partial\Delta$, and if $e^{i\theta} \in B$, then $\Phi(e^{i\theta}) \in \partial Q \cap \Delta$.

Since P is a Jordan domain, Ψ has a homeomorphic extension to the boundary. As a matter of fact that extension is Hölder continuous with an exponent $\alpha(\varepsilon)$ which tends to 1 as $\varepsilon \rightarrow 0$. To see this one can verify that the mapping Υ defined below is a quasiconformal mapping from the whole plane onto itself whose maximal dilatation $K(\varepsilon)$ is $K(\varepsilon) = 1 + o(1)$, as $\varepsilon \rightarrow 0$, and which maps the unit disk onto P . By standard results [A2, p. 74-76], [LV, p. 70] the Riemann mapping Ψ admits a $K(\varepsilon)^2$ -quasiconformal extension to the whole plane and, in particular, is Hölder continuous with exponent $\alpha(\varepsilon) = 1/K(\varepsilon)^2$ in a neighborhood of the closed unit disk.

The definition of $\Upsilon : \mathbb{C} \rightarrow \mathbb{C}$ is as follows:

$$\Upsilon(re^{i\theta}) = \begin{cases} re^{i\theta}, & \text{if } e^{i\theta} \in E = \partial\Delta \setminus \bigcup_{j=1}^{\infty} I_j, \ r > 0, \\ rR(\theta)e^{i\theta}, & \text{if } e^{i\theta} \in I_j, \ r > 0, \\ 0, & \text{if } r = 0, \end{cases}$$

where $R(\theta)$ is given by $\omega(R(\theta)e^{i\theta}, I_j) = 1 - \varepsilon$.

Let us consider now the holomorphic function $g : \Delta \rightarrow \Delta$ given by

$$g = \Psi^{-1} \circ b \circ \Phi.$$

Now, g is inner. To see this, suppose not. Then there exists a set $L \subset \partial\Delta \setminus N$ of positive measure such that if $e^{i\theta} \in L$, then $|g(e^{i\theta})| < 1$; and observe that $L \subset G$

and $\Phi(L) \subset \partial\Delta \cap \partial Q$ and that, by Löwner's lemma, $\Phi(L)$ also has positive length. At every point of $\Phi(L)$ the function b has an asymptotic value in Δ which implies by Lindelöf's lemma that $|b(e^{i\theta})| < 1$; but recall that b is inner.

Using Lemmas 1 and 2 and the above remarks about the conformal mapping Ψ we see that

$$(5) \quad \text{Dim}\left(\Phi\left(g^{-1}(\Psi^{-1}(E)) \setminus N\right)\right) \geq \alpha(\varepsilon) \text{Dim}(E).$$

Now, if $e^{i\theta} \in g^{-1}(\Psi^{-1}(E)) \setminus N$, we let $\gamma(t) = \Phi(te^{i\theta})$. Since $\Psi(g(te^{i\theta})) = b(\gamma(t))$ and since $\lim_{t \rightarrow 1} \Psi(g(te^{i\theta}))$ exists and belongs to E , one obtains first that $\lim_{t \rightarrow 1} \gamma(t) = \Phi(e^{i\theta})$ exists and then, by Lindelöf's lemma, that b has radial limit at $\Phi(e^{i\theta})$ and that this limit belongs to E . But $\gamma \subset Q$ and therefore by Lemma 3, $b(r\Phi(e^{i\theta})) \in P(E, \varepsilon')$, for each $\varepsilon' < \varepsilon/2$, and each r , with $0 \leq r < 1$. Consequently, by Lemma 4 and the fact that F is Lipschitz with constant 1 in the Poincaré metric, $d_f(\Phi(e^{i\theta})) \leq k + h(\varepsilon/2)$. We conclude from (5) that

$$\text{Dim}\{\varphi \in [0, 2\pi] : d_f(e^{i\varphi}) < +\infty\} \geq \alpha(\varepsilon) \text{Dim}(E).$$

Keeping E fixed, letting $\varepsilon \rightarrow 0$, and using (3) one gets

$$\text{Dim}\{\varphi \in [0, 2\pi] : d_f(e^{i\varphi}) < +\infty\} \geq \delta(\mathcal{R}) - \eta.$$

Finally, one gets (2) by letting $\eta \rightarrow 0$. □

4

We say that a nonconstant holomorphic function f from the disk into a planar domain Ω is an *inner function into* Ω if the set of θ 's such that $\lim_{r \rightarrow 1} f(re^{i\theta})$ exists and belongs to Ω has zero length. If $\Omega = \Delta$ this agrees with the usual definition. Also, covering maps of domains of \mathbb{C} are always inner functions. One has the following

Corollary of proof. *For a domain Ω , not conformal to the punctured disk, and a function f inner into Ω ,*

$$\text{Dim}(\{\theta \in [0, 2\pi] : \lim_{r \rightarrow 1} f(re^{i\theta}) \text{ does not exist}\}) \geq \delta(\Omega).$$

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