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## CURVATURE AND FINITE DOMINATION

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ABSTRACT. Upper bounds obtained by Gromov on the Betti numbers of certain closed Riemannian manifolds are shown to be upper bounds on the minimum number of cells in  $CW$ -spaces dominating such manifolds.

In [Gro1], Gromov obtains a bound on the sum of the Betti numbers of a closed Riemannian manifold  $V$  in terms of a lower bound on the sectional curvature and an upper bound on the diameter. In more detail: Fix a field  $F$ , and let  $\beta_i = \beta_i(V; F)$  be the dimension over  $F$  of  $H_i(V; F)$ . Let  $D = D(V)$  be the diameter of  $V$ .

**Theorem A** [Gro1]. *There exists a constant  $\mathcal{C} = \mathcal{C}(n)$  such that every closed connected Riemannian  $n$ -manifold  $V$  satisfies*

$$\sum_0^n \beta_i \leq \mathcal{C}^{1+\kappa D}$$

*provided the sectional curvature of  $V$  is bounded from below by  $-\kappa^2$ , where  $\kappa \geq 0$ .*

**Corollary.** *If  $V$  has non-negative sectional curvature, then the sum of the Betti numbers is  $\leq \mathcal{C}$ .*

A stronger theorem can be obtained with a small change in Gromov's proof.  
*Terminology:* For spaces  $X$  and  $Y$ , we say that  $X$  dominates  $Y$  if there exist maps

$$Y \xrightarrow{i} X \xrightarrow{r} Y$$

such that  $ri$  is homotopic to the identity.

**Theorem B.** *There exists a constant  $\mathcal{C} = \mathcal{C}(n)$  such that every closed connected Riemannian  $n$ -manifold  $V$  can be dominated by a  $CW$ -space  $X$  having at most*

$$\mathcal{C}^{1+\kappa D}$$

*cells. (Assume as before that the sectional curvature of  $V$  is bounded from below by  $-\kappa^2$ , where  $\kappa > 0$ .)*

Note that Theorem B implies an upper bound for the minimum number of generators of  $\pi_1(V)$ . This is in agreement with [Gro2], but less explicit.

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The referee has asked me to point out that Theorem B explains better than Theorem A how a lower bound on the sectional curvatures of  $V$  restricts the topological complexity of  $V$ . It gives an upper bound on the minimum *number* of cells in a CW-space  $X$  dominating  $V$ , but there is no bound for the complexity of the attaching maps for the cells of  $X$ . For example, in dimension 3 there are infinitely many different homology types of compact Riemannian manifolds of constant sectional curvature 1 (lens spaces). In dimension 7, there are infinitely many different homology types of compact *simply connected* Riemannian manifolds having strictly positive sectional curvature (the examples of Allof and Wallach [AlWa]).

The referee has also drawn my attention to [Abr1] and [Abr2]. Abresch extended Gromov's result to asymptotically non-negatively curved manifolds (which are complete by definition, but not always closed). He obtained more explicit bounds on the Betti numbers. His result is

$$(*) \quad \sum_i \beta_i(V^n) \leq c(n) \cdot \exp\left(\frac{15n-13}{4} \cdot b_1(V^n)\right)$$

where  $b_1(V^n)$  is a real number (not a Betti number) measuring to some extent the “amount” of negative curvature in  $V$  and, in Abresch's own words,

*the function  $c(n)$  can be effectively estimated by  
an expression which grows exponentially in  $n^3$ .*

If  $V$  is closed, with diameter  $D$  and sectional curvature bounded from below by  $-\kappa^2$  everywhere, then  $b_1(V) \leq \kappa D$  by [Abr1, 2.3].

Again, a small change in Abresch's proof shows that inequality  $(*)$  and the estimate for  $c(n)$  remain correct if the sum of the Betti numbers is replaced by the minimum number of cells in a CW-space dominating  $V$ .

The changes should be made in §2.3 of [Gro1], and in §1 of [Abr2]. This is where the *Leray spectral sequence* appears. Gromov refers to [Groth] for details. Grothendieck's account is of course “homological”. A more geometric explanation of the Leray spectral sequence (using *homotopy direct limits*) is available. This is where we start.

## 1. THE LERAY SPECTRAL SEQUENCE

Let  $\mathcal{A}$  be a simplicial complex. We shall regard  $\mathcal{A}$  as a category: objects are the simplices of  $\mathcal{A}$ , and morphisms are the inclusion maps. For a contravariant functor  $\mathcal{Z}$  from  $\mathcal{A}$  to the category of spaces, let

$$|\mathcal{Z}| := \left( \coprod_{s \subset \mathcal{A}} \mathcal{Z}(s) \times s \right) / \sim$$

where the coproduct runs over all simplices  $s \subset \mathcal{A}$  and  $\sim$  stands for the “usual” relations,  $(f^*a, b) \sim (a, b)$  whenever  $a \in \mathcal{Z}(s')$ ,  $b \in s$ , and  $f : s \hookrightarrow s'$ . Note that  $|\mathcal{Z}|$  projects to  $\mathcal{A}$  by  $(a, b) \mapsto b$ . The inverse image of the  $k$ -skeleton of  $\mathcal{A}$  under this map is the *vertical  $k$ -skeleton* of  $|\mathcal{Z}|$ , denoted by  $|\mathcal{Z}|(k)$ .

The construction  $|\mathcal{Z}|$  is a special case of a *homotopy direct limit*. The notion goes back to [Se] and the standard reference is [BK].

With  $\mathcal{A}$  and  $\mathcal{Z}$  as above, let  $X$  be a space and let  $\tau : \mathcal{Z} \longrightarrow X$  be a natural transformation (where we think of  $X$  as a constant functor on  $\mathcal{A}$ ). This induces a map

$$\tau_* : |\mathcal{Z}| \longrightarrow X; \quad (a, b) \mapsto \tau(a).$$

**Example 1.2.** Let  $X$  be a *CW-space*, and let  $\{X_\gamma \mid \gamma \in \Gamma\}$  be a collection of *CW*-subspaces of  $X$ . Let  $\mathcal{A}$  be the *nerve* of the collection of subspaces: i.e.,  $\mathcal{A}$  has one  $k$ -simplex for each finite subset  $\Gamma' \subset \Gamma$  such that

$$\bigcap_{\gamma \in \Gamma'} X_\gamma \neq \emptyset.$$

Of course, we let  $\mathcal{Z}(s) = \bigcap X_\gamma$  where the intersection is taken over all  $\gamma$  which are vertices of  $s$ . The inclusions  $\mathcal{Z}(s) \subset X$  define a natural transformation

$$\tau : \mathcal{Z} \longrightarrow X.$$

If  $X$  is the union of the  $X_\gamma$ , then  $\tau_* : |\mathcal{Z}| \longrightarrow X$  is a homotopy equivalence.

The proof consists essentially in showing that the fibers of  $\tau_*$  are contractible spaces. For  $x \in X$ , the fiber of  $\tau_*$  over  $x$  is homeomorphic to the full simplex spanned by vertices  $\gamma$  such that  $x \in X_\gamma$ .

**Example 1.3.** Let  $X$  be a smooth  $n$ -manifold, and let  $\{X_\gamma \mid \gamma \in \Gamma\}$  be a collection of open subsets of  $X$ . Define  $\mathcal{A}$ ,  $\mathcal{Z}$  and  $\tau$  as before. If  $X$  is the union of the  $X_\gamma$ , then  $\tau_* : |\mathcal{Z}| \longrightarrow X$  is a homotopy equivalence.

The proof is by reduction to the previous example (use triangulations of  $X$ ). Details are left to the reader. The smoothness assumption is unnecessary, but it makes the proof easier.

In the situation of 1.2 or 1.3, assuming that  $X$  is the union of the  $X_\gamma$ , we have the canonical filtration of  $|\mathcal{Z}|$  by vertical skeletons  $|\mathcal{Z}|(k)$ . Now a filtration of a space *always* gives rise to a filtration of its singular chain complex, and then to a spectral sequence converging to the homology of the space. Here we obtain a spectral sequence converging to the homology of  $|\mathcal{Z}|$ , which is the homology of  $X$ . This is the *Leray* spectral sequence. We are not going to use it. We will use the filtration of  $|\mathcal{Z}|$  by vertical skeletons.

## 2. CELL CONTENT

The reader should now have [Gro1] before his/her eyes, more specifically, §2.3 of [Gro1]. First a remark on terminology: As I understand it, Gromov means by a *ball* in the Riemannian manifold  $V$  a certain open subset  $B = B(x, R)$  of  $V$ , equipped with the (sometimes additional) structure of a center  $x$  and radius  $R$ . For example, if  $V$  is closed and  $D$  is the diameter of  $V$ , then  $B(x, 2D)$  and  $B(x, 10D)$  must be regarded as different balls in  $V$ , although the underlying subsets of  $V$  are both equal to  $V$ . If  $B = B(x, R)$  is a ball in  $V$ , and  $\lambda$  is a positive real number, then  $\lambda B$  denotes the ball  $B = B(x, \lambda R)$ .

In §2.3, Gromov defines the *content* of a ball  $B$  in  $V$  as the rank of the inclusion homomorphism

$$H_*(\frac{1}{5}B; F) \longrightarrow H_*(B; F).$$

Further, he writes:

**Quotation 2.1.** “Take a ball  $B$  and cover the concentric ball  $\frac{1}{5}B$  by some open balls  $B_i$ , where  $i = 1, \dots, N$ , all of the same radius. Consider also the concentric coverings  $\{\lambda_j B_i\}$ , where  $j = 0, 1, \dots, n+1$  and  $\lambda_j = 10^j$ . Suppose that all balls

$5\lambda_j B_i$  (where  $j = 0, \dots, n+1$  and  $i = 1, \dots, N$ ) are contained in  $B$ , and let the contents of these balls be bounded by a constant  $p$ , that is

$$\text{Cont}(5\lambda_j B_i) \leq p.$$

Denote by  $J$  the index [Gro1,2.2] of the system  $\{5\lambda_{n+1} B_i\}$ , where  $i = 1, \dots, N$ .

The content of  $B$  satisfies the following inequality:

$$\text{Cont}(B) \leq (n+1)pJ.$$

**Definition 2.2.** The *cell content* of a ball  $B$  in  $V$  is  $\leq q$  if there exists a CW-space  $Y$  with at most  $q$  cells, and maps

$$\frac{1}{5}B \xrightarrow{f} Y \xrightarrow{g} B$$

such that  $gf$  is homotopic to the inclusion.

**Lemma 2.3.** Keeping the hypotheses of 2.1 in all other respects, suppose that the cell contents of the balls  $5\lambda_j B_i$  (where  $j = 0, \dots, n$  and  $i = 1, \dots, N$ ) are bounded by a constant  $p$ . Then the cell content of  $B$  is not greater than  $pJ$ .

(This will be proved in the next section.) Now return to the assumptions and notation of Theorem B above; in particular, let  $D$  be the diameter of  $V$ . Lemma 2.3 implies, by arguments identical with Gromov's, that for any  $x \in V$  the cell content of  $B(x, 10D)$  is bounded by  $C^{1+\kappa D}$  for suitable  $C$  independent of  $V$  (but depending on the dimension  $n$ ). Since

$$B(x, 10D) = \frac{1}{5}B(x, 10D) = V \quad \text{"as sets"},$$

this means that  $V$  can be dominated by a cell complex with at most  $C^{1+\kappa D}$  cells.

### 3. PROOF OF THE LEMMA

Using Example 1.3, we can deduce Lemma 2.3 from the following statement.

**Proposition 3.1.** Let  $\mathcal{A}$  be a compact simplicial complex with  $J$  simplices. Let

$$\mathcal{Z}_0 \xrightarrow{T_0} \mathcal{Z}_1 \xrightarrow{T_1} \mathcal{Z}_2 \rightarrow \dots \rightarrow \mathcal{Z}_n \xrightarrow{T_n} \mathcal{Z}_{n+1}$$

be a diagram of functors (contravariant, from  $\mathcal{A}$  to spaces) and natural transformations. Assume that, for each simplex  $s \subset \mathcal{A}$  and each  $j \in \{0, 1, \dots, n\}$ , the map from  $\mathcal{Z}_j(s)$  to  $\mathcal{Z}_{j+1}(s)$  given by  $T_j$  has a (strict) factorization

$$(*) \quad \mathcal{Z}_j(s) \xrightarrow{\alpha_{j,s}} Y_{j,s} \xrightarrow{\beta_{j,s}} \mathcal{Z}_{j+1}(s)$$

where  $Y_{j,s}$  is homotopy equivalent to a CW-space with not more than  $p$  cells. Then the map from  $|\mathcal{Z}_0|(n)$  to  $|\mathcal{Z}_{n+1}|$  induced by  $T_n T_{n-1} \dots T_1 T_0$  has a factorization

$$(**) \quad |\mathcal{Z}_0|(n) \longrightarrow W \longrightarrow |\mathcal{Z}_{n+1}|$$

where  $W$  is homotopy equivalent to a CW-space with not more than  $pJ$  cells.

**Interpretation 3.2.** Let  $\mathcal{A}$  be the nerve of the collection of open sets  $\{\lambda_{n+1}B_i\}$  (notation of 2.1 and 2.3 above). Define  $\mathcal{Z}_j$  by

$$\mathcal{Z}_j(s) = \bigcap_{i \text{ vertex of } s} \lambda_j B_i \quad \text{for } s \subset \mathcal{A} \text{ and } 0 \leq j \leq n+1.$$

The natural transformations  $T_j$  are given by inclusion for  $0 \leq j \leq n$ . Gromov's interpolation argument (in [Gro1, 2.3]) and the assumptions in 2.2 above imply that the factorizations (\*) exist. (They can be made strict by converting certain maps into fibrations.) Therefore the factorization (\*\*) exists. Now

$$|\mathcal{Z}_0| \simeq \bigcup_{1 \leq i \leq N} B_i$$

by 1.3, and the right-hand side contains  $\frac{1}{5}B$ . Similarly

$$|\mathcal{Z}_{n+1}| \simeq \bigcup_{1 \leq i \leq N} \lambda_{n+1} B_i$$

and the right-hand side is contained in  $B$ . We conclude that the inclusion

$$\frac{1}{5}B \hookrightarrow B$$

has a factorization

$$\frac{1}{5}B \longrightarrow W \longrightarrow B$$

where  $W$  is homotopy equivalent to a CW-space with not more than  $pJ$  cells. (Never mind the difference between  $|\mathcal{Z}_0|$  and  $|\mathcal{Z}_0|(n)$ : the inclusion of  $|\mathcal{Z}_0|(n)$  in  $|\mathcal{Z}_0|$  is  $n$ -connected, so any map from an  $n$ -manifold such as  $\frac{1}{5}B$  to  $|\mathcal{Z}_0|$  can be deformed into  $|\mathcal{Z}_0|(n)$ .) This shows that the cell content of  $B$  is at most  $pJ$ .

*Proof of 3.1.* Without loss of generality,  $\dim(\mathcal{A}) \leq n$ , and then  $|\mathcal{Z}_0|(n) = |\mathcal{Z}_0|$ . Define a new contravariant functor  $\mathcal{Y}$  from  $\mathcal{A}$  to spaces by

$$\mathcal{Y}(s) = Y_{i,s} \quad \text{where } i = n - \dim(s).$$

Induced maps are defined as follows. For simplices  $s \subset t$  (proper inclusion) and  $i = n - \dim(s)$  and  $j = n - \dim(t)$  use the composition

$$Y_{j,t} \xrightarrow{\beta_{j,t}} \mathcal{Z}_{j+1}(t) \longrightarrow \mathcal{Z}_i(t) \xrightarrow{\mathcal{Z}_i(s \subset t)} \mathcal{Z}_i(s) \xrightarrow{\alpha_{i,s}} Y_{i,s}$$

where the unlabelled arrow is a specialization of  $T_{i-1} \cdots T_{j+1}$ , or the identity if  $i$  equals  $j+1$ . (Check that this gives a functor.) There are obvious natural transformations

$$\mathcal{Z}_0 \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z}_{n+1}$$

with composition equal to  $T_n T_{n-1} \cdots T_0$ . Hence we have a factorization

$$|\mathcal{Z}_0| \longrightarrow |\mathcal{Y}| \longrightarrow |\mathcal{Z}_{n+1}|$$

of the map induced by  $T_n \cdots T_0$ . To complete the proof, apply the next lemma.

**Lemma 3.3.** *Let  $\mathcal{A}$  be a simplicial complex with  $J$  simplices, and let  $\mathcal{Y}$  be a contravariant functor from  $\mathcal{A}$  to spaces. Assume that each  $\mathcal{Y}(s)$  is homotopy equivalent to a CW-space with at most  $p$  cells. Then  $|\mathcal{Y}|$  is homotopy equivalent to a CW-space with at most  $pJ$  cells.*

*Proof.* Use induction on  $J$ . For the induction step, let  $s$  be a simplex of maximal dimension in  $\mathcal{A}$ , and let  $\mathcal{A}'$  be the complement of the interior of  $s$  in  $\mathcal{A}$ . Let  $\mathcal{Y}'$  be the restriction of  $\mathcal{Y}$  to  $\mathcal{A}'$ . Note that  $|\mathcal{Y}'|$  is the pushout of a diagram

$$\mathcal{Y}(s) \times s \hookrightarrow \mathcal{Y}(s) \times \partial s \rightarrow |\mathcal{Y}'|. \quad \square$$

## 4. CELL RANK

Now switch to §1 of [Abr2]. (I shall use somewhat different notation to be consistent, starting with  $V, U, U_0$  where Abresch writes  $M, X, Y$ , respectively.) For open subsets  $U_0 \subset U$  of the manifold  $V^n$  and  $t > 0$ , Abresch defines

$$\begin{aligned} \text{rk}_j(U, U_0) &:= \text{rank}(H_j(U_0; F) \rightarrow H_j(U; F)) \\ \text{rk}_*^t(U, U_0) &:= \sum_{j \geq 0} \text{rk}_j(U, U_0) \cdot t^j. \end{aligned}$$

Supposing that  $B_i^0 \subset B_i^1 \subset \cdots \subset B_i^{n+1}$ , for  $1 \leq i \leq N$ , are open subsets of  $V$  such that

$$U_0 \subset \bigcup_{i=1}^N B_i^0 \quad \text{and} \quad U \supset \bigcup_{i=1}^N B_i^{n+1},$$

he states the following lemma (which replaces 2.1).

**Quotation 4.1.** *Let  $t > 0$ ,  $t^{-1} \in \mathbb{N}$ , and suppose that any  $B_i^n$  intersects at most  $t^{-1}$  distinct sets  $B_k^n$ ,  $i \neq k$ ; then there holds the following inequality:*

$$\begin{aligned} \text{rk}_*^t(U, U_0) &\leq \text{rk}_*^t\left(\bigcup_{i=1}^N B_i^{n+1}, \bigcup_{i=1}^N B_i^0\right) \\ &\leq (e-1)N \cdot \sup\left\{\text{rk}_*^t\left(\bigcap_{i \in S} B_i^{j+1}, \bigcap_{i \in S} B_i^j\right) \mid 0 \leq j \leq n, \emptyset \neq S \subset \{1, \dots, N\}\right\}. \end{aligned}$$

**Definition 4.2.** For a compact CW-space  $Y$  and  $t > 0$ , define

$$\sharp^t(Y) := \sum_{j \geq 0} (\text{number of } j\text{-cells in } Y) \cdot t^j.$$

For open subsets  $U_0 \subset U$  in  $V$  and  $t > 0$ , let  $\text{crk}_*^t(U, U_0)$  be the minimum of all numbers  $q \in \mathbb{N}$  such that there exist maps

$$U_0 \xrightarrow{f} Y \xrightarrow{g} U$$

where  $Y$  is a compact CW-space with  $\sharp^t(Y) \leq q$  and  $gf$  is homotopic to the inclusion. If there is no such  $q$  let  $\text{crk}_*^t(U, U_0) = \infty$ .

**Lemma 4.3.** *With  $\text{rk}_*$  replaced by  $\text{crk}_*$  throughout, the inequality in 4.1 remains correct.*

*Proof.* Let  $\mathcal{A}$  be the nerve of the collection of open sets  $\{B_i^{n+1}\}$ , where  $1 \leq i \leq N$ . As in 3.2, define contravariant functors  $\mathcal{Z}_j$  from  $\mathcal{A}$  to spaces:

$$\mathcal{Z}_j(s) = \bigcap_{i \text{ vertex of } s} B_i^j$$

where  $0 \leq j \leq n+1$ . As in 3.2, there are natural transformations  $\mathcal{Z}_j \rightarrow \mathcal{Z}_{j+1}$  for  $0 \leq j \leq n$ , given by inclusion. Let

$$p = \sup\left\{\text{crk}_*^t\left(\bigcap_{i \in S} B_i^{j+1}, \bigcap_{i \in S} B_i^j\right) \mid 0 \leq j \leq n, \emptyset \neq S \subset \{1, \dots, N\}\right\}.$$

As in 3.1 (\*\*) and the proof of 3.1, we can construct a factorization

$$|\mathcal{Z}_0|(n) \longrightarrow W \longrightarrow |\mathcal{Z}_{n+1}|$$

where  $W = |\mathcal{Y}|$  is the geometric realization of a contravariant functor  $\mathcal{Y}$  from the  $n$ -skeleton  $\mathcal{A}^n$  to spaces, and  $\mathcal{Y}(s)$  is homotopy equivalent to a CW-space  $X(s)$  such that

$$\sharp^t(X(s)) \leq p$$

for every face  $s \subset \mathcal{A}^n$ . We now have to show that

$$\text{crk}_*^t(|\mathcal{Y}|) \leq (e-1)Np.$$

To this end we show first that  $\sharp^t(\mathcal{A}^n) \leq (e-1)N$ , using the hypotheses in , 4.1 ; this is actually carried out in [Abr2, p.479]. (Beware that our  $t$  is Abresch's  $t^{-1}$ .) Then we finish with a variation on 3.3:

**Lemma 4.4.** *Let  $\mathcal{B}$  be a compact simplicial complex with  $\sharp^t(\mathcal{B}) = J$  and let  $\mathcal{Y}$  be a contravariant functor from  $\mathcal{B}$  to spaces. Assume that each  $\mathcal{Y}(s)$  is homotopy equivalent to a CW-space  $X(s)$  with  $\sharp^t(X(s)) \leq p$ . Then  $|\mathcal{Y}|$  is homotopy equivalent to a CW-space  $X$  with  $\sharp^t(X) \leq pJ$ .*

The proof is by induction on the number of simplices in  $\mathcal{B}$ , like that of 3.3.  $\square$

## 5. BIG SPACES WITH SMALL HOMOLOGY

Here is an example showing that Theorem B is stronger than Theorem A. Let  $M$  be a square matrix (size  $k \times k$ ) with integer entries such that both  $M$  and  $M - I_k$  have determinant  $\pm 1$ ; for instance,  $k = 2$  and

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let  $\pi = \mathbb{Z}$ , and let the generator of  $\pi$  act on  $\mathbb{Z}^k$  by  $M$ . Let  $\mu$  be the minimum number of generators of

$$E = \bigoplus_{i=1}^s \mathbb{Z}^k$$

as a  $\pi$ -module. Then  $\mu k \geq s$ , because  $\text{hom}_\pi(E, \mathbb{Z}^k)$  contains a free abelian group of rank  $s$ . Hence  $\mu \geq s/k$ . Let  $X$  be a wedge of  $sk$  spheres of dimension  $d > 1$ , and let  $f : X \rightarrow X$  be a homotopy equivalence such that  $H_d(X)$ , with the action of  $\pi$  determined by  $f_*$ , is isomorphic to  $E$  as a  $\pi$ -module. Finally let  $Y$  be the mapping torus of  $f$ ,

$$Y = X \times [0, 1] / (x, 1) \sim (f(x), 0).$$

Then  $\pi_1(Y) = \pi = \mathbb{Z}$ , and  $H_*(Y; F) \cong H_*(S^1; F)$  for any field  $F$ . But the number of cells in any CW-space dominating  $Y$  is  $\geq \mu$ , which is  $\geq s/k$ , which is as large as we please.

To obtain closed manifold examples of the same type, just make sure that  $Y$  embeds in a high-dimensional euclidean space. Then take a smooth regular neighbourhood and double along the boundary.

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