PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 2, February 1996

SOBOLEV IMBEDDING THEOREMS IN BORDERLINE CASES

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(Communicated by Barbara Lee Keyfitz)

Abstract. An imbedding theorem is given for functions whose gradient belongs to a class slightly larger than $L^n(\Omega)$, $\Omega \subset \mathbb{R}^n$.

1. Introduction

We wish here to come back to the well-known imbedding theorem for $W^{1,n}(\Omega)$ functions due to N. Trudinger [T] and J. Moser [Mo].

One way to shed some light on the phenomena involved consists in extending this result to other spaces closely related to $W^{1,n}$, as in [ALT].

We consider here spaces of functions that are larger than $W^{1,n}$, but are contained in $\bigcap_{1<p<n} W^{1,p}$. More precisely, we consider functions u whose gradient Du satisfies for some $\sigma > 0$

$$\int_\Omega |Du|^n \log^{-\sigma}(e + |Du|)\, dx < \infty, \tag{1}$$

a class of functions motivated by recent work on the regularity properties of Jacobians (see [BFS], [CLMS], [Mu], [IS], [Gr]).

In this paper we prove, in particular, that if $u \in W_0^{1,1}$ and (1) holds, then

$$\int_\Omega \exp\left(\frac{|u(x)|^\alpha}{\lambda}\right) dx < \infty \tag{2}$$

for any $\lambda > 0$, where $\alpha = \frac{n}{n-1+\sigma}$.

This result relies on the simple observation that condition (2) is equivalent to saying that

$$\lim_{p\to\infty} \frac{1}{p}\left(\fint_\Omega |u|^{\alpha p}\, dx\right)^{1/p} = 0$$

(where $\fint_\Omega$ stands for $\frac{1}{|\Omega|}\int_\Omega$).

More generally, our results apply to functions such that

$$\sup_{\varepsilon>0} \varepsilon^\sigma \fint_\Omega |Du|^{n-\varepsilon}\, dx < \infty$$

(see Theorem 2).

Received by the editors July 7, 1993 and, in revised form, September 12, 1994.
1991 *Mathematics Subject Classification.* Primary 46E35.

2. Some preliminary results

Let Ω be an open set in $\mathbb{R}^n$, with finite measure $|\Omega|$. We denote by EXP $=$ EXP(Ω) the set of functions $g : \Omega \to \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\fint_\Omega \exp\left(\frac{|g|}{\lambda}\right) dx < \infty.$$

One way to test if a function f belongs to EXP is given by the following proposition, whose proof is an easy exercise (see [G, Chapter VI, Example 17]).

Proposition 1. *Let $g : \Omega \to \mathbb{R}$ be a measurable function. Set*

$$E(g) = e \limsup_{p\to\infty} \frac{1}{p} \left(\fint_\Omega |g|^p \, dx \right)^{1/p}.$$

Then

$$E(g) = \inf \left\{ \lambda > 0 : \fint_\Omega \exp\left(\frac{|g|}{\lambda}\right) dx < \infty \right\}.$$

Remark. This proposition says that $g \in$ EXP if and only if $E(g) < \infty$. In particular, if $f \in L^\infty(\Omega)$ we have $E(f) = 0$ and $E(g - f) = E(g)$ for any $g \in$ EXP.

We recall that EXP is a Banach space under the norm (see [RR])

$$\|g\|_{\mathrm{EXP}} = \inf \left\{ \lambda > 0 : \fint_\Omega \exp\left(\frac{|g|}{\lambda}\right) dx \le 2 \right\}.$$

We also remark that L^∞ is not a dense subspace of EXP. Indeed one can easily prove the following

Proposition 2. *$E(g) = 0$ if and only if there exists a sequence $f_h \in L^\infty$ such that $\|f_h - g\|_{\mathrm{EXP}} \to 0$.*

Proof. If f_h verifies $\|f_h - g\|_{\mathrm{EXP}} \to 0$, then by the previous remark:

$$E(g) = E(f_h - g) \le \|f_h - g\|_{\mathrm{EXP}}.$$

Conversely, assume that $E(g) = 0$, and let $\varepsilon > 0$. By definition, we have

$$\fint_\Omega \exp\left(\frac{|g|}{\varepsilon}\right) dx < \infty;$$

then there exists h_ε such that, for $h > h_\varepsilon$,

$$\frac{1}{|\Omega|} \int_{\{|g|>h\}} \exp\left(\frac{|g|}{\varepsilon}\right) dx \le 1.$$

If we set

$$f_h(x) = \begin{cases} g(x) & \text{if } |g(x)| \le h, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\fint_\Omega \exp\left(\frac{|f_h - g|}{\varepsilon}\right) dx = \frac{1}{|\Omega|} \int_{\{|g|>h\}} \exp\left(\frac{|g|}{\varepsilon}\right) dx + \frac{|\{|g| \le h\}|}{|\Omega|} \le 2.$$

Therefore we have

$$\|f_h - g\|_{\mathrm{EXP}} \le \varepsilon \quad \text{for } h > h_\varepsilon.$$

3. The main results

The first Sobolev type result we want to deduce is the following

Theorem 1. *If $u \in W_0^{1,1}(\Omega)$ satisfies for some $\sigma > 0$ the condition*

$$\lim_{\varepsilon\to 0^+} \varepsilon^\sigma \int_\Omega |Du|^{n-\varepsilon}\,dx = 0, \tag{3}$$

then, for any $c > 0$, we have

$$\int_\Omega \exp\left(\frac{|u|^\alpha}{c}\right) dx < \infty \tag{4}$$

with $\alpha = \frac{n}{n-1+\sigma}$, i.e. $E(|u|^\alpha) = 0$.

Let us recall that the Riesz potential If of a function $f \in L^1(\Omega)$ is defined as

$$If(x) = \int_\Omega f(y)|x-y|^{1-n}\,dy.$$

We will use the following theorem ([GT, Theorem 7.34]).

Theorem. *Let $1 \le p, q \le \infty$ and assume that $0 \le \delta = \frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. Then*

$$\|If\|_{L^q(\Omega)} \le \left(\frac{1-\delta}{\frac{1}{n}-\delta}\right)^{1-\delta} \omega_n^{1-1/n} |\Omega|^{1/n-\delta} \|f\|_{L^p(\Omega)} \tag{5}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

We can now pass to the

Proof of Theorem 1. Since

$$|u(x)| \le \frac{1}{n\omega_n} I(|Du|)(x),$$

using Proposition 1, it will be enough to prove that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} \left(\int_\Omega [I(|Du|)]^{1/\varepsilon}\,dx\right)^\varepsilon = 0. \tag{6}$$

In (5) let us take $q = \frac{1}{\varepsilon}, p = n - \varepsilon$, and note that, if $0 < \varepsilon \le \frac{1}{n}$, then certainly $\delta < \frac{1}{n}$; hence we have

$$\varepsilon^{1/\alpha}\left(\int_\Omega [I(|Du|)]^{1/\varepsilon}\,dx\right)^\varepsilon \le \varepsilon^{1/\alpha}\left(\frac{1-\delta}{\frac{1}{n}-\delta}\right)^{1-\delta} \omega_n^{1-1/n}|\Omega|^{1/n-\delta-\varepsilon}\|Du\|_{L^{n-\varepsilon}}(\Omega).$$

It is easy to check that, if $0 < \varepsilon \le \frac{1}{n}$, then

$$\left(\frac{1-\delta}{\frac{1}{n}-\delta}\right)^{1-\delta} \le c(n)\varepsilon^{-(n-1)/n};$$

so we have

$$\begin{aligned}
&\varepsilon^{1/\alpha}\left(\int_\Omega [I(|Du|)]^{1/\varepsilon}\,dx\right)^\varepsilon \\
&\quad\le c(n,|\Omega|)\varepsilon^{1/\alpha-(n-1)/n}\varepsilon^{-\sigma/(n-\varepsilon)}\left(\varepsilon^\sigma \int_\Omega |Du|^{n-\varepsilon}\,dx\right)^{1/(n-\varepsilon)} \\
&\quad\le c(n,|\Omega|)\varepsilon^{1/\alpha-(n-1)/n-\sigma/n}\left(\varepsilon^\sigma \int_\Omega |Du|^{n-\varepsilon}\,dx\right)^{1/(n-\varepsilon)}
\end{aligned}$$

which proves (6), since $\frac{1}{\alpha} - \frac{n-1}{n} - \frac{\sigma}{n} = 0$.

Remark 1. If $g \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega |g|^n \log^{-\sigma}(e+g)\,dx < +\infty,$$

then (see [BFS, Lemma 3])

$$\lim_{\varepsilon\to 0^+} \varepsilon^\sigma \int_\Omega |g|^{n-\varepsilon}\,dx = 0.$$

From this it follows that if $u \in W^{1,1}_0(\Omega)$ and $|Du|^n \log^{-\sigma}(e+|Du|) \in L^1(\Omega)$, then

$$\int_\Omega \exp\left(\frac{|u|^\alpha}{c}\right) < \infty$$

for any $c > 0$, where α is given by Theorem 1. Note that, in any dimension, if $\sigma = 1$, then $\alpha = 1$.

Remark 2. Theorem 1 is optimal in the sense that the exponent α cannot be improved. In fact, if

$$u(x) = \frac{\log|x|}{|\log|\log|x|\,|\,|^\vartheta}$$

for $|x|$ small, with $\vartheta > 1/n$, then $|Du|^n \log^{-1}(e+|Du|)$ is in L^1, while, for any $c > 0, \delta > 0$

$$\int \exp\left(\frac{|u|^{1+\delta}}{c}\right) = \infty.$$

We conclude by proving an imbedding theorem in terms of the quantity on the left-hand side of (3), which can be regarded as an extension of Trudinger's imbedding theorem.

Theorem 2. *Let $u \in W^{1,1}_0(\Omega)$ satisfy for some $\sigma \geq 0$*

$$M = \sup_{0<\varepsilon\leq 1}\left(\varepsilon^\sigma \int_\Omega |Du|^{n-\varepsilon}\,dx\right)^{1/(n-\varepsilon)} < \infty. \tag{7}$$

Then, if $\alpha = \frac{n}{n-1+\sigma}$, there exist $c_1 = c_1(n,\sigma)$, $c_2 = c_2(n,\sigma)$ such that

$$\int_\Omega \exp\left(\frac{|u|}{c_1 M|\Omega|^{1/n}}\right)^\alpha dx \leq c_2. \tag{8}$$

Proof. Arguing as in the proof of Theorem 1, we have for any $0 < \varepsilon \leq \frac{1}{n}$

$$\begin{aligned}\varepsilon^{1/\alpha}\left(\int_\Omega |u(x)|^{1/\varepsilon}\right)^\varepsilon &\leq \frac{\varepsilon^{1/\alpha}}{n\omega_n}\left(\int_\Omega [I(|Du|)]^{1/\varepsilon}\,dx\right)^\varepsilon \\ &\leq c(n)\varepsilon^{1/\alpha-(n-1)/n}|\Omega|^{1/n}\left(\varepsilon^\sigma \int_\Omega |Du|^{n-\varepsilon}\,dx\right)^{1/(n-\varepsilon)} \\ &\leq c(n)\varepsilon^{1/\alpha+1/n-1-\sigma/n}|\Omega|^{1/n}M \\ &= c(n)|\Omega|^{1/n}M.\end{aligned}$$

From this inequality, taking $\alpha\varepsilon = \frac{1}{p}$, we get

$$\sup_{p\geq \frac{n}{\alpha}} \frac{1}{p} \left(\ {}_{\Omega} (|u|^\alpha)^p \, dx \right)^{1/p} \leq [c(n)|\Omega|^{1/n} M]^\alpha.$$

The result now follows, noting that if

$$\lambda = \sup_{p\geq p_0} \frac{1}{p} \left(\ {}_{\Omega} |u|^p \, dx \right)^{1/p},$$

where $p_0 \geq 1$, then

$${}_{\Omega} \exp\left(\frac{|u|}{2e\lambda} \right) dx \leq c(p_0).$$

Remark 3. If $|Du| \in L^{n,\infty} = \text{weak-}L^n$, then (7) holds with $\sigma = 1$ (see [IS]) and then (8) holds with $\alpha = 1$. For this and similar results, see [ALT]. Note that Theorem 2 holds true also in the case $\sigma = 0$. In this case the above result reduces to Trudinger's theorem (see [T]).

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