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THE μ-PIP AND INTEGRABILITY OF A SINGLE FUNCTION

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ABSTRACT. Two examples are given that answer in the negative the following question asked by E. M. Bator: If $f : \Omega \to X^*$ is bounded and weakly measurable and for each x^{**} in X^{**} there is a bounded sequence (x_n) in X such that $x^{**}f = \lim_n fx_n$ a.e., does it follow that f is Pettis integrable?

1. INTRODUCTION

Given a finite (complete) measure space (Ω, Σ, μ), a Banach space X is said to have the μ-Pettis Integrability Property (μ-PIP) if every weakly measurable bounded function $f : \Omega \to X$ is Pettis integrable. In [1] E. Bator shows that a dual space X^* has the μ-PIP with respect to a perfect measure μ if and only if for every bounded weakly measurable $f : \Omega \to X^*$, $\|w^* - \int f\,d\mu\| = \|D - \int f\,d\mu\|$. In [2] and [4], it is shown how the above statement can be strengthened by dropping the assumption that the measure space be perfect. In fact,

Theorem 1. *Let X^* be a dual space. The following are equivalent:*
(1) *X^* has the μ-PIP.*
(2) [2] *For every weakly measurable bounded function $f : \Omega \to X^*$,*

$$\left\|w^* - \int f\,d\mu\right\| = \left\|D - \int f\,d\mu\right\|.$$

(3) [4] *For every weakly measurable bounded function $f : \Omega \to X^*$,*

$$w^* - \int f\,d\mu = 0 \text{ implies } D - \int f\,d\mu = 0.$$

The following corollary, proven for perfect measures in [1], and in general in [2], follows easily:

Corollary 1. *A dual space X^* has the μ-PIP if and only if*

$$(*)\begin{cases}\text{For every bounded weakly measurable function } f : \Omega \to X^* \\ \text{and each } x^{**} \text{ in } X^{**}, \text{ there exists a bounded sequence } (x_n) \text{ in } X \\ \text{such that } fx_n \to x^{**}f \text{ almost everywhere.}\end{cases}$$

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Consider the following property of a particular weakly measurable and bounded function $g : \Omega \to X^*$:

$$(**) \begin{cases} \textit{For each } x^{**} \textit{ in } X^{**} \textit{ there exists a bounded sequence } (x_n) \textit{ in } X \\ \textit{such that } g x_n \to x^{**} g \textit{ almost everywhere.} \end{cases}$$

In [1] Bator asks if property $(**)$ implies Pettis integrability of such a function g. In [2], this question is answered in the negative. The example given is based on one by Talagrand and assumes the existence of two-valued measurable cardinals.

As noted in [5, p. 189], it is consistent with the usual axioms of set theory (ZFC) that there are no two-valued measurable cardinals and it is impossible to prove that their existence is consistent with ZFC. The purpose of this note is to show, assuming only the existence of a first uncountable ordinal, that property $(**)$ does not imply Pettis integrability. The first example shows that certain functions that are weak*-zero, but not weakly-zero, give rise to families of functions that satisfy $(**)$ but fail to be Pettis integrable. It also shows that functions with strong measurability properties (universally weakly measurable functions, see [5]) can satisfy $(**)$ and still fail to be Pettis integrable. The second example shows that even in the best possible situation, where X^* is a dual of a separable space, Bator's question has a negative answer.

Let us fix some notation and terminology. The dual of a Banach space X will be denoted by X^*. Given a complete finite measure space (Ω, Σ, μ), a function $f : \Omega \to X^*$ is called *weakly measurable* (resp. *weak* measurable*) if for all x^{**} in X^{**} (resp. all x in X) the scalar-valued function $x^{**} f$ (resp. fx) is measurable.

If E is in Σ, the *Dunford integral* of f over E, denoted by $D - \int_E f \, d\mu$, is the member of X^{***} defined by the equation $(D - \int_E f \, d\mu)(x^{**}) = \int_E x^{**} f \, d\mu$ for all $x^{**} \in X^{**}$. If $D - \int_E f \, d\mu$ belongs to X^* for all E in Σ, then f is said to be *Pettis integrable.*

The *weak* integral* of f over E, denoted by $w^* - \int_E f \, d\mu$, is the member of X^* defined by the equation $(w^* - \int_E f \, d\mu)(x) = \int_E fx \, d\mu$ for all $x \in X$.

A function $f : \Omega \to X^*$ is said to be *weakly equivalent to zero* (resp. *weak* equivalent to zero*) if for all x^{**} in X^{**} (resp. all x in X), $x^{**} f = 0$ μ-a.e. (resp. $fx = 0$ μ-a.e.).

2. Examples

When considering property $(**)$, we must pay close attention to the restrictions made on the range-space, that is, the space in which the function is valued. Consider for example a function g into X^*. When viewed as a function into X^{***}, it satisfies $(**)$, by default, but certainly does not have to be Pettis integrable.

Example 1. This example is tailored after a well-known example of Phillips; see [3]. Let ω_1 be the first uncountable ordinal, Σ be the σ-algebra of all countable and co-countable subsets of $[0, \omega_1]$, and define $\mu : \Sigma \to \{0, 1\}$ by the equation

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A^c \text{ is countable.} \end{cases}$$

Define a function $f : [0, \omega_1] \to l_\infty[0, \omega_1] = (l_1[0, \omega_1])^*$ by the equation

$$[f(s)](t) = \begin{cases} 0 & \text{if } t < s, \\ 1 & \text{if } t \geq s. \end{cases}$$

Claim 1. f is weakly measurable.

The dual of $l_\infty[0,\omega_1]$ is the space of all bounded and additive measures on $2^{[0,\omega_1]}$. Fix such a measure β.

There exists a countable subset R of $[0,\omega_1]$ and a unique decomposition $\beta = \beta_1 + \beta_2$ into bounded additive measures such that for any $A, \beta_1(A) = \beta_1(A \cap R)$ and β_2 vanishes on countable sets. As

$$\beta_1 f(s) = \int_{[0,\omega_1]} [f(s)](t) d\beta_1(t) = \beta_1(R \cap [s,\omega_1])$$

and

$$\beta_2 f(s) = \int_{[0,\omega_1]} [f(s)](t) d\beta_2(t) = \beta_2([s,\omega_1]),$$

it follows that $\beta f = \beta_1 f + \beta_2 f = \beta_2([0,\omega_1])$ μ-almost everywhere.

Claim 2. f is not Pettis integrable.

Indeed, the weak* integral of f is identically zero, but for any $\beta = \beta_1 + \beta_2$, and any set E, $\int_E \beta f(s)\, d\mu(s) = \beta_2([0,\omega_1])\mu(E)$.

Now, define $\tilde{f} : [0,\omega_1] \to l_\infty[0,\omega_1]$ by the equation

$$\tilde{f}(s) = f(s) + \chi_{[0,\omega_1]}(s).$$

Then $\tilde{f}$ is weakly measurable, not Pettis integrable, but satisfies property $(**)$, in fact, for any $\beta = \beta_1 + \beta_2$ in the dual of $l_\infty[0,\omega_1]$,

$$\beta\tilde{f} = \beta([0,\omega_1]) + \beta_2([0,\omega_1]) \quad \mu\text{-a.e.}$$

Consequently,

$$\beta\tilde{f} = \{\beta\tilde{f}\} \cdot \gamma\tilde{f},$$

where γ is any positive norm-one element of $l_1[0,\omega_1]$. Note that $\|\omega^* - \int \tilde{f}\, d\mu\| = 1$ while $\|D - \int \tilde{f}\, d\mu\| = 2$.

Remark 1. The above example shows that for any finite measure space (Ω, Σ, μ), any bounded function $f : \Omega \to X^*$ which is weakly a constant, that is, for every $x^{**}, x^{**}f = c_{x^{**}}$ (= constant) a.e., and weak* equivalent to zero but not weakly equivalent to zero, gives rise to a family of functions satisfying $(**)$.

Indeed, if f is such a function, the family $\{f + x^* : x^* \in X^* \backslash \{0\}\}$ satisfies $(**)$, but none of the functions are Pettis integrable.

Remark 2. A function $f : \Omega \to X$ defined on a compact Hausdorff space Ω is said to be *universally weakly measurable* if for every Radon measure μ on Ω, the scalar functions $x^*f, x^* \in X^*$, are μ-measurable.

Phillips [3] has constructed a bounded function $f : [0,1] \to l_\infty[0,1]$ such that x^*f is Borel measurable for all x^* in $l_\infty[0,1]^*$, and hence, f is universally weakly measurable. With respect to the Lebesgue measure on $[0,1]$, f is weak*, but not weakly, equivalent to zero. Furthermore, f is weakly constant in the sense of the above remark, and again by the same remark, property $(**)$ fails to imply Pettis integrability even in the case where f satisfies the stronger assumption of being universally weakly measurable.

Example 2. There exists a function f with values in $l_\infty(\mathbb{N})$ that satisfies $(**)$, but fails to be Pettis integrable.

Let $\Omega = (\{0,1\}^{\mathbb{N}}, \overline{\Sigma}, \overline{\mu})$ be as in [5, Theorem 13-2-1] and let $f : \{0,1\}^{\mathbb{N}} \to l_\infty(\mathbb{N})$ be the function that assigns to each point $a \in \{0,1\}^{\mathbb{N}}$ its characteristic function χ_a.

Write $l_\infty(\mathbb{N})^* = l_1(\mathbb{N}) \oplus c_0^\perp$. In [5, Theorem 13-3-3] it is shown that for any γ in $c_0^\perp$,

$$\gamma f = k_\gamma \ (= \text{constant}) \quad \overline{\mu}\text{-a.e.}$$

Hence, for $x^* = (\alpha_i) \oplus \gamma$ in $l_1(\mathbb{N}) \oplus c_0^\perp$,

$$x^* f = (\alpha_i) f + \gamma = (\alpha_i) f + k_\gamma \quad \overline{\mu}\text{-a.e.}$$

Let $\{e_i : i \in \mathbb{N}\}$ be the standard basis for $l_\infty(\mathbb{N})$ and define $S : l_\infty(\mathbb{N}) \to l_\infty(\mathbb{N})$ by the equation

$$S(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

Define a function $\tilde{f} : \{0,1\}^{\mathbb{N}} \to l_\infty(\mathbb{N})$ by the equation

$$\tilde{f}(a) = e_1 + S(f(a)).$$

If $x^* = (\alpha_i) \oplus \gamma \in l_\infty(\mathbb{N})^*$ and we write $(\beta_i) \oplus \delta$ to denote $S^*(x^*)$, then

$$\begin{aligned} x^* \tilde{f} &= x^*(e_1) + x^* S(f) = \alpha_1 + S^*(x^*) f = \alpha_1 + \{(\beta_i) \oplus \delta\} f \\ &= (\alpha_1 + k_\delta) + (\beta_i) f = (\alpha_1 + k_\delta, \beta_1, \beta_2, \beta_3, \dots) \tilde{f}. \end{aligned}$$

Hence, $\tilde{f}$ satisfies property $(**)$, but is not Pettis integrable since f is not.

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