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ON A POLYNOMIAL INEQUALITY OF KOLMOGOROFF'S TYPE

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ABSTRACT. We prove an inequality of the form

$$\|f^{(j)}\|^2 \le A\|f^{(m)}\|^2 + B\|f\|^2$$

for polynomials of degree n and any fixed $0 < j < m \le n$. Here $\|\cdot\|$ is the L_2-norm on $(-\infty, \infty)$ with a weight e^{-t^2}. The coefficients A and B are given explicitly and depend on j, m and n only. The equality is attained for the Hermite orthogonal polynomials $H_n(t)$.

1. INTRODUCTION

Kolmogoroff established in [1] an inequality of the form

$$\|f^{(j)}\|_\infty \le K(j,n)\|f^{(n)}\|_\infty^{j/n}\|f\|_\infty^{1-j/n} \tag{1}$$

for every sufficiently smooth function f on $(-\infty, \infty)$ with the best possible constant $K(j, n)$. Similar inequalities have been proved later for other norms and intervals (see Korneichuk [2] and Tikhomirov [4] for references). As much as we know there is not a Kolmogoroff type result for algebraic polynomials of fixed degree. One may consider (1) as an estimation of the norm of $f^{(j)}$ in terms of the norms of $f^{(n)}$ and f. Then it is natural to look for some other, say linear, expression of the bound. Such inequalities for algebraic polynomials of degree n were studied by Varma [5], where exact bounds of the form

$$\|f^{(j)}\|^2 \le A\|f^{(m)}\|^2 + B\|f\|^2 \tag{2}$$

were found for $m = 2, 3, 4$ and $0 < j < m \le n$. Here, as everywhere in this paper,

$$\|f\| := \left\{\int_{-\infty}^{\infty} e^{-t^2} f^2(t)\,dt\right\}^{1/2}.$$

The purpose of this note is to give a family of inequalities of form (2) for each $m \le n$.

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2. The result

The Hermite polynomials

$$H_n(t) := (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$$

play an important role in our study. Recall that

$$\int_{-\infty}^{\infty} e^{-t^2} H_k(t) H_i(t)\,dt = 0 \quad \text{for } i \neq k,$$

$$\int_{-\infty}^{\infty} e^{-t^2} H_k^2(t)\,dt = \sqrt{\pi} \cdot 2^k k!, \tag{3}$$

$$H_k'(t) = 2k H_{k-1}(t). \tag{4}$$

Denote by π_n the set of all algebraic polynomials of degree n. We shall prove here the following.

Theorem 1. *Let j, m and n be arbitrary integer numbers satisfying the requirement $0 < j < m \leq n$. Then the inequality*

$$\|f^{(j)}\|^2 \leq A \|f^{(m)}\|^2 + \left\{ 2^j \binom{n}{j} j! - A 2^m \binom{n}{m} m! \right\} \|f\|^2 \tag{5}$$

holds for every $f \in \pi_n$ and any A such that

$$A \leq \frac{j}{m 2^{m-j}} \cdot \frac{1}{(n-j) \cdots (n-m+1)}.$$

Moreover, by choosing $f = H_n$ we obtain equality in (5).

The proof is based on the following simple fact.

Lemma 1. *Suppose that the inequality* (2) *holds for $f = H_k$, $k = 0, 1, \ldots, n$. Then it holds for every $f \in \pi_n$.*

Proof. Note first that in view of (4), $\{H_k^{(i)}(t)\}_{k=0}^n$ $(0 \leq i < n)$ form an orthogonal system on $(-\infty, \infty)$ with the same weight e^{-t^2}. Then, for $f \in \pi_n$,

$$\begin{aligned}
\|f^{(i)}\|^2 &= \int_{-\infty}^{\infty} e^{-t^2} \left[\sum_{k=0}^n a_k(t) H_k^{(i)}(t) \right]^2 dt \\
&= \int_{-\infty}^{\infty} e^{-t^2} \sum_{k=0}^n a_k^2(t) [H_k^{(i)}(t)]^2\,dt \\
&= \sum_{k=0}^n a_k^2(f) \|H_k^{(i)}\|^2.
\end{aligned}$$

Assume now that

$$\|H_k^{(j)}\|^2 \leq A \|H_k^{(m)}\|^2 + B \|H_k\|^2$$

for $k = 0, \ldots, n$. Multiplying by $a_k^2(t)$ and summing from 0 to n, we get

$$\sum_{k=0}^n a_k^2(t) \|H_k^{(j)}\|^2 \leq A \sum_{k=0}^n a_k^2(t) \|H_k^{(m)}\|^2 + B \sum_{k=0}^n a_k^2(t) \|H_m^{(m)}\|^2$$

which is just (2), according to our first observation. The proof is complete.

For the sake of simplicity, we separate an important part of the proof of our result in the following lemma.

Set

$$\varphi_j(x) := x(x-1)\dots(x-j+1).$$

Lemma 2. *Suppose that* $0 < j < m \le n$. *Then*

$$R(x) := \frac{\varphi_j(n) - \varphi_j(x)}{\varphi_m(n) - \varphi_m(x)}$$

is a decreasing function of x *in* $[m-1, n]$.

Proof. We have to show that $R(x) < R(x-\delta)$ for every $x \in (m-1, n]$ and any sufficiently small $\delta > 0$. Note first that for positive a, b, a_1 and b_1, the inequality

$$\frac{a + a_1}{b + b_1} > \frac{a}{b}$$

holds if and only if $\frac{b_1}{a_1} < \frac{b}{a}$. Therefore, choosing

$$\begin{aligned} a &:= \varphi_j(n) - \varphi_j(x), \qquad & a_1 &= \varphi_j(x) - \varphi_j(x-\delta), \\ b &:= \varphi_m(n) - \varphi_m(x), \qquad & b_1 &= \varphi_m(x) - \varphi_m(x-\delta), \end{aligned}$$

we conclude that $R(x) < R(x-\delta)$ if $\frac{b_1}{a_1} < \frac{b}{a}$. Note now that the zeros of $\varphi_m(t)$ and $\varphi_j(t)$ lie in $[0, m-1]$. Hence $\varphi'_m(t) \ne 0$, $\varphi'_j(t) \ne 0$ for $t > m-1$. Then we can apply the Generalized Mean Value Theorem and get

$$\begin{aligned} \frac{\varphi_m(n) - \varphi_m(x)}{\varphi_j(n) - \varphi_j(x)} &= \frac{\varphi'_m(\xi)}{\varphi'_j(\xi)} \quad \text{for some } \xi \in (x, n), \\ \frac{\varphi_m(x) - \varphi_m(x-\delta)}{\varphi_j(x) - \varphi_j(x-\delta)} &= \frac{\varphi'_m(\eta)}{\varphi'_j(\eta)} \quad \text{for some } \eta \in (x-\delta, x). \end{aligned}$$

So, the lemma will be proved if we show that the ratio

$$r(t) := \frac{\varphi'_m(t)}{\varphi'_j(t)}$$

is an increasing function of t on $(m-1, n)$. Since

$$r'(t) = \frac{1}{[\varphi'_j(t)]^2} \left[\frac{\varphi''_m(t)}{\varphi'_m(t)} - \frac{\varphi''_j(t)}{\varphi'_j(t)} \right] \varphi'_m(t) \cdot \varphi'_j(t),$$

$r'(t)$ will be positive on $(m-1, n)$ if

$$\frac{\varphi''_m(t)}{\varphi'_m(t)} > \frac{\varphi''_j(t)}{\varphi'_j(t)}. \tag{6}$$

Now let us prove this inequality. Denote by $\{x_{m,i}\}_{i=1}^{m-1}$ and $\{x_{j,i}\}_{i=1}^{j-1}$ the zeros of $\varphi'_m(t)$ and $\varphi'_j(t)$, respectively. It follows from the Rolle theorem that

$$\begin{gathered} 0 < x_{m1} < 1 < x_{m2} < 2 < \dots < m-2 < x_{m,m-1} < m-1, \\ 0 < x_{j1} < 1 < x_{j2} < 2 < \dots < j-2 < x_{j,j-1} < j-1. \end{gathered}$$

Therefore

$$x_{j,i} < x_{m,i+1} \quad \text{for } i = 1, \dots, j-1.$$

Then, for $t > m-1$,

$$t - x_{j,i} > t - x_{m,i+1}$$

and consequently

$$\frac{1}{t - x_{j,i}} < \frac{1}{t - x_{m,i+1}}.$$

Summing these inequalities for $i = 1, \dots, j-1$, we get

$$\frac{\varphi_j''(t)}{\varphi_j'(t)} = \sum_{i=1}^{j-1} \frac{1}{t - x_{j,i}} < \sum_{i=1}^{j-1} \frac{1}{t - x_{m,i+1}} < \sum_{i=1}^{m-1} \frac{1}{t - x_{m,i}} = \frac{\varphi_m''(t)}{\varphi_m'(t)}$$

and (6) is established. This completes the proof of the lemma. □

Proof of Theorem 1. Set

$$B := 2^j \varphi_j(n) - A 2^m \varphi_m(n).$$

According to Lemma 1, it is sufficient to prove that

$$\|H_k^{(j)}\|^2 \le A \|H_k^{(m)}\|^2 + B \|H_k\|^2 \tag{7}$$

for $k = 0, \dots, n$. But in view of the properties (3) and (4) of the Hermite polynomials,

$$\frac{\|H_k^{(i)}\|^2}{\|H_k\|^2} = 2^i k(k-1)\dots(k-i+1) = 2^i \varphi_i(k).$$

Thus if we divide both sides of (7) by $\|H_k\|^2$ and insert there the expression for B we will get the following equivalent form of (7)

$$A[\varphi_m(n) - \varphi_m(k)] \le 2^{j-m}[\varphi_j(n) - \varphi_j(k)]. \tag{8}$$

We should remark here that (7) takes this form only for $k = m, \dots, n$ since $H_k^{(m)}(t) \equiv 0$ for $k < m$.

It is clear now that (8), and hence (7), turns into an equality for $k = n$. Thus A must satisfy (8) for $k = m, m+1, \dots, n-1$. Observe that $\varphi_m(n) > \varphi_m(x)$ for $x \in (m-1, n)$ since the zeros of $\varphi_m(t)$ are at the points $0, 1, \dots, m-1$. Then we can rewrite (8) as

$$A \le \frac{1}{2^{m-j}} \frac{\varphi_j(n) - \varphi_j(k)}{\varphi_m(n) - \varphi_m(k)} = \frac{R(k)}{2^{m-j}}, \qquad k = m, \dots, n-1. \tag{9}$$

But according to Lemma 2, $R(n-1) < R(n-2) < \dots < R(m)$. Thus (9) will be satisfied for $k = m, \dots, n-1$ if

$$A \le \frac{R(n-1)}{2^{m-j}}. \tag{10}$$

This is one of the restrictions on A. Let us see what happens if $k = 0, \dots, m-1$. If $k = 0, 1, \dots, j-1$, then (7) simply implies $B \ge 0$, which yields

$$A \le \frac{1}{2^{m-j}} \cdot \frac{\varphi_j(n)}{\varphi_m(n)}.$$

For $k = j, j+1, \dots, m-1$ we get

$$\|H_k^{(j)}\|^2 \le B \|H_k\|^2,$$

which can be written as

$$B = 2^j \varphi_j(n) - A2^m \varphi_m(n) \geq 2^j \varphi_j(k).$$

This implies the next restriction on A,

$$A \leq \frac{1}{2^{m-j}} \cdot \frac{\varphi_j(n) - \varphi_j(k)}{\varphi_m(n)}, \qquad k = j, \dots, m-1. \tag{11}$$

Since $\varphi_m(m-1) = 0$ and $\varphi_j(k) \leq \varphi_j(m-1)$, if $k \leq m-1$, we have

$$\begin{aligned}\frac{\varphi_j(n) - \varphi_j(k)}{\varphi_m(n)} &= \frac{\varphi_j(n) - \varphi_j(k)}{\varphi_m(n) - \varphi_m(m-1)} \geq \frac{\varphi_j(n) - \varphi_j(m-1)}{\varphi_m(n) - \varphi_m(m-1)} \\ &= R(m-1) \geq R(n-1).\end{aligned}$$

Thus the restrictions (11), together with the previous one corresponding to $k = 0, \dots, j-1$, are weaker than (10). Therefore (7) will hold for each A satisfying the condition

$$A \leq \frac{R(n-1)}{2^{m-j}} = \frac{j}{m2^{m-j}} \cdot \frac{1}{(n-j)\dots(n-m+1)}$$

and $B = 2^j \varphi_j(n) - A2^m \varphi_m(n) = 2^j \binom{n}{j} j! - A2^m \binom{n}{m} m!$. The proof is completed. □

3. Particular cases

Let us mention here some interesting particular cases of the inequality (5).

(a) Choose $A = 0$. Then (5) becomes

$$\|f^{(j)}\| \leq \sqrt{2^j \varphi_j(n)} \|f\|, \qquad j = 0, 1, \dots, n,$$

which is the known inequality of Schmidt [3].

(b) In case

$$A = \frac{1}{2^{m-j}} \cdot R(n-1),$$

which is the greatest possible value of the parameter A, we get

$$\|f^{(j)}\|^2 \leq \frac{j}{m2^{m-j}} \cdot \frac{1}{(n-j)\dots(n-m+1)} \cdot \|f^{(m)}\|^2 + 2^j \binom{n}{j} j! \left\{1 - \frac{j}{m}\right\} \|f\|^2.$$

(c) Let us choose

$$A = \frac{R(n)}{2^{m-j}}.$$

Such a choice is admissible since $R(n) < R(n-1)$. We have

$$R(n) = \lim_{x \to n} \frac{\varphi_j(n) - \varphi_j(x)}{\varphi_m(n) - \varphi_m(x)} = \frac{\varphi_j'(n)}{\varphi_m'(n)}$$

and thus

$$\|f^{(j)}\|^2 \leq \frac{1}{2^{m-j}} \frac{\varphi_j'(n)}{\varphi_m'(n)} \|f^{(m)}\|^2 + \left\{2^j \varphi_j(n) - 2^j \varphi_m(n) \cdot \frac{\varphi_j'(n)}{\varphi_m'(n)}\right\} \|f\|^2.$$

Inequalities of this form have been obtained by Varma [5] for $m = 2, 3, 4$.

(d) Let $m = n$. Assume that $f \in \pi_{n-1}$. Then $\|f^{(m)}\| = 0$ and for the maximal value of the parameter A the estimation (5) takes the form

$$\|f^{(j)}\|^2 \le 2^j \varphi_j(n-1)\|f\|^2,$$

which is again the exact inequality of Schmidt for $f \in \pi_{n-1}$.

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