PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 2, February 1996

POLYNOMIAL RINGS OVER GOLDIE-KERR COMMUTATIVE RINGS II

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(Communicated by Wolmer V. Vasconcelos)

In memory of Pere Menal

Abstract. An overlooked corollary to the main result of the stated paper (Proc. Amer. Math. Soc. **120** (1994), 989–993) is that any Goldie ring R of Goldie dimension 1 has Artinian classical quotient ring Q, hence is a Kerr ring in the sense that the polynomial ring $R[X]$ satisfies the *acc* on annihilators ($= acc\perp$).

More generally, we show that a Goldie ring R has Artinian Q when every zero divisor of R has essential annihilator (in this case Q is a local ring; see Theorem 1′).

A corollary to the proof is Theorem 2: A commutative ring R has Artinian Q iff R is a Goldie ring in which each element of the Jacobson radical of Q has essential annihilator.

Applying a theorem of Beck we show that any $acc\perp$ ring R that has Noetherian local ring R_p for each associated prime P is a Kerr ring and has Kerr polynomial ring $R[X]$ (Theorem 5).

Introduction

Throughout, R denotes a commutative ring.

It is convenient to state the corollary in generalized form as follows:

1. **Theorem.** *If R is a Goldie ring in which each zero divisor x has essential annihilator $x^\perp$, then R has Artinian quotient ring Q.*

In this case, it follows from Small's theorem [S] that $R[X]$ has Artinian quotient ring.

A ring R has *finite Goldie* (or uniform) *dimension n* if n is the maximal number of nonzero ideals in a direct sum contained in R. Furthermore, R is *Goldie* if R has $acc\perp$ and finite Goldie dimension. *Uniform ring* is another term for a ring with Goldie dimension 1, equivalently, 0 is an irreducible ideal. The *singular ideal* of R is the set

$$Z(R) = \{x \in R \mid x^\perp \text{ is essential}\}.$$

Then $Z(R)$ is contained in the set $z(R)$ of zero divisors. Obviously, when R is uniform, we have that $z(R) = Z(R)$, and then $Z(Q)$ is the set of non-units of Q.

We can sharpen Theorem 1 in this terminology:

Received by the editors April 25, 1994 and, in revised form, August 5, 1994.

1991 *Mathematics Subject Classification.* Primary 13B25, 13CO5, 13EO5, 13H99, 13J10; Secondary 16D90, 16P60, 16S50.

$1'$. **Theorem.** *If R is a Goldie ring, and if $z(R) \subseteq Z(R)$, then Q is an Artinian local ring (hence R is Kerr). Conversely.*[1]

Proof. Q also is a Goldie ring (see [F3]), and $Z(Q)$ is a nil ideal in any $acc\perp$ ring (see, e.g., [F1]). Moreover, every element x in the Jacobson radical $J(Q)$ is a nonunit; hence $x = rs^{-1}$, where $r \in z(R) \subseteq Z(R)$ and $s \in R^{\star}$. Then, x has essential right annihilator $x^{\perp}$ in Q, since $r^{\perp} \cap Q$ is essential in R, a fact that shows that $J(Q) \subseteq Z(Q)$. Thus, $J(Q)$ is nil, so Q is Artinian by Theorem 1.1 of [F2]. Since $J(Q)$ is the set of non-units of Q, it follows that Q is local.

The converse hinges on the fact that if Q is Artinian, then Q is Noetherian and hence Goldie, so R is Goldie. Furthermore, $J(Q)$ is nilpotent, and every nilpotent element x has essential annihilator. (Let I be any nonzero ideal, and $x^n = 0$ where $x^{n-1} \neq 0$. If $x^{\perp} \cap I = 0$, then $xI \neq 0$. Suppose i is least such that $x^i I \neq 0$. Then $x^{\perp} \cap I \supseteq x^i I \neq 0$, a contradiction which shows that $x^{\perp}$ is an essential ideal.) Since $Z(R)$ is nilpotent in an $acc\perp$ ring (loc. cit.), then $J(Q) = Z(Q)$. Since Q is local and $J(Q)$ is nilpotent, then every zero divisor x of R lies in

$$J(Q) \cap R = Z(Q) \cap R = Z(R),$$

so $z(R) \subseteq Z(R)$ as needed. □

The proof of Theorem $1'$ has the corollary.

2. **Theorem.** *A ring R has Artinian Q iff R is a Goldie ring and the Jacobson radical of Q coincides with its singular ideal, that is, $J(Q) = Z(Q)$.*

Proof. If $J(Q) = Z(Q)$, then R $acc\perp \Rightarrow J(Q)$ is nil, so R Goldie $\Rightarrow Q$ is Artinian by Theorem 1.1 of [F2]. Conversely, Q Artinian $\Rightarrow J(Q)$ is nil, hence $J(Q) \subseteq Z(Q)$. But $Z(Q)$ is nil in an $acc\perp$ ring, hence $Z(Q) = J(Q)$. □

In a uniform ring every nonzero ideal is essential, so the theorems each imply that any uniform $acc\perp$ ring R has Artinian Q, but Q is in fact then quasi-Frobenius since Q has simple socle. With this fact as a motivator, we next derive a more general theorem with the same conclusion (Theorem 2).

A ring R is *F-injective* ($= \aleph_0$-injective) if every map $I \to R$ of a finitely generated ideal I is extendable to $R \to R$. (Any FP-injective ring R is F-injective; cf. [F3], p. 189.) Any F-injective ring R coincides with its quotient ring Q. Any valuation ring R has FP-injective Q by a theorem of Facchini ([F-P], p. 96, Corollary 6-10; cf. [F-F]).

3. **Theorem.** *If R is an $acc\perp$ ring with F-injective (e.g. FP or self-injective) quotient ring Q, then R is Kerr, in fact Q is quasi-Frobenius ($= QF$).*

Proof. Every finitely generated ideal I of an F-injective ring R is an annihilator (see, e.g., [F3], p. 189, Prop. 23.21.2). The $acc\perp$ in R implies the $acc\perp$ in Q, and hence Q satisfies the *acc* on finitely generated ideals, so Q is Noetherian. But a Noetherian F-injective ring is self-injective, hence QF. □

4. **Corollary.** *Any uniform $acc\perp$ ring R, e.g. any $acc\perp$ valuation ring, is a Kerr ring. Furthermore Q is Artinian in fact QF.*

Proof. Q is Artinian by Theorem 1, and has Goldie dim$= 1$, hence has simple socle, which by classical ideal theory (cf. Corollary 2 of [F1]) implies that Q is QF. □

[1] Classically, it is known that a ring R has local Q iff the set $z(R)$ is an ideal P. In this case, P is a prime ideal and $Q = R_P$ is the local ring at P. Any Artinian ring R is a finite product of local Artinian rings. See Theorem 2 in this connection.

When is $R[X]$ Kerr?

We raised the question in [F2]: If R is Kerr, is $R[X]$?[2]

We cited some obvious examples in [F2], e.g. any subring of a Noetherian ring, and mentioned the Camillo-Guralnick theorem which yields an affirmative answer for an algebra over an uncountable field. We next show that Beck's theorem [B] yields another affirmative answer.

5. **Theorem.** *If R is an acc$\perp$ ring and if R_p is Noetherian for every associated prime P, then the same is true of $R[X]$. Furthermore, both R and $R[X]$ have (flat) embeddings into Noetherian rings, hence each is a Kerr ring.*

Proof. In any acc$\perp$ ring R, the set Ass$R$ of associated prime ideals is finite (see [F4], Corollary 3.7 and Theorem 3.6), and obviously $\bigcup_{P\in \text{Ass}R} P$ is the set $z(R)$ of zero divisors (i.e., every $x \in z(R)$ is contained in some $P \in \text{Ass}R$). We can now apply Beck's theorem ([B], Theorem 5.1) to conclude that R has a flat embedding in a Noetherian ring T, and hence both R and $R[X]$ are Kerr, since $R[X]$ is contained in a Noetherian ring $T[X]$.

Next, contraction induces a 1-1 correspondence Ass$R[X] \to$ Ass$R$ under various conditions including *acc*$\perp$ in R ([F4], Theorem 3.12 B; any *acc*$\perp$ ring is trivially a zip ring in the terminology employed there).

Thus, if $P \in \text{Ass}R[X]$, then

$$P_0 = P \cap R \in \text{Ass}R$$

and

$$R[X]_P = R_{P_0}[X]_{PR_P[X]} = R_M[X]_{PR_M[X]}$$

which holds for *any* prime ideal P of $R[X]$, and M any maximal ideal containing $P_o = P \cap R$. (See, e.g., [H], p.73, Lemma 13.1.)

In particular, this shows that

$$R_{P_0} \text{ Noetherian} \Rightarrow R[X]_P \text{ Noetherian},$$

so $R[X]$ has the stated property (this also follows from the theorem of Beck since $R[X] \hookrightarrow T[X]$ is a flat embedding in a Noetherian ring). □

Since a Kerr ring need not embed in a Noetherian ring, this shows that a Kerr ring R does not in general localize to Noetherian rings at associated primes.

6. **Corollary.** *If R satisfies the hypothesis of Theorem 5, then so does the infinite polynomial ring $S = R[x_1, \dots, x_n, \dots]$, that is, S has a flat embedding in a Noetherian ring, hence is Kerr.*

Proof. By Theorem 5, R has a flat embedding in a Noetherian ring T, and by a remark of D. D. Anderson (cited in [C], p. 75, Remark 2), $A = T[x_1, \dots, x_n, \dots]$ localized at the ideal P consisting of all polynomials in A of content 1 is a Noetherian ring. Thus, since $A \hookrightarrow A_P$ is a flat embedding in a Noetherian ring, $S \hookrightarrow A_P$ is, also. □

[2] In the meanwhile Cedó and Herbera have found a ring R over which the polynomial ring in n variables is Kerr but that in $n+1$ variables is not (Fax of November 1994). See [C-H].

Note added in proof

The author has discovered an error in the proof of Theorem 2.2 in [F2]. The first sentence should read:

"If I is an ideal of R, then I is an annihilator of R iff IQ is an annihilator of Q and $IQ \cap R = I$."

The third sentence should read:

"This also implies that if K is an ideal of Q, then $K \cap R \in \operatorname{Ass} R$ iff $K \in \operatorname{Ass} Q$."

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