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A PROOF OF THE TRACE THEOREM OF SOBOLEV SPACES ON LIPSCHITZ DOMAINS

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ABSTRACT. A complete proof of the trace theorem of Sobolev spaces on Lipschitz domains has not appeared in the literature yet. The purpose of this paper is to give a complete proof of the trace theorem of Sobolev spaces on Lipschitz domains by taking advantage of the intrinsic norm on $H^s(\partial\Omega)$. It is proved that the trace operator is a linear bounded operator from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial\Omega)$ for $\frac{1}{2} < s < \frac{3}{2}$.

1. INTRODUCTION

Sobolev spaces play an important role in the study of partial differential equations on smooth and nonsmooth domains and their boundary value problems. So far the studies of these problems have shown that such spaces are quite appropriate and natural. In recent years, boundary value problems on Lipschitz domains and the method of layer potentials for their solutions have attracted attention in both pure and applied mathematics (see [6], [7], [9] and the references therein). On the other hand, there are a lot of engineering problems such as point sensor placements at vertex points of the boundary in the mechanical engineering, corrosive engineering and spacecraft, point controllers at vertex points of the boundary in the stabilization of structural dynamics, etc., which require very careful local analysis around vertex points of the boundary (see [3] and the references therein). Thus Sobolev spaces on Lipschitz domains play a very important role in those studies. Most properties of Sobolev spaces on Lipschitz domains are rigorously proved (see [1], [5], [8]). But a complete proof of the trace theorem of Sobolev spaces on Lipschitz domains has not appeared in the literature, to the best of the author's knowledge. On a bounded Lipschitz domain Ω with boundary $\partial\Omega$, we can only define $H^s(\partial\Omega)$ in a unique invariant way for $|s| \leq 1$. Thus the trace properties are different from that of Sobolev spaces on smooth domains. For Lipschitz domains, E. Gagliardo [4] (1957) proved the trace theorem for $H^s(\Omega)$ where $\frac{1}{2} < s \leq 1$. D. S. Jerison and C. E. Kenig [6] (1982) stated the trace theorem for the case $s = \frac{3}{2}$ without any proof. M. Costabel [2] (1988) proved a trace theorem on special Lipschitz domains for the range $1 < s < \frac{3}{2}$. However, he did not use the natural intrinsic norm on $H^s(\partial\Omega)$. It is not obvious that the trace norm in (4.17) of Costabel's paper [2] is equivalent to the natural intrinsic trace norm. Because of the needs in studies

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of boundary value problems on Lipschitz domains, optimization and stabilization of structural dynamics on Lipschitz domains and other contemporary engineering problems, it is necessary to give a complete proof of the trace theorem of Sobolev spaces on Lipschitz domains, which is the purpose of this paper. The idea of our proof is to use Costabel's approach, the interpolation theorem [8] and the intrinsic norm of $H^s(\partial\Omega)$. In this paper the trace theorem of Sobolev spaces on Lipschitz domains for the range $\frac{1}{2} < s < \frac{3}{2}$ is proved. For $s > \frac{3}{2}$, the trace operator is a bounded linear operator from $H^s(\Omega)$ to $H^1(\partial\Omega)$, whose proof is analogous to the proof given in this paper. For $s = \frac{3}{2}$, no proof is available.

2. Definition of Sobolev spaces on Lipschitz domains

Definition 1 (see [9])**.** A bounded simply connected open subset $\Omega \subset \Re^N$ is called a Lipschitz domain if $\forall P \in \partial\Omega$, there exist a rectangular coordinate system (x, s) $(x \in \Re^{N-1}, s \in \Re)$, a neighborhood $V(P) \subset \Re^N$ of P, and a function $\varphi_P : \Re^{N-1} \to \Re$ such that

(a) $|\varphi_P(x) - \varphi_P(y)| \leq C_P|x - y|, \quad C_P < +\infty, \quad \forall x, y \in \Re^{N-1}$;
(b) $V(P) \cap \Omega = \{(x, s) \in \Re^N | s > \varphi(x)\} \cap V(P)$.

The coordinate system (x, s) in Definition 1 may be chosen as a combination of rotation and translation of the standard rectangular coordinate system for $\Re^N$. The neighborhood $V(P)$ may be chosen as a cylinder $Z(P, r_P)$, which is an open right circular, doubly truncated cylinder centered at $P \in \Re^N$ with radius r_P satisfying the following properties:

(a) the bases of $Z(P, r_P)$ are some positive distance from $\partial\Omega$;
(b) the s-axis contains the axis of $Z(P, r_P)$;
(c) φ_P may be taken to have compact support in $\Re^{N-1}$, and $\text{supp}(\varphi_P) \subset B(0, 2r_P)$;
(d) $P = (0, \varphi_P(0))$.

The pair $(Z(P, r_P), \varphi_P)$ will be called a cylinder coordinate pair at $P \in \partial\Omega$. By the compactness of $\partial\Omega$, it is possible to cover $\partial\Omega$ with a finite number of cylinder coordinate pairs $\{(Z(P_k, r_k), \varphi_k)\}$, $1 \leq k \leq N_0$. For the Lipschitz domain Ω and a given covering of cylinder coordinate pairs, $\{(Z(P_k, r_k), \varphi_k)\}$, $1 \leq k \leq N_0$, there is a number $M > 0$ such that all ϕ_{P_k}, $1 \leq k \leq N_0$, have Lipschitz norms less than or equal to M. The smallest of such numbers is called the Lipschitz constant for $\partial\Omega$.

Definition 2 (see [1])**.** Let $m \geq 0$ be an integer. Denote by $H^m(\Omega)$, the Sobolev space, the space of all distributions u defined in Ω such that

$$D^\alpha u \in L^2(\Omega), \quad \forall |\alpha| \leq m.$$

$H^m(\Omega)$ is equipped with the norm

$$\|u\|_{m,\Omega} = \left\{ \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2 dx \right\}^{\frac{1}{2}}.$$

It is easy to verify that $H^m(\Omega)$ is a Banach space. If we define the inner product generated from $\|\cdot\|_{m,\Omega}$ in a natural way, then $H^s(\Omega)$ is a Hilbert space. The Sobolev space of non-integer order, $H^s(\Omega)$, is defined by the real interpolation method (see

[1], [8]). According to [1] and [5], when $s = m + \sigma$ and $0 < \sigma < 1$, $H^s(\Omega)$ can be equipped with an equivalent and intrinsic norm

$$\|u\|_{s,\Omega} = \left\{ \|u\|_{m,\Omega}^2 + \sum_{|\alpha|=m} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+2\sigma}} dxdy \right\}^{\frac{1}{2}}.$$

Definition 3 (see [5])**.** Denote by $H^s(\partial\Omega)$ $(0 < s < 1)$, the Sobolev trace space, the space of all distributions u defined on $\partial\Omega$ such that

$$\int_{\partial\Omega} |u(x)|^2 d\sigma_x + \int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N-1+2s}} d\sigma_x d\sigma_y < +\infty.$$

$H^s(\partial\Omega)$ is equipped with the norm

$$\|u\|_{s,\partial\Omega} = \left\{ \int_{\partial\Omega} |u(x)|^2 d\sigma_x + \int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N-1+2s}} d\sigma_x d\sigma_y \right\}^{\frac{1}{2}}.$$

According to [5] (page 20) the trace spaces defined above are the same as the trace spaces introduced in [1] and [8].

3. Proof of the trace theorem

Definition 4. For any $u \in C^\infty(\Omega)$, define the trace operator $\gamma|_{\partial\Omega}$ by

$$\gamma|_{\partial\Omega} u(x) = u(x), \quad x \in \partial\Omega.$$

When $\Omega = \Re_+^N$, denote by γ_0 the trace operator $\gamma_{\Re_0^N}$ where

$$\Re_0^N = \partial\Re_+^N = \left\{ x \in \Re^N |\, x = (x', 0),\ x' \in \Re^{N-1} \right\}.$$

The trace theorem of Sobolev spaces on Lipschitz domains is as follows.

Theorem 1. *Let Ω be a bounded simply connected Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}$. Then the trace operator $\gamma|_{\partial\Omega}$ is a bounded linear operator from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial\Omega)$.*

Before we prove this theorem, we need to establish several lemmas.

Definition 5. $\Omega \subset \Re^N$ is called a special Lipschitz domain if there exists a function $\varphi : \Re^{N-1} \to \Re$ such that

$$|\varphi(x') - \varphi(y')| \le M|x' - y'|, \quad x', y' \in \Re^{N-1},$$
$$\Omega = \left\{ x \in \Re^N |\ x_N > \varphi(x'),\ x' \in \Re^{N-1} \right\}$$

and

$$\partial\Omega = \left\{ x \in \Re^N |\ x_N = \varphi(x'), x' \in \Re^{N-1} \right\}.$$

Lemma 1. *Let Ω be a special Lipschitz domain. Define a linear operator T_φ : $L^2(\Omega) \to L^2(\Re_+^N)$, where $\Re_+^N = \left\{ x \in \Re^N |\ \ x_N > 0 \right\}$, by*

$$(T_\varphi u)(x) = u(x', x_N + \varphi(x')), \quad a.e.\ x \in \Re_+^N.$$

Then T_φ is a bounded linear operator from $H^s(\Omega)$ to $H^s(\Re_+^N)$ for $0 \le s \le 1$.

Proof. From the definition of T_φ, it is easy to verify that $T_\varphi \in \mathcal{L}(L^2(\Omega), L^2(\Re^N_+))$, where $\mathcal{L}(X,Y)$ always denotes the space of all linear bounded operators from X to Y. If $u \in H^1(\Omega)$, then a simple computation yields

$$\frac{\partial}{\partial x_i}(T_\varphi u)(x) = \frac{\partial}{\partial x_i}u(x', x_N + \varphi(x')) + \frac{\partial}{\partial x_N}u(x', x_N + \varphi(x'))\frac{\partial \varphi}{\partial x_i}(x'), \quad 1 \le i \le N-1,$$

$$\frac{\partial}{\partial x_N}(T_\varphi u)(x) = \frac{\partial}{\partial x_N}u(x', x_N + \varphi(x')).$$

Since φ is uniformly Lipschitz continuous with Lipschitz constant M, $T_\varphi u \in H^1(\Re^N_+)$ and

$$\|T_\varphi u\|_{H^1(\Re^N_+)} \le C\|u\|_{H^1(\Omega)}, \tag{1}$$

where C is only dependent of $\partial\Omega$. Thus $T_\varphi \in \mathcal{L}(H^1(\Omega), H^1(\Re^N_+))$. By the interpolation theorem [8], it follows that for $0 \le s \le 1$

$$T_\varphi \in \mathcal{L}(H^s(\Omega), H^s(\Re^N_+)). \quad \square$$

Any $u \in L^2(\Re^N_+)$ can be understood as a function from $\Re_+$ to $L^2(\Re^{N-1})$, i.e.

$$x_N \to u(\cdot, x_N),$$

thus $u \in L^2(\Re_+, L^2(\Re^{N-1}))$.

Lemma 2. *Let Ω be a special Lipschitz domain. Let T_φ be as defined in Lemma 1 and $1 \le s \le 2$. Then T_φ is a bounded linear operator from $H^s(\Omega)$ to*

$$H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1}))$$

Proof. By Lemma 1,

$$T_\varphi \in \mathcal{L}(H^1(\Omega), H^1(\Re_+; L^2(\Re^{N-1}))) \tag{2}$$

and

$$T_\varphi \in \mathcal{L}(H^1(\Omega), L^2(\Re_+; H^1(\Re^{N-1}))). \tag{3}$$

Let $u \in H^2(\Omega)$; then we have

$$\frac{\partial^k}{\partial x_N^k}(T_\varphi u)(x) = \frac{\partial^k}{\partial x_N^k}u(x', x_N + \varphi(x')), \quad k = 1, 2, \quad a.e.\ x \in \Re^N_+.$$

Thus $T_\varphi u \in H^2(\Re_+; L^2(\Re^{N-1}))$ and

$$\|T_\varphi u\|_{H^2(\Re_+; L^2(\Re^{N-1}))} \le \|u\|_{H^2(\Omega)}.$$

Hence

$$T_\varphi \in \mathcal{L}(H^2(\Omega), H^2(\Re_+; L^2(\Re^{N-1}))). \tag{4}$$

Note that for $u \in H^2(\Omega)$,

$$\frac{\partial}{\partial x_i}(T_\varphi u)(x) = \frac{\partial}{\partial x_i}u(x', x_N + \varphi(x')) + \frac{\partial}{\partial x_N}u(x', x_N + \varphi(x'))\frac{\partial}{\partial x_i}\varphi(x'),$$

$$1 \le i \le N-1, \quad a.e.\ x \in \Re^N_+;$$

and

$$\frac{\partial^2}{\partial x_N \partial x_i}(T_\varphi u)(x) = \frac{\partial^2}{\partial x_N \partial x_i} u(x', x_N + \varphi(x')) + \frac{\partial^2}{\partial^2 x_N} u(x', x_N + \varphi(x')) \frac{\partial}{\partial x_i}\varphi(x')$$

$$1 \le i \le N-1, a.e.\ x \in \Re^N_+.$$

Hence $T_\varphi u \in H^1(\Re_+; H^1(\Re^{N-1}))$, and

$$\|T_\varphi u\|_{H^1(\Re_+; H^1(\Re^{N-1}))} \le C \|u\|_{H^2(\Omega)},$$

where C is only dependent on the Lipschitz constant of $\partial\Omega$. Therefore

$$T_\varphi \in \mathcal{L}(H^2(\Omega), H^1(\Re_+; H^1(\Re^{N-1}))). \tag{5}$$

By applying the interpolation theorem [8], we obtain from (2) and (4) that

$$T_\varphi \in \mathcal{L}(H^s(\Omega), H^s(\Re_+; L^2(\Re^{N-1}))), \quad 1 \le s \le 2,$$

and from (3) and (5) that

$$T_\varphi \in \mathcal{L}(H^s(\Omega), H^{s-1}(\Re_+; H^1(\Re^{N-1}))), \quad 1 \le s \le 2.$$

Therefore we have

$$T_\varphi \in \mathcal{L}(H^s(\Omega), H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1}))). \quad \square$$

Lemma 3. *Let Ω be a special Lipschitz domain and $0 \le s < 1$. Define a linear operator $S_\varphi : L^2(\partial\Omega) \to L^2(\Re^{N-1})$ by*

$$S_\varphi u(x') = u(x', \varphi(x')), \quad a.e.\ x' \in \Re^{N-1}.$$

Then S_φ is a linear bounded invertible operator from $H^s(\partial\Omega)$ to $H^s(\Re^{N-1})$.

Proof. Since Ω is a special Lipschitz domain,

$$d\sigma_x = \sqrt{\sum_{i=1}^{N-1} \left| \frac{\partial \varphi}{\partial x_i}(x') \right|^2 + 1}\ dx'$$

and

$$1 \le \sqrt{\sum_{i=1}^{N-1} \left| \frac{\partial \varphi}{\partial x_i}(x') \right|^2 + 1} \le \sqrt{1 + (N-1)M^2},$$

where $x = (x', \varphi(x')) \in \partial\Omega$. Thus

$$\int_{\Re^{N-1}} |S_\varphi u(x')|^2 dx' \le \int_{\partial\Omega} |u(x)|^2 d\sigma_x \le \sqrt{1 + (N-1)M^2} \int_{\Re^{N-1}} |S_\varphi u(x')|^2 dx', \tag{6}$$

where M is the Lipschitz constant of $\partial\Omega$. Therefore S_φ is bounded and injective from $L^2(\partial\Omega)$ to $L^2(\Re^{N-1})$. For any $f \in L^2(\Re^{N-1})$, define

$$u(x', x_N) = f(x')\phi(x_N - \varphi(x')), \quad a.e.\ \ x \in \Re^N,$$

where $\phi \in C_0^\infty(\Re)$ and $\phi(0) = 1$. By (6), we have $\gamma|_{\partial\Omega} u \in L^2(\partial\Omega)$ and $S_\varphi(\gamma|_{\partial\Omega} u) = f$. Hence S_φ is surjective from $L^2(\partial\Omega)$ to $L^2(\Re^{N-1})$. Thus by (6), S_φ is a linear bounded invertible operator from $L^2(\partial\Omega)$ to $L^2(\Re^{N-1})$.

Notice that $\forall x, y \in \partial\Omega$, $x = (x', \varphi(x'))$ and $y = (y', \varphi(y'))$,

$$|x' - y'| \le |x - y| \le \sqrt{1 + M^2}|x' - y'|.$$

For any $u \in H^s(\partial\Omega)$ and $0 < s < 1$,

$$\begin{aligned}
\|u\|^2_{s,\partial\Omega} &= \int_{\partial\Omega} |u(x)|^2 d\sigma_x + \int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N-1+2s}} d\sigma_x d\sigma_y \\
&\ge \int_{\Re^{N-1}} |u(x', \varphi(x'))|^2 dx' \\
&+ \frac{1}{(1 + M^2)^{\frac{N-1}{2}+s}} \int_{\Re^{N-1}\times\Re^{N-1}} \frac{|u(x', \varphi(x')) - u(y', \varphi(y'))|^2}{|x' - y'|^{N-1+2s}} dx'dy' \\
&\ge \frac{1}{(1 + M^2)^{\frac{N-1}{2}+s}} \|S_\varphi u\|^2_{s,\Re^{N-1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|u\|^2_{s,\partial\Omega} &= \int_{\partial\Omega} |u(x)|^2 d\sigma_x + \int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N-1+2s}} d\sigma_x d\sigma_y \\
&\le \sqrt{1 + (N-1)M^2} \int_{\Re^{N-1}} |u(x', \varphi(x'))|^2 dx' \\
&+ \{1 + (N-1)M^2\} \int_{\Re^{N-1}\times\Re^{N-1}} \frac{|u(x', \varphi(x')) - u(y', \varphi(y'))|^2}{|x' - y'|^{N-1+2s}} dx'dy' \\
&\le \{1 + (N-1)M^2\} \|S_\varphi u\|^2_{s,\Re^{N-1}}.
\end{aligned}$$

Hence

$$\frac{1}{(1 + M^2)^{\frac{N-1+2s}{4}}} \|S_\varphi u\|_{s,\Re^{N-1}} \le \|u\|_{s,\partial\Omega} \le \sqrt{1 + (N-1)M^2} \|S_\varphi u\|_{s,\Re^{N-1}}. \tag{7}$$

Hence S_φ is bounded and injective from $H^s(\partial\Omega)$ to $H^s(\Re^{N-1})$. For any $f \in H^s(\Re^{N-1})$, define

$$u(x', x_N) = f(x')\phi(x_N - \varphi(x')), \quad a.e.\ \ x \in \Re^N,$$

where $\phi \in C_0^\infty(\Re)$ and $\phi(0) = 1$. It is easy to check that $\gamma|_{\partial\Omega} u \in H^s(\partial\Omega)$ and $\gamma|_{\partial\Omega} u = f$. Therefore S_φ is surjective from $H^s(\partial\Omega)$ to $H^s(\Re^{N-1})$. Hence S_φ is a linear bounded invertible operator from $H^s(\partial\Omega)$ to $H^s(\Re^{N-1})$. □

Theorem 2. *Let Ω be a special Lipschitz domain and $\frac{1}{2} < s \le 1$. Then the trace operator $\gamma|_{\partial\Omega} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is continuous.*

Proof. For $\frac{1}{2} < s \le 1$, applying the trace theorem on half-space $\Re^N_+$ [[1], [8]], we have

$$\gamma_0 \text{ is a bounded linear operator from } H^s(\Re^N_+) \text{ onto } H^{s-\frac{1}{2}}(\Re^N_0).$$

Thus for any $u \in H^s(\Omega)$, applying Lemma 1 and Lemma 3, we have

$$\begin{aligned}\|\gamma|_{\partial\Omega}u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &= \|S_\varphi^{-1}(S_\varphi\gamma|_{\partial\Omega}u)\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\le C_1\|S_\varphi\gamma|_{\partial\Omega}u\|_{H^{s-\frac{1}{2}}(\Re^{N-1})} \\ &= C_1\|\gamma_0T_\varphi u\|_{H^{s-\frac{1}{2}}(\Re^{N-1})} \\ &\le C_2\|T_\varphi u\|_{H^s(\Re^N_+)} \\ &\le C_3\|u\|_{H^s(\Omega)},\end{aligned}$$

where we have used the fact that

$$S_\varphi\gamma|_{\partial\Omega} = \gamma_0T_\varphi,$$

and the C_i's always denote constants dependent on $\partial\Omega$. Thus $\gamma|_{\partial\Omega}$ is a bounded linear operator from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial\Omega)$. □

Lemma 4. *Let $1 < s < \frac{3}{2}$. The trace operator γ_0 is a bounded linear operator from $H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1}))$ onto $H^{s-\frac{1}{2}}(\Re^{N-1})$.*

Proof. For $1 < s < \frac{3}{2}$, let

$$W(\Re_+) = H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1})),$$

and

$$W(\Re) = H^s(\Re; L^2(\Re^{N-1})) \cap H^{s-1}(\Re; H^1(\Re^{N-1})).$$

Define the extension operator P_0 from $W(\Re_+)$ to $W(\Re)$ by

$$P_0u(x) = \begin{cases} u(x', x_N), & \text{if } x_N > 0, \\ 6u(x', -x_N) - 8u(x', -2x_N) + 3u(x', -3x_N), & \text{if } x_N < 0. \end{cases}$$

It is easy to verify that P_0 is well defined and $P_0 \in \mathcal{L}(W(\Re_+), W(\Re))$. For any $u \in W(\Re)$, define the norm of u by

$$\|u\|_* = \left\{\int_{\Re^N} ((1+|\xi_N|)^s + (1+|\xi_N|)^{s-1}(1+|\xi'|))^2|\hat{u}(\xi)|^2 d\xi\right\}^{\frac{1}{2}},$$

where $\hat{u}(\xi)$ denotes the Fourier transform of u,

$$\hat{u}(\xi) = \int_{\Re^N} e^{-2\pi i\langle x,\xi\rangle}u(x)dx.$$

$\|\cdot\|_*$ is equivalent to the usual norm on $W(\Re)$. Therefore $u \in W(\Re)$ if and only if

$$\{(1+|\xi_N|)^s + (1+|\xi_N|)^{s-1}(1+|\xi'|)\}\hat{u}(\xi) \in L^2(\Re^N),$$

and there exists an absolute constant C such that

$$\|P_0u\|_* \le C\|u\|_{W(\Re_+)}, \tag{8}$$

for any $u \in W(\Re_+)$. Let

$$w(\xi', \xi_N) = (1+|\xi_N|)^s + (1+|\xi_N|)^{s-1}(1+|\xi'|).$$

Thus for $1 < s < \frac{3}{2}$, we have

$$\int_{\Re} \frac{1}{w^2(\xi', \xi_N)} d\xi_N = 4 \int_1^{+\infty} \frac{1}{(y^s + (1+|\xi'|)y^{s-1})^2} dy$$

$$= 4 \frac{1}{(1+|\xi'|)^{2s-1}} \int_{\frac{1}{1+|\xi'|}}^{+\infty} \frac{1}{(y^s + y^{s-1})^2} dy$$

$$\leq 4 \frac{1}{(1+|\xi'|)^{2s-1}} \int_0^{+\infty} \frac{1}{y^{2s} + y^{2s-2}} dy \leq \frac{C(s)}{(1+|\xi'|)^{2s-1}},$$

where $C(s) = 2\pi + \dfrac{4}{3-2s}$. Therefore for any $u \in W(\Re_+)$, we have

$$\|\gamma_0 u\|^2_{H^{s-\frac{1}{2}}(\Re^{N-1})} = \|(P_0 u)(\cdot, 0)\|^2_{H^{s-\frac{1}{2}}(\Re^{N-1})}$$

$$= \int_{\Re^{N-1}} (1+|\xi'|)^{2s-1} \left| \widehat{P_0 u}(\xi', 0) \right|^2 d\xi'$$

$$= \int_{\Re^{N-1}} (1+|\xi'|)^{2s-1} \left| \int_{\Re} \widehat{P_0 u}(\xi', \xi_N) d\xi_N \right|^2 d\xi'$$

$$\leq \int_{\Re^{N-1}} (1+|\xi'|)^{2s-1} \cdot \left\{ \int_{\Re} \frac{1}{w^2(\xi', \xi_N)} d\xi_N \int_{\Re} w^2(\xi', \xi_N) \left| \widehat{(P_0 u)}(\xi', \xi_N) \right|^2 d\xi_N \right\} d\xi'$$

$$\leq C(s) \int_{\Re^N} w^2(\xi', \xi_N) \left| \widehat{P_0 u}(\xi', \xi_N) \right|^2 d\xi$$

$$= C(s) \|P_0 u\|_*^2 \leq C_1 \|u\|^2_{W(\Re_+)}.$$

Thus γ_0 is bounded from

$$H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1}))$$

to $H^{s-\frac{1}{2}}(\Re^{N-1})$. For any $f \in H^{s-\frac{1}{2}}(\Re^{N-1})$, define

$$\hat{u}(\xi', x_N) = \hat{f}(\xi') \phi(x_N(1+|\xi'|)),$$

where $\phi \in C_0^\infty(\Re)$ and $\phi(0) = 1$. Thus we have

$$\gamma_0 u(x') = f(x'), \quad x' \in \Re^{N-1},$$

$$\hat{u}(\xi) = \frac{1}{1+|\xi'|} \hat{f}(\xi') \hat{\phi}(\frac{\xi_N}{1+|\xi'|}).$$

Hence

$$\|u\|_*^2 = \int_{\Re^N} ((1+|\xi_N|)^s + (1+|\xi_N|)^{s-1}(1+|\xi'|))^2 |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{\Re^{N-1}} \frac{|\hat{f}(\xi')|^2}{(1+|\xi'|)^2} \cdot \left\{ \int_{\Re} ((1+|\xi_N|)^s + (1+|\xi_N|)^{s-1}(1+|\xi'|))^2 \left| \hat{\phi}(\frac{\xi_N}{1+|\xi'|}) \right|^2 d\xi_N \right\} d\xi'$$

$$\leq \int_{\Re^{N-1}} (1+|\xi'|)^{2s-1} |\hat{f}(\xi')|^2 d\xi' \int_{\Re} ((1+|\xi_N|)^s + (1+|\xi_N|)^{s-1})^2 |\hat{\phi}(\xi_N)|^2 d\xi_N$$

$$\leq C \int_{\Re^{N-1}} (1+|\xi'|)^{2s-1} |\hat{f}(\xi')|^2 d\xi'.$$

Therefore $u \in H^s(\Re_+; L^2(\Re^{N-1})) \cap H^{s-1}(\Re_+; H^1(\Re^{N-1}))$. So γ_0 is surjective. Hence the lemma is proved. □

Theorem 3. *Let Ω be a special Lipschitz domain and $1 < s < \frac{3}{2}$. Then the trace operator $\gamma|_{\partial\Omega} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is bounded.*

Proof. Notice that

$$S_\varphi \gamma|_{\partial\Omega} = \gamma_0 T_\varphi.$$

Thus applying Lemma 2, Lemma 3 and Lemma 4, we obtain

$$\begin{aligned} \|\gamma|_{\partial\Omega} u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &= \|S_\varphi^{-1}(S_\varphi \gamma|_{\partial\Omega} u)\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\leq C_1 \|S_\varphi \gamma|_{\partial\Omega} u\|_{H^{s-\frac{1}{2}}(\Re^{N-1})} \\ &= C_1 \|\gamma_0 T_\varphi u\|_{H^{s-\frac{1}{2}}(\Re^{N-1})} \\ &\leq C_2 \|T_\varphi u\|_{H^s(\Re_+; L^2(\Re^{N-1})) \bigcap H^{s-1}(\Re_+; H^1(\Re^{N-1}))} \\ &\leq C_3 \|u\|_{H^s(\Omega)}, \end{aligned}$$

Thus the theorem is proved. □

Now we are ready to prove the trace theorem on general Lipschitz domains.

Proof of Theorem 1. Since Ω is a bounded simply connected Lipschitz domain, there exists a finite number of cylinder coordinate pairs $\{(Z(P_k, r_k), \varphi_k)\}$, $1 \leq k \leq m$, such that $\partial\Omega \subset \bigcup_{k=1}^m Z(P_k, r_k)$. Let $\{\phi_k\}_{k=1}^m$ be a unit decomposition of $\partial\Omega$, i.e.

(a) $\text{supp}(\phi_k) \subset Z(P_k, r_k)$, $1 \leq k \leq m$.
(b) $\sum_{k=1}^m \phi_k(x) = 1$, $x \in \partial\Omega$.

Thus for any $u \in H^s(\Omega)$,

$$\phi_k u \in H^s(\Omega \cap Z(P_k, r_k)), \quad 1 \leq k \leq m,$$

$$\gamma|_{\partial\Omega} u(x) = \sum_{k=1}^m (\phi_k u)(x), \quad x \in \partial\Omega.$$

By the zero-extension technique and $\text{supp}(\phi_k) \subset Z(P_k, r_k)$, $\phi_k(x)u(x)$ may be considered as a function on $\{x \in \Re^N | x_N > \varphi_k(x')\}$. Hence Theorem 2 and Theorem 3 can be applied. Therefore $\forall u \in H^s(\Omega)$, applying Theorem 2 and Theorem 3, we have

$$\begin{aligned}
\|\gamma|_{\partial\Omega} u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &= \|\sum_{k=1}^{m} \phi_k u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\
&\leq \sum_{k=1}^{m} \|\phi_k u\|_{H^{s-\frac{1}{2}}(\partial(\Omega \bigcap Z(P_k, r_k)))} \\
&\leq \sum_{k=1}^{m} C_k \|\phi_k u\|_{H^s(\Omega \bigcap Z(P_k, r_k))} \\
&\leq C\|u\|_{H^s(\Omega)}.
\end{aligned}$$

Thus the theorem is proved. □

As a final remark, it must be pointed out that, for $s > \frac{3}{2}$, the trace operator is a bounded linear operator from $H^s(\Omega)$ to $H^1(\partial\Omega)$. The proof is similar to the above proof, the detail is omitted. When $s = \frac{3}{2}$, no proof is available.

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References

1. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1978. MR **56:**9247
2. M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results*, SIAM J. Math. Anal. **19** (1988), 613–626. MR **89h:**35090
3. Z. Ding and J. Zhou, *Constrained LQR problems governed by the potential equation on Lipschitz domain with point observations*, J. Math. Pures Appl. **74** (1995), 317–344.
4. E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Ren. Sem. Mat. Univ. Padova **27** (1957), 284–305. MR **21:**1525
5. P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Advanced Publishing Program, Boston, 1985. MR **86m:**35044
6. D. S. Jerison and C. E. Kenig, *Boundary value problems on Lipschitz domains*, in Studies in Partial Differential Equations (W. Littmann, ed.), MAA Studies in Math., vol. 23, Math. Assoc. of America, 1982, pp. 1–68. MR **85f:**35057
7. C. E. Kenig, *Recent progress on boundary-value problems on Lipschitz domains*, Proc. Sympos. Pure Math., vol. 43, Amer. Math. Soc., Providence, RI, 1985, pp. 175–205. MR **87e:**35029
8. J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, vol. 1, Springer-Verlag, New York, 1972. MR **50:**2670
9. G.Verchota, *Layer potentials and regularity for the Dirichlet problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611. MR **86e:**35038

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