

## A NOTE ON THE WEIGHTED NORM INEQUALITY FOR THE ONE-SIDED MAXIMAL OPERATOR

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ABSTRACT. Let  $M_g^+$  be the one-sided maximal function. In this note we obtain some necessary and sufficient conditions in order that the weighted weak type inequality holds for  $M_g^+$ . Meanwhile, some necessary or sufficient conditions for the weighted inequality for  $M_g^+$  are given.

### 1. INTRODUCTION

Given a positive and locally integrable function  $g$  on the real line  $R$ , the one-sided maximal function  $M_g^+$  is defined by

$$(1.1) \quad M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x, x+h)} \int_x^{x+h} |f(y)|g(y) dy$$

for  $f \in L_{\text{loc}}^1(g(x) dx)$ , where  $g(x, x+h) = \int_x^{x+h} g(y) dy$ .

Symmetrically, the left maximal function  $M_g^-$  can be defined in the obvious way.

Recently, weighted inequalities for the operators  $M_g^+$  and  $M_g^-$  have been extensively studied (see [9], [6], [5], [4]). In this note we will characterize the pairs of weights  $(w, v)$  such that  $M_g^+$  (or  $M_g^-$ ) is of weak type  $(1, q)$  with  $1 \leq q$ . Secondly, we will give a necessary condition on the pairs  $(w, v)$  in order that the weak type  $(p, q)$  inequality with  $0 < p \leq q < \infty$  holds for  $M_g^+$  (or  $M_g^-$ ) with respect to the measures  $w dx$  and  $v dx$ . This result shows that most weight functions do not verify the weighted weak type  $(p, q)$  inequality when  $1 < p < q < \infty$ . In the case of  $1 \leq p$  and  $0 < q < p$  we will introduce a  $B_g^+(p, q)$  (or  $B_g^-(p, q)$ ) condition and prove that this condition is sufficient for the weak type  $(p, q)$  inequality and necessary for the strong type  $(p, q)$  inequality for the operator  $M_g^+$  (or  $M_g^-$ ). Finally, we will prove analogies of Sawyer's theorem in [10] and Verbitsky's theorem in [11]. They characterize the weak type  $(p, q)$  inequality when  $1 \leq p < \infty$  and  $0 < q < p$ .

Throughout this paper,  $w$  and  $v$  always are weight functions by which we mean nonnegative measurable functions taking values in  $[0, \infty]$ . For a given weight  $w$  and measurable set  $E$ ,  $w(E) = \int_E w(x) dx$  and  $\chi_E(x)$  denotes the characteristic function of  $E$ . Particularly,  $|E|$  is the Lebesgue measure of  $E$  and  $w((a, b))$  is written by  $w(a, b)$ . For  $p \geq 1$ , let  $p' = \frac{p}{p-1}$  be its conjugate index, and we adopt the usual

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conventions for multiplication in  $[0, \infty]$ , namely,  $\infty \cdot t = t \cdot \infty = \infty$  for  $0 < t \leq \infty$ ,  $0 \cdot \infty = \infty \cdot 0 = 0$ ,  $\infty^{-1} = 0$  and  $0^{-1} = \infty$ .

Now we state our results. We just present our theorems concerning  $M_g^+$ , since their counterparts for  $M_g^-$  can be obtained similarly.

**Theorem 1.** *Suppose  $1 \leq q < \infty$ . Then the weak type  $(1, q)$  inequality*

$$(1.2) \quad (\lambda^q w(\{x \in R: M_g^+ f(x) > \lambda\}))^{1/q} \leq C \int_R |f(x)|v(x) dx$$

holds for all  $\lambda > 0$  and measurable  $f$ , if and only if  $(w, v) \in A_g^+(1, q)$ , that is,

$$(1.3) \quad \sup_{a < b < c} \left( \frac{w(a, b)^{1/q}}{g(a, c)} \operatorname{ess\,sup}_{x \in (b, c)} \left( \frac{g}{v} \right) \right) = A < \infty.$$

Moreover, for the best constants  $A$  and  $C$ , we have  $A \leq C \leq 4A$ .

*Remark 1.* The good weights for the weak type  $(p, p)$  inequalities ( $p \geq 1$ ) were established in [6], [9] and [5]. However, in those articles the condition  $A_g^+(1, 1)$  was represented by

$$(1.4) \quad M_g^- \left( \frac{w}{g} \right) (x) \leq C \frac{v(x)}{g(x)} \quad \text{for a.e. } x \in R.$$

Later on, in [4], we established the weak type inequalities for  $M_g^+$  in Orlicz classes. In the particular case of  $1 < p \leq q < \infty$  we obtained the  $A_g^+(p, q)$  condition, i.e.

$$\sup_{a < b < c} \frac{w(a, b)^{1/q} \sigma(b, c)^{1/p'}}{g(a, c)} = A < \infty,$$

where  $\sigma = g^{p'} v^{1-p'}$ . Our  $A_g^+(1, q)$  condition can be considered as the limiting case of  $A_g^+(p, q)$  as  $p \rightarrow 1$ . Furthermore, it has an extended version in the case  $0 < q < p = 1$  (cf. Theorem 4). This sort of  $A_g^+(1, q)$  conditions for  $M_g^+$  are new in the literature, although the parallel conditions for usual Hardy-Littlewood maximal operator are well known.

**Theorem 2.** *Let  $0 < p \leq q < \infty$  and  $(w, v)$  is a pair of weight functions. Suppose the weak type  $(p, q)$  inequality*

$$(1.5) \quad (\lambda^q w(\{x \in R: M_g^+ f(x) > \lambda\}))^{1/q} \leq C \left( \int_R |f(x)|^p v(x) dx \right)^{1/p}$$

holds for all  $\lambda > 0$  and measurable  $f$ . Then we have:

- (i) For a.e.  $x \in R$ , either  $w(x) = 0$  or  $v(x) = \infty$  if  $p < q$ .
- (ii) There exists a constant  $B$  such that

$$(1.6) \quad w(x) \leq Bv(x) \quad \text{a.e. } x \in R$$

if  $p = q$ . Moreover,  $B \leq (2C)^p$  for the best constants  $B$  and  $C$  in (1.6) and (1.5) respectively.

*Remark 2.* Theorem 2 indicates that if  $v$  is finite a.e. (e.g. locally integrable) and  $p < q$ , then the weight  $w$  must be zero a.e. But, the  $A_g^+(p, q)$  ( $1 \leq p \leq q < \infty$ ) always contain nontrivial pairs of weight functions, i.e. neither  $w(x) \equiv 0$  nor  $v(x) \equiv \infty$ . Here we shall give an example.

**Example 1.** Since  $g$  is locally integrable, there exists  $M > 0$  such that

$$|E| = |\{x \in \mathbf{R}: g(x) \leq M\}| > 0.$$

Choose an interval  $(a_1, a_2)$  satisfying  $|(a_1, a_2) \cap E| > 0$ , and fix  $a_3 < a_4 < a_1$  arbitrary. Let  $w(x) = \chi_{[a_3, a_4]}(x), v(x) = 1$  for  $x \in (a_1, a_2) \cap E$  and  $v(x) = \infty$  elsewhere. Then  $(w, v) \in A_g^+(p, q)$  for all  $1 \leq p \leq q < \infty$ .

Indeed, we only need to verify the  $A_g^+(p, q)$  conditions for  $a < b < c$  with  $(a, b) \cap (a_3, a_4) \neq \emptyset$  and  $(b, c) \cap (a_1, a_2) \neq \emptyset$ . In these cases, it is obvious that  $g(b, c) \geq g(a_4, a_1), w(a, b) \leq a_4 - a_3, \sigma(b, c) \leq M^{p'}(a_2 - a_1)(p > 1)$  and  $\text{ess sup}_{x \in (b, c)} \frac{g(x)}{v(x)} \leq M(p = 1)$ . Then the conclusion follows easily.

**Definition 1.** Suppose  $1 \leq p < \infty$  and  $0 < q < p$ . We say that a pair of weight functions  $(w, v) \in B_g^+(p, q)$  if

$$(1.7) \quad \left[ \sum_j \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r \right]^{1/r} \leq B < \infty \quad (\text{when } p > 1),$$

$$(1.8) \quad \left[ \sum_j \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \text{ess sup}_{x \in (b_j, c_j)} \left( \frac{g(x)}{v(x)} \right) \right)^r \right]^{1/r} \leq B < \infty \quad (\text{when } p = 1)$$

for all sequences  $\{(a_j, c_j)\}$  of pairwise disjoint intervals and  $b_j \in (a_j, c_j)$ . Where  $\sigma = \sigma_{g,v,p} = g^{p'} v^{1-p'}$  and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . We shall keep these notations throughout this paper.

**Theorem 3.** Suppose  $1 \leq p < \infty, 0 < q < p$  and  $(w, v)$  is a pair of weight functions. If  $(w, v) \in B_g^+(p, q)$ , then the weak type  $(p, q)$  inequality (1.5) holds for all  $\lambda > 0$  and measurable  $f$ . Moreover, for the best constants  $B$  in Definition 1 and  $C$  in (1.5) we have  $C \leq 2^{2+1/p'} B$ .

On the other hand, if the strong type  $(p, q)$  inequality

$$(1.9) \quad \left( \int_{\mathbf{R}} [M_g^+ f(x)]^q w(x) dx \right)^{1/q} \leq C \left( \int_{\mathbf{R}} |f(x)|^p v(x) dx \right)^{1/p}$$

holds for all measurable  $f$ , then  $(w, v) \in B_g^+(p, q)$  with  $B \leq C$ .

*Remark 3.* We will give some examples of  $B_g^+(p, q)$  functions, which shows that the  $B_g^+(p, q)$  condition is not sufficient for the strong type inequality.

**Example 2.** Suppose  $(u, v) \in A_g^+(p, p)(p \geq 1)$ . Choose  $\Omega \in \mathbf{R}$  with  $u(\Omega) < \infty$ . Let  $w = u\chi_\Omega$ . Then  $(w, v) \in B_g^+(p, q)$  for all  $0 < q < p$ .

**Example 3.** Choose  $\Omega \subset \mathbf{R}$  bounded. Let  $w = g$ , and  $v = g$  on  $\Omega$  and  $\infty$  elsewhere. Then  $(w, v) \in B_g^+(p, q)$  for all  $1 \leq q < p$ . Particularly, setting  $g = 1$  and  $\Omega = [0, 1]$ , we obtain a simple example which satisfies  $B_g^+(p, 1)$  (therefore the weak type  $(p, 1)$  inequality), but the function  $f = \chi_\Omega$  causes the strong type  $(p, 1)$  inequality to fail.

For  $q > 1$ , an example follows from the corresponding example given by Muckenhoupt in the case  $p = q = 2$  (see [7], p. 218).

**Example 4.** Set  $g = 1$ . Let  $w(x) = x \log(-x)$  on  $[-\frac{1}{2}, 0)$  and 0 elsewhere, and  $v(x) = x^2(\log(-x))^4$  on  $[-\frac{1}{2}, 0)$  and  $\infty$  elsewhere. Then  $(w, v) \in B_g^+(3, 2)$ , but the function  $f(x) = -\chi_{[-\frac{1}{2}, 0)}(x)/(x \log^2(-x))$  violates the strong type  $(3, 2)$  inequality.

**Definition 2.** Suppose  $1 \leq p < \infty$  and  $0 < q < p$ . For every  $(a, b) \subset \mathbf{R}$ , let

$$\begin{aligned} \rho_{(a,b)} &= \sup_{c>b} \frac{w(a,b)^{1/p} \sigma(b,c)^{1/p'}}{g(a,c)} & (p > 1), \\ \rho_{(a,b)} &= \sup_{c>b} \frac{w(a,b)}{g(a,c)} \operatorname{ess\,sup}_{x \in (b,c)} \left( \frac{g(x)}{v(x)} \right) & (p = 1). \end{aligned}$$

Define  $\Phi(x) = \Phi_{p,q,w,v}(x) = \sup\{\rho_{(a,b)} \chi_{(a,b)}(x)\}$ .

**Theorem 4.** Suppose  $1 \leq p < \infty$  and  $0 < q < p$ . The following statements are equivalent.

- (i) The weak type inequality (1.5) holds.
- (ii) There exists a constant  $B > 0$  such that for every sequence  $\{(a_j, b_j)\}$  of pairwise disjoint  $(a_j, b_j)$  and  $c_j > b_j$  arbitrary, the inequality

$$w \left( \bigcup_j (a_j, b_j) \right)^{1/q} \leq B \left\{ \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p \right\}^{1/p} \quad (p > 1)$$

or

$$w \left( \bigcup_j (a_j, b_j) \right)^{1/q} \leq B \sum_j \frac{g(a_j, c_j)}{\operatorname{ess\,sup}_{x \in (b_j, c_j)} \left( \frac{g(x)}{v(x)} \right)} \quad (p = 1)$$

holds.

- (iii) The function  $\Phi(x)$  defined in Definition 2 is in  $L^{r,\infty}(w)$ , i.e.

$$\|\Phi\|_{r,\infty,w} = \sup_{\lambda>0} \lambda w(\{|\Phi(x)| > \lambda\})^{1/r} < \infty.$$

Moreover, for the best constants  $B$  and  $C$  we have

$$2^{-1/r} \|\Phi\|_{r,\infty,W} \leq B \leq C \leq (\|\Phi\|_{r,\infty,W}^r + 4^p)^{1/r}.$$

In spite of Theorem 4, Theorem 3 is still worthwhile, because the  $B_g^+(p, q)$  condition is more convenient in applications. For instance, it is not easy to test Examples 3 and 4 by condition (ii) or (iii).

*Remark 4.* The equivalence between (i) and (ii) holds even for arbitrary  $0 < q < \infty$ , and this is a Sawyer's type characterization for the one-sided maximal function (cf. [10]). The condition (iii) is a Verbitsky's type condition (cf. [11]). However, the case of  $p = 1$  is new and the proof is new.

## 2. PROOFS OF THE THEOREMS

*Proof of Theorem 1. Necessity.* The argument is based on the following elementary fact about the weak type  $(p, q)$  inequality (1.5).

For every  $a < b < c$  and measurable set  $S \subset (b, c)$ , it follows from the weak type inequality (1.5) easily that

$$(2.1) \quad \frac{g(S)}{g(a,c)} w(a,b)^{1/q} \leq C v(S)^{1/p},$$

on setting  $f(x) = \chi_S(x)$ .

Given  $-\infty < a < b < c < \infty$ , without loss of generality, we may assume  $0 < \text{ess sup}_{x \in (b,c)} (\frac{g}{v}) < \infty$ . Then  $0 < \alpha_{g,v,(b,c)} = \alpha = \text{ess inf}_{x \in (b,c)} (\frac{v}{g}) < \infty$ . Let  $\lambda > \alpha$ . It is obvious that

$$|S_\lambda| = \left| \left\{ x \in (b, c) : \frac{v(x)}{g(x)} < \lambda \right\} \right| > 0.$$

It follows from the inequality (2.1) that

$$\frac{w(a, b)^{1/q}}{g(a, c)} \leq C \frac{v(S_\lambda)}{g(S_\lambda)} < C\lambda.$$

Letting  $\lambda \rightarrow \alpha$ , we get

$$(2.2) \quad \frac{w(a, b)^{1/q}}{g(a, c)} \leq C\alpha.$$

*Sufficiency.* We may assume that  $f \geq 0$  is bounded and has compact support. Let  $\lambda > 0$  and  $\Omega_\lambda = \{x : M_g^+ f(x) > \lambda\}$ . It is well known that (see [6])  $\Omega_\lambda = \bigcup_j (a_j, b_j)$  where the intervals  $(a_j, b_j)$  are bounded pairwise disjoint and

$$(2.3) \quad \lambda \leq \frac{1}{g(x, b_j)} \int_x^{b_j} f(x)g(x) dx \quad \text{for every } x \in [a_j, b_j].$$

Following the idea used in [5], we set  $x_j^0 = a_j$  and  $x_j^k \in (a_j, b_j)$  such that

$$\int_{x_j^k}^{b_j} f(x)g(x) dx = \frac{1}{2^k} \int_{a_j}^{b_j} f(x)g(x) dx \quad (k = 1, 2, \dots).$$

Then  $\{x_j^k\}_k$  is increasing,  $\lim_{k \rightarrow \infty} x_j^k = b_j$  and

$$(2.4) \quad \lambda \leq \frac{4}{g(x_j^k, x_j^{k+2})} \int_{x_j^{k+1}}^{x_j^{k+2}} f(x)g(x) dx.$$

Then we have

$$\begin{aligned} \lambda^q w(\Omega_\lambda) &= \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^k, x_j^{k+1}) \\ &\leq 4^q \sum_j \sum_{k=0}^\infty \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q} \left( \int_{x_j^{k+1}}^{x_j^{k+2}} fg \right)^q \quad (\text{by (2.4)}) \\ &\leq 4^q \sum_j \sum_{k=0}^\infty \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q} \left( \text{ess sup}_{x \in (x_j^{k+1}, x_j^{k+2})} \left( \frac{g}{v} \right) \right)^q \left( \int_{x_j^{k+1}}^{x_j^{k+2}} fv \right)^q \\ &\leq (4A)^q \sum_j \sum_{k=0}^\infty \left( \int_{x_j^{k+1}}^{x_j^{k+2}} fv \right)^q \quad (\text{by } A_g^+(1, q) \text{ condition (1.3)}) \\ &\leq \left( 4A \int f(x)v(x) dx \right)^q. \end{aligned}$$

This completes the proof of Theorem 1. □

*Remark 5.* The necessity of Theorem 1 is still true when  $0 < q < 1$ . An immediate consequence is this: if the weak type  $(1, q)$  inequality holds for  $M_g^+$  with respect to  $(w, v)$  and  $\text{ess sup}_{(a,b)} (\frac{g}{v}) = \infty$ , then  $w(x) = 0$  a.e.  $x \in (-\infty, a)$ .

*Proof of Theorem 2.* The result is derived from the following propositions.

**Proposition 1.** *Suppose weak type  $(p, q)$  inequality (1.5) holds for given  $0 < p, q < \infty$ . If  $w$  is not locally integrable, then there exists  $a \in [-\infty, \infty]$  such that  $w$  is locally integrable on  $(-\infty, a)$  and  $v(x) = \infty$  a.e.  $x \in (a, \infty)$ . On the other hand, if  $v(x) = 0$  on  $S$  with  $|S| > 0$ , then there exists  $b \in [-\infty, \infty]$  such that  $w(x) = 0$  a.e.  $x \in (-\infty, b)$  and  $v(x) > 0$  a.e.  $x \in (b, \infty)$ .*

**Proposition 2.** *Under the same conditions as those in Theorem 2, suppose  $v(x) > 0$  a.e. and  $w(x)$  is locally integrable on an interval  $(a, b)$ . Then we have*

- (i) *For a.e.  $x \in (a, b)$ , either  $w(x) = 0$  or  $v(x) = \infty$  if  $p < q$ .*
- (ii)  *$w(x) \leq (2C)^p v(x)$  a.e.  $x \in (a, b)$  if  $p = q$ . (Where  $C$  is the best constant in (1.6).)*

*Proof of Proposition 1.* Let

$$E = \{x \in R: \text{there exists } \delta > 0 \text{ such that } w(x - \delta, x + \delta) < \infty\}$$

and put  $a = \inf\{R \setminus E\}$ . It is obvious that  $w$  is locally integrable on  $(-\infty, a)$ .

Meanwhile, for  $a < c < d$  and measurable  $S \subset (c, d)$  arbitrary, there exists  $e \in R \setminus E$  with  $a \leq e < c$ . It follows from the definition of  $R \setminus E$  that  $w(e - \delta, c) = \infty$  for any  $\delta > 0$ . Hence the inequality (2.1) shows that  $v(S) = \infty$ , which implies  $v(x) = \infty$  a.e.  $x \in (a, \infty)$ , since  $S \subset (c, d)$  and  $(c, d) \subset (a, \infty)$  are arbitrary.

In order to prove the rest, we set  $\tilde{E} = \{x \in R: v(x) = 0\}$ . Let

$$b = \text{ess sup}(I(x)\chi_{\tilde{E}}(x)),$$

where  $I(x) = x$ . For every  $x < b$ , we have  $|\tilde{E} \cap (x, b)| > 0$ . Then the inequality (2.1) implies  $w(x_1, x) = 0$  for all  $x_1 < x$ . The required conclusion follows.  $\square$

*Proof of Proposition 2.* In order to produce a contradiction to (i) we assume that there exists a constant  $M > 0$  such that

$$|E| = \left| \left\{ x \in (a, b): w(x) > \frac{1}{M}, v(x) < M \right\} \right| > 0.$$

Write  $v_E(x) = v(x)\chi_E(x)$ , then  $v_E$  is locally integrable. Choose  $x_0 \in E$  being a Lebesgue point of  $g(x), w(x), v_E(x)$  and a point of density of  $E$ .

Since  $x_0$  is a point of density of  $E$ , for every positive integer  $n$ , there exists a  $h_n > 0$  such that

$$1 - \frac{1}{n} < \frac{|E \cap (x_0 - h_n, x_0 + h_n)|}{2h_n} \leq 1$$

and  $h_n \rightarrow 0$  as  $n \rightarrow \infty$ . For these  $h_n$  we have

$$\begin{aligned} 1 - \frac{1}{n} &< \frac{|E \cap (x_0 - h_n, x_0)|}{2h_n} + \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n} \\ &\leq \frac{1}{2} + \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n}, \end{aligned}$$

therefore

$$(2.5) \quad \frac{1}{2} - \frac{1}{n} < \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n}.$$

Let  $S = E \cap (x_0, x_0 + h)$ . It follows from (2.1) that

$$\frac{g(E \cap (x_0, x_0 + h_n))}{g(x_0 - h_n, x_0 + h_n)} w(x_0 - h_n, x_0)^{1/q} \leq C v_E(x_0, x_0 + h_n)^{1/p}.$$

By use of (2.5) we have

$$\begin{aligned} & \left(\frac{1}{2} - \frac{1}{n}\right) \frac{2h_n}{g(x_0 - h_n, x_0 + h_n)} \frac{g(E \cap (x_0, x_0 + h_n))}{|E \cap (x_0, x_0 + h_n)|} \left(\frac{w(x_0 - h_n, x_0)}{h_n}\right)^{1/q} \\ & \leq C h_n^{1/p-1/q} \left(\frac{v_E(x_0, x_0 + h_n)}{h_n}\right)^{1/p}. \end{aligned}$$

Inequality (2.5) shows that the sequence  $\{E \cap (x_0, x_0 + h_n)\}$  shrinks to  $x_0$  nicely (see [8], p. 140). On taking  $n \rightarrow \infty$ , it follows from the Lebesgue differential theorem, more precisely its slightly generalized version (see [8], p. 141), that

$$\frac{1}{2} w(x_0) \leq 0.$$

This contradicts the fact that  $w(x_0) > \frac{1}{M}$ .

Now we prove (ii). For contradiction we suppose

$$|S| = |\{x \in (a, b) : w(x) > (2C)^p v(x)\}| > 0.$$

Write  $v_s(x) = v(x)\chi_s(x)$ . Choose  $x_0 \in S$  in the same way as that in the proof of (i). Then the same argument as above shows

$$\frac{1}{2} w(x_0)^{1/p} \leq C v(x_0)^{1/p}.$$

This contradicts the fact that  $(2C)^p v(x_0) < w(x_0)$ . □

*Remark 6.* It follows from Proposition 1 that we may assume that  $w(x)$  is locally integrable on  $R$  and  $v(x) > 0$  a.e.  $x \in R$  in advance, when we study the weak type or strong type inequality for the one-sided maximal operator.

*Proof of Theorem 3.* In order to prove the first part, without loss of generality we may assume that  $f \geq 0$  is bounded and has compact support. For  $\lambda > 0$ , let  $\Omega_\lambda, (a_j, b_j)$  and  $(x_j^k, x_j^{k+1})$  be the same as those in the proof of the sufficiency of Theorem 1. Then we can write

$$\begin{aligned} \lambda^q w(\Omega_\lambda) &= \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^k, x_j^{k+1}) \\ (2.6) \quad &= \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^{2k}, x_j^{2k+1}) + \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^{2k+1}, x_j^{2k+2}) \\ &= \text{I} + \text{II}, \quad \text{say.} \end{aligned}$$

Suppose  $p > 1$ . We shall only estimate term I in (2.6), and the rest is similar.

(2.7)

$$\begin{aligned}
\text{I} &\leq 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})^q} \left( \int_{x_j^{2k+1}}^{x_j^{2k+2}} fg \right)^q \quad (\text{by (2.4)}) \\
&\leq 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})^q} \left( \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{q/p} \sigma(x_j^{2k+1}, x_j^{2k+2})^{q/p'} \\
&\hspace{15em} (\text{by Hölder's inequality}) \\
&\leq 4^q \left[ \sum_j \sum_{k=0}^{\infty} \left( \frac{w(x_j^{2k}, x_j^{2k+1})^{1/q} \sigma(x_j^{2k+1}, x_j^{2k+2})^{1/p'}}{g(x_j^{2k}, x_j^{2k+2})} \right)^r \right]^{q/r} \\
&\hspace{10em} \times \left( \sum_j \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{p/q} \quad (\text{by Hölder's inequality again}) \\
&\leq (4B)^q \left( \sum_j \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{q/p}.
\end{aligned}$$

Thus we get

$$(\lambda^q w(\Omega_\lambda))^{1/q} \leq 2^{1/p'} 4B \left( \int_R r^p v \right)^{1/p}.$$

With a few obvious changes (cf. the proof of the sufficiency of Theorem 1) the foregoing procedure is still available for the case of  $p = 1$ . The first part of Theorem 3 is proved.

Now we prove the rest of Theorem 3. Let  $p > 1$ . Given  $\{(a_j, c_j)\}$  pairwise disjoint and  $b_j \in (a_j, c_j)$ , we may assume  $0 < \sigma(b_j, c_j) < \infty$  for all  $j$ . Indeed, if  $\sigma(b_j, c_j) = \infty$  for some  $j$ , which means  $\int_R \left(\frac{g}{v}\chi_{(b_j, c_j)}\right)^{p'} v dx = \infty$ , then there exists  $f$  such that  $\int_R |f(x)|^p v(x) dx$  and  $\int_{b_j}^{c_j} fg = \infty$ , which implies  $M_g^+ f(x) = \infty$  on  $(-\infty, b_j)$ . Therefore the strong type  $(p, q)$  inequality holds only if  $w(x) = 0$  a.e.  $x \in (-\infty, b_j)$ , and we have  $w(a_j, b_j) = 0$ . Let

$$f_j(x) = \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/q'}}{g(a_j, c_j)} \right)^{r/p} \left( \frac{g(x)}{v(x)} \right)^{p'/p} \chi_{(b_j, c_j)}(x)$$

and  $f_N(x) = \sum_{j=1}^N f_j(x)$  for given natural number  $N$ . Observe that

$$(2.8) \quad M_g^+ f(x) \geq \frac{1}{g(a_j, c_j)} \int_{b_j}^{c_j} fg \quad \text{for every } f \geq 0 \text{ and } x \in (a_j, b_j).$$

We have

$$\begin{aligned}
C^p \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r &= C^p \int_R f_N(x)^p v(x) dx \\
&\geq \left( \int_R (M_g^+ f_N(x))^q w(x) dx \right)^{p/q} \\
&\geq \left( \sum_{j=1}^N \left[ \frac{\int_{b_j}^{c_j} f_N g}{g(a_j, c_j)} \right]^q w(a_j, b_j) \right)^{p/q} \quad (\text{by (2.8)}). \\
&= \left\{ \sum \frac{w(a_j, b_j)}{g(a_j, c_j)^q} \left[ \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/q'}}{g(a_j, c_j)} \right)^{r/p} \sigma(b_j, c_j) \right]^q \right\}^{p/q} \\
&= \left\{ \sum \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r \right\}^{p/q}.
\end{aligned}$$

This is

$$\left[ \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r \right]^{1/r} \leq C.$$

Letting  $N \rightarrow \infty$ , we get  $(w, v) \in B_g^+(p, q)$  with  $B \leq C$ .

Suppose  $p = 1$ . Keeping Remark 5 in mind, we assume

$$0 < \beta_j = \text{ess sup}_{(b_j, c_j)} \left( \frac{g}{v} \right) < \infty \quad \text{for every } j.$$

Given  $\lambda > 1$ , set  $E_j = \{x \in (b_j, c_j) : \frac{g(x)}{v(x)} > \frac{\beta_j}{\lambda}\}$ . Then  $v(E_j) < \infty$  and  $|E_j| > 0$  which implies  $v(E_j) > 0$  (cf. Remark 6 we may assume  $v(x) > 0$  a.e.). Put

$$f_j(x) = \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r \frac{\chi_{E_j}(x)}{v(E_j)}$$

and  $f_N(x) = \sum_{j=1}^N f_j(x)$ . Then

$$\int_R f_N(x) v(x) dx = \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r$$

and

$$\begin{aligned}
\left[ \frac{\int_{b_j}^{c_j} f_N g}{g(a_j, c_j)} \right]^q w(a_j, b_j) &= \frac{w(a_j, b_j)}{g(a_j, c_j)^q} \left[ \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r \frac{g(E_j)}{v(E_j)} \right]^q \\
&\geq \left( \frac{1}{\lambda} \right)^q \left[ \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r \frac{\beta_j}{g(a_j, c_j)} \right]^q w(a_j, b_j) \\
&\quad \left( \text{since } g(E_j) > \frac{\beta_j}{\lambda} v(E_j) \right) \\
&= \left( \frac{1}{\lambda} \right)^q \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r \left( \text{since } \frac{1}{r} = \frac{1}{q} - 1 \right).
\end{aligned}$$

The argument similar to that used above shows

$$\left[ \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \beta_j \right)^r \right]^{1/r} \leq \lambda C.$$

Then we obtain  $(w, v) \in B_g^+(1, q)$  with  $B \leq C$ . This completes the proof of Theorem 3.  $\square$

*Proof of Theorem 4.* We shall prove the theorem only in the case  $p > 1$ , since the necessary modifications for the case  $p = 1$  are indicated in the proof of Theorems 1 and 3.

(i)  $\Rightarrow$  (ii) Similarly to the proof of Theorem 3, the condition  $\sigma(a, b) = \infty$  together with the weak type inequality (1.5) implies that  $w(x) = 0$  for a.e.  $x \in (-\infty, b)$ . Therefore we may assume that  $0 < \sigma(b_j, c_j) < \infty$  for all  $j$ . Let

$$f = \left\{ \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)} \right)^p \frac{\sigma(x)}{v(x)} \chi_{(b_j, c_j)} \right\}^{1/p}.$$

It follows from (2.8) that  $M_g^+ f(x) \geq 1$  on  $\bigcup (a_j, b_j)$  and the weak type inequality (1.5) yields

$$w \left( \bigcup (a_j, b_j) \right)^{p/q} \leq C \int_R f^p v = C \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p.$$

This is (ii) with  $B \leq C$ .

(ii)  $\Rightarrow$  (iii) For  $\lambda > 0$ , write  $E_\lambda = \{\Phi(x) > \lambda\}$ . For each  $x \in E_\lambda$  there exist  $a_x < b_x < c_x$  such that  $x \in (a_x, b_x)$  and  $\frac{w(a_x, b_x)^{1/p} \sigma(b_x, c_x)^{1/p'}}{g(a_x, c_x)} > \lambda$ . Then for arbitrary compact set  $K \subset E_\lambda$ , from  $\{(a_x, b_x)\}_{x \in K}$  we can choose a finite subfamily of open intervals  $\{I_i\}$ , which still covers  $K$ . By use of the covering lemma in [3] (Lemma 4.4, p. 25), one can select a finite subsequence  $\{(a_j, b_j)\}$  from  $\{I_i\}$  so that  $\{(a_j, b_j)\}$  are pairwise disjoint and  $w(\bigcup_i I_i) \leq 2 \sum_j w(a_j, b_j)$ . For the set  $\{(a_j, b_j)\}$ , we have

$$\begin{aligned} w \left( \bigcup_j (a_j, b_j) \right)^{p/q} &\leq B^p \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p \quad (\text{by (ii)}) \\ &\leq B^p \lambda^{-p} \sum_j w(a_j, b_j). \end{aligned}$$

Thus we get

$$w(K) \leq w \left( \bigcup_i I_i \right) \leq 2 \sum_j w(a_j, b_j) \leq 2B^p \lambda^{-p},$$

which implies  $\|\Phi\|_{r, \infty, w} \leq 2^{1/r} B$ .

(iii)  $\Rightarrow$  (i) Let  $f$ ,  $\Omega_\lambda$  and  $\{(x_j^k, x_j^{k+1})\}$  be the same as those in the proof of Theorem 1. In addition, we may assume  $\int_R f^p v = 1$ . Write

$$\begin{aligned} (2.9) \quad w(\Omega_\lambda) &= w(\Omega_\lambda \cap \{\Phi(x) > \lambda^{q/r}\}) + w(\Omega_\lambda \cap \{\Phi(x) \leq \lambda^{q/r}\}) \\ &= \text{I} + \text{II}, \quad \text{say.} \end{aligned}$$

For item I in (2.9) we have

$$(2.10) \quad \text{I} \leq w(\{\Phi(x) > \lambda^{q/r}\}) \leq \|\Phi\|_{r,\infty,w}^r \lambda^{-q}.$$

To estimate II in (2.9), we consider  $(x_j^k, x_j^{k+1})$  having non-empty intersection with  $\{\Phi(x) \leq \lambda^{q/r}\}$ . It follows from (2.4) and Hölder's inequality that

$$(2.11) \quad \begin{aligned} w(x_j^k, x_j^{k+1})^{1/p} \lambda &\leq 4 \frac{w(x_j^k, x_j^{k+1})^{1/p} \sigma(x_j^{k+1}, x_j^{k+2})^{1/p'}}{g(x_j^k, x_j^{k+2})} \left( \int_{x_j^{k+1}}^{x_j^{k+2}} f^p v \right)^{1/p} \\ &\leq 4 \lambda^{q/r} \left( \int_{x_j^{k+1}}^{x_j^{k+2}} f^p v \right)^{1/p}, \end{aligned}$$

since  $(x_j^k, x_j^{k+1}) \cap \{\Phi(x) \leq \lambda^{q/r}\} \neq \emptyset$ . Raising (2.11) to the power of  $p$  and then summing it over  $j$  and  $k$ , we have

$$(2.12) \quad \text{II} \leq 4^p \lambda^{-q}.$$

Combining (2.10) and (2.12), we obtain (1.5) with  $C \leq (\|\Phi\|_{r,\infty,w}^r + 4^p)^{1/q}$ . The proof of Theorem 4 is complete.  $\square$

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