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REPRESENTING THE AUTOMORPHISM GROUP OF AN ALMOST CRYSTALLOGRAPHIC GROUP

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ABSTRACT. Let E be an almost crystallographic (AC-) group, corresponding to the simply connected, connected, nilpotent Lie group L and with holonomy group F. If $L^F = \{1\}$, there is a faithful representation $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aff}(L)$. In case E is crystallographic, this condition $L^F = \{1\}$ is known to be equivalent to $Z(E) = 1$ or $b_1(E) = 0$. We will show (Example 2.2) that, for AC-groups E, this is no longer valid and should be adapted. A generalised equivalent algebraic (and easier to verify) condition is presented (Theorem 2.3). Corresponding to an AC-group E and by factoring out subsequent centers we construct a series of AC-groups, which becomes constant after a finite number of terms. Under suitable conditions, this opens a way to represent $\mathrm{Aut}(E)$ faithfully in $\mathrm{Gl}(k,\mathbb{Z}) \times \mathrm{Aff}(L_1)$ (Theorem 4.1). We show how this can be used to calculate $\mathrm{Out}(E)$. This is of importance, especially, when E is almost Bieberbach and, hence, $\mathrm{Out}(E)$ is known to have an interesting geometric meaning.

1. Preliminaries

Let us fix some notation here. If G is a group and $x, y \in G$, we will use the following conventions: $[x,y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$. The following commutator identities are rather well known, and will be used later on.

(1)

$$\forall x, y \in G: \ \forall m \in \mathbb{N}_0: \quad [x^m, y] = \prod_{j=1}^{m} [x,y]^{x^{m-j}} \ \text{ and } \quad [x, y^m] = \prod_{j=0}^{m-1} [x,y]^{y^j}.$$

We will write $\mu(x)$ for the inner automorphism of G determined by x; i.e. $\mu(x)(y) = xyx^{-1}$.

Recall that the lower central series of G is defined inductively by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ $(n \in \mathbb{N}_0)$. G is said to be c-step nilpotent (or nilpotent of class c) if and only if $\gamma_c(G) \neq \{1\}$ and $\gamma_{c+1}(G) = \{1\}$. It then follows that $\gamma_c(G) \subseteq Z(G)$. Furthermore, its upper central series is defined inductively by $Z_0(G) = \{1\}$ and $Z_{n+1}(G)/_{Z_n(G)} = Z(G/_{Z_n(G)})$ $(n \in \mathbb{N})$.

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We recall also briefly the concept of the isolator (sometimes called the root set) of a subgroup in a group.

Definition 1.1 ([Pas77], [Seg83])**.** Let G be a group and H a subgroup of G. The **isolator** of H in G is defined by

$$\sqrt[G]{H} = \{g \in G \parallel g^k \in H \text{ for some } k \geq 1\}.$$

It is well known that, for every group G, $\sqrt[G]{\gamma_k(G)}$ is a characteristic subgroup of G and

$$[\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \subseteq \sqrt[G]{\gamma_{k+l}(G)}.$$

The first Betti number of a finitely generated group G, written as $b_1(G)$, is defined as the torsion-free rank of its abelianised group $G/_{\gamma_2(G)}$. Now, it is clear that, if G is a finitely generated group and $\mathbb{Z}^k$ is a trivial G-module, then

$$Z^1(G, \mathbb{Z}^k) = H^1(G, \mathbb{Z}^k) \cong \mathbb{Z}^{kb_1(G)}. \tag{2}$$

Consequently,

Lemma 1.2. *Let G be a finitely generated group with torsion-free center $Z(G)$. If $b_1(G/_{Z(G)}) = 0$, then $Z(G/_{Z(G)}) = \{1\}$.*

Proof. Take $xZ(G) \in Z(G/_{Z(G)})$, $x \in G$. The inner automorphism $\mu(x)$ induces the identity on $Z(G)$ and on $G/_{Z(G)}$. The subgroup of all such automorphisms of G is isomorphic to $Z^1(G/_{Z(G)}, Z(G))$ (e.g. see [IM94]). Since $Z(G)$ is a torsion-free, trivial $G/_{Z(G)}$-module and $b_1(G/_{Z(G)}) = 0$, we conclude that $Z^1(G/_{Z(G)}, Z(G))$ is trivial (use (2)) or $x \in Z(G)$. □

2. Automorphisms of almost crystallographic groups I

Let L be a connected, simply connected, nilpotent Lie group. We write $\mathrm{Aff}(L)$ for the semi-direct product $L \rtimes \mathrm{Aut}(L)$. $\mathrm{Aff}(L)$ is called the group of affine diffeomorphisms of L and acts in a natural way on L; for $x, y \in L$ and $\alpha \in \mathrm{Aut}(L)$, ${}^{(x,\alpha)}y = x\ \alpha(y)$. Let C be a compact subgroup of $\mathrm{Aut}(L)$. A uniform, discrete subgroup E of $L \rtimes C \subset \mathrm{Aff}(L)$ is called an almost crystallographic (AC-) group (of L). It is well known that $N = E \cap L$ ([Aus60]) is a torsion-free , finitely generated, nilpotent normal subgroup of finite index in E, which is maximal nilpotent in E. The finite group $F = E/_{E \cap L}$, which is sometimes called the holonomy group, acts faithfully on L. The Hirsch length (rank) of N is often referred to as the dimension of E.

As an abstract group, a group E is AC if and only if it contains a torsion-free, finitely generated, nilpotent normal subgroup N of finite index, which is maximal nilpotent in E. In this case, N equals the Fitting subgroup $\mathrm{Fitt}(E)$ of E, which is defined as the subgroup generated by all nilpotent normal subgroups of E (see [Seg83]). In this case, the Lie group L is the Mal'cev completion of N ([Mal51]). Consequently, it is clear at once that isomorphic AC-groups correspond to the same Lie group L.

A torsion-free AC-group is called an almost Bieberbach (AB-) group. AC-(resp. AB-) groups have been studied intensively as generalisations of classical crystallographic (resp. Bieberbach) groups (i.e. the situation with $L = \mathbb{R}^k$). In this perspective, the following theorem, which is a generalisation of the classical second Bieberbach theorem, can be found in [LR85].

Theorem 2.1. *If $f : E \to E'$ is an isomorphism of two AC-groups (of L), then f can be realised as conjugation in* $\mathrm{Aff}(L)$*.*

Let L^F be the subset of L consisting of points fixed under the action of F. One easily verifies that

$$C_{\mathrm{Aff}(L)}(E) = \{(x, \mu(x^{-1})) \in \mathrm{Aff}(L) \parallel x \in L^F\} \cong L^F.$$

Consequently, if $L^F = \{1\}$, an isomorphism of two AC-groups (of L) can be realised in a unique way as an affine conjugation. Applied to a given AC-group E, one obtains a representation $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aff}(L)$.

Unfortunately, in the usual algebraic setting, verifying if $L^F = \{1\}$ is often very hard, as one does not know the action of F on L explicitly. In the crystallographic case, however, a simple algebraic equivalent condition is known. To see this, note that L^F contains the abelian, normal subgroup $Z(L)^F$. Also, $Z(E)$ is a uniform lattice of $Z(L)^F$. Therefore, E is centerless if and only if $Z(L)^F = \{1\}$. Consequently, if E is a centerless, k-dimensional, *crystallographic* group (in which case $L = Z(L)$) (or equivalently, if E is crystallographic with first Betti number zero ([HS86])), each isomorphism between E and another crystallographic group E' can be realised as conjugation by a *unique* element of $\mathrm{Aff}(\mathbb{R}^k)$. So, if E is crystallographic with $b_1(E) = 0$, there is a well-defined representation $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aff}(\mathbb{R}^k)$.

As a first observation, we show with an example (more general for centerless AC-groups (L non-abelian)) that a similar *faithful* representation does not hold anymore; i.e. being centerless is not any more sufficient to imply that an automorphism of E is realised as conjugation by a *unique* element in $\mathrm{Aff}(L)$. Remark that this does not imply that a representation $\mathrm{Aut}(E) \to \mathrm{Aff}(L)$ does not exist.

Example 2.2. Consider the Heisenberg group

$$H = \{\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}\}$$

which is connected, simply connected and nilpotent of class 2. In H consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In a connected, simply connected nilpotent Lie group G, it makes sense to speak about $g^x = \exp\,(x \ \log g)$, where $g \in G$ and $x \in \mathbb{R}$. So, we can say that $H = \{A^x B^y C^z \parallel x, y, z \in \mathbb{R}\}$. Observe that, in H, the relations $[B, A] = C, [C, A] = 1$ and $[C, B] = 1$ hold. Also, remark that $Z(H) = \{C^z \parallel z \in \mathbb{R}\} \cong \mathbb{R}$.

Take the uniform lattice N in H generated by $a = A$, $b = B$ and $c = \sqrt{C}$. As a presentation for N, we have

$$N : \langle a, b, c \parallel [b, a] = c^2,\ [c, a] = [c, b] = 1\rangle.$$

Take $F \cong \mathbb{Z}_2$, given as $\{1, \alpha\}$, and let F act on N via the homomorphism $\varphi : F \to \mathrm{Aut}(N)$ given by

$$\varphi(\alpha) : N \to N : a \mapsto a\ ,\ b \mapsto b^{-1}\ ,\ c \mapsto c^{-1}.$$

This can be lifted uniquely to an action $\widetilde{\varphi} : F \to \mathrm{Aut}(H)$; then α sends $A \mapsto A$, $B \mapsto B^{-1}$ and $C \mapsto C^{-1}$. Clearly, $H^F = \{A^x \parallel x \in \mathbb{R}\} \cong \mathbb{R}$ while $Z(H)^F$ is trivial.

Let $1 \to N \to E \cong N \rtimes F \to F \to 1$ be the semi-direct product determined by φ. E.g. a presentation of E could be

$$E : \langle a, b, c, \alpha \parallel [b,a] = c^2,\ [c,a] = [c,b] = 1,$$
$$\alpha a = a\alpha,\ \alpha b = b^{-1}\alpha,\ \alpha c = c^{-1}\alpha,\ \alpha^2 = 1\rangle.$$

It is not hard to see that E is a 3-dimensional, centerless AC-group having N as the 2-step nilpotent maximal nilpotent subgroup. An embedding $\imath$ of E into $\mathrm{Aff}(H) = H \rtimes \mathrm{Aut}(H)$ is given by

$$\imath : E \hookrightarrow \mathrm{Aff}(H) : a \mapsto (A, 1)\ ,\ b \mapsto (B, 1)\ ,\ c \mapsto (\sqrt{C}, 1)\ ,\ \alpha \mapsto (1, \widetilde{\varphi}(\alpha)).$$

The following automorphism of E

$$\sigma : E \to E : a \mapsto ac\ ,\ b \mapsto b\ ,\ c \mapsto c\ ,\ \alpha \mapsto b\alpha$$

can be realised as conjugation by the affinities $(h_\sigma, \alpha_\sigma) \in H \rtimes \mathrm{Aut}(H)$, where $h_\sigma = A^x B^{\frac{1}{2}} C^{\frac{x}{2}}$ and $\alpha_\sigma : H \to H$ sends $A \mapsto A$, $B \mapsto BC^x$ and $C \mapsto C$, for each $x \in \mathbb{R}$.

Our next aim is to present a necessary and sufficient condition for the situation $L^F = \{1\}$, given an AC-group E of L with holonomy F.

From now on, we assume that $1 \to N \to E \to F \to 1$ is a short exact sequence of groups where N is finitely generated, torsion-free , c-step nilpotent of finite index in E and maximal nilpotent in E. These extensions have been called "essential" ([Lee88]) or "strict normal" ([GS92]).

Define the following quotients:

$$\tau_i(N) = N/\sqrt[N]{\gamma_{i+1}(N)} \text{ and } \tau_i(E) = E/\sqrt[N]{\gamma_{i+1}(N)}, 1 \le i \le c.$$

It is known that $1 \to \tau_{c-1}(N) \to \tau_{c-1}(E) \to F \to 1$ is again essential ([DIM93]) and that $\tau_{c-1}(N)$ is nilpotent of class $c-1$. Obviously, $\tau_j(\tau_{c-1}(E))$ $(\tau_j(\tau_{c-1}(N)))$ is isomorphic to $\tau_j(E)$ $(\tau_j(N)))$ $(1 \le j \le c-1)$, and hence, by induction, all extensions $1 \to \tau_i(N) \to \tau_i(E) \to F \to 1$ $(1 \le i \le c)$ are essential.

Theorem 2.3. *Let E be an AC-group given by an essential extension $1 \to N \to E \to F \to 1$ where N is c-step nilpotent, and assume L is the Mal'cev completion of N. Then, $L^F = \{1\}$ if and only if all AC-groups $\tau_i(E)$ $(1 \le i \le c)$ are centerless.*

Proof. In the abelian case $(c = 1)$, we already observed that E is centerless if and only if $L^F = \{1\}$.

We proceed by induction on c. As mentioned above, $\tau_{c-1}(E)$ is an AC-group with Fitting subgroup $\tau_{c-1}(N)$. This is a $(c-1)$-step nilpotent group with Mal'cev completion $L/_{\gamma_c(L)}$. Remark that all AC-groups $\tau_j(\tau_{c-1}(E)) \cong \tau_j(E)$ $(1 \le j \le c-1)$ are centerless. Hence, by induction, $(L/_{\gamma_c(L)})^F = \{1\}$ or $L^F = \gamma_c(L)^F$. Now, because L is c-step nilpotent $(\gamma_c(L) \subset Z(L))$ and E is centerless $(Z(L)^F = \{1\})$, we can conclude that L^F is trivial.

To prove the converse, first observe that it will be sufficient to show that, if $L^F = \{1\}$, then $(L/_{\gamma_c(L)})^F = \{1\}$. This will imply, by induction, that all $\tau_j(\tau_{c-1}(E)) \cong \tau_j(E)$ $(1 \le j \le c-1)$ are centerless. Added to the fact that, if L^F is trivial, then also $\tau_c(E) = E$ is centerless, this then finishes the claim.

Let k be the order of F and assume $\ell\gamma_c(L)$ is a fixed point for the action of F on $L/_{\gamma_c(L)}$. Define a (normalised) 1-cochain $\lambda : F \to \gamma_c(L) \subset Z(L)$ as follows: for $x \in F$, ${}^x\ell = \ell\ \lambda(x)$. It is easily verified that $\lambda : F \to Z(L)$ is a 1-cocycle.

As L and $Z(L)$ are both divisible with unique roots, one can consider the element $\ell_0 = \prod_{y\in F} \lambda(y)^{\frac{1}{k}} \in Z(L)$. Now, verify that, for $x \in F$,

$$^{x}\ell_0 = \prod_{y\in F} {}^{x}\lambda(y)^{\frac{1}{k}} = \prod_{y\in F} (\lambda(xy)\ \lambda(x)^{-1})^{\frac{1}{k}} = \prod_{y\in F} \lambda(xy)^{\frac{1}{k}} \prod_{y\in F} \lambda(x)^{-\frac{1}{k}} = \ell_0\ \lambda(x)^{-1}.$$

Consequently $\lambda : F \to Z(L)$ is a 1-coboundary. Hence, for all $x \in F$,

$$^{x}(\ell\ell_0) = \ell\ \lambda(x)\ {}^{x}\ell_0 = \ell\ \ell_0$$

and there is a fixed point for the action of F on L. This contradicts the assumption. We conclude that $(L/_{\gamma_c(L)})^F = \{1\}$. ☐

Corollary 2.4. *Let E be an AC-group given by an essential extension $1 \to N \to E \to F \to 1$ as above. If all AC-groups $\tau_i(E)$ $(1 \le i \le c)$ are centerless, then there is a well-defined faithful representation* $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aff}(L)$.

3. The AC-series of an almost crystallographic group

It is well known (e.g. [DIM93, Corollary 5.5]) that, if $1 \to N \to E \to F \to 1$ is an essential extension, the corresponding abstract kernel $\psi : F \to \mathrm{Out}(N)$ is injective. For such an extension it then follows that $C_E(N) = Z(N)$ and, hence, that $Z(E)$ is a normal subgroup of N. This allows us to state the following

Proposition 3.1. *If $1 \to N \to E \to F \to 1$ is an essential extension, then $1 \to N/_{Z(E)} \to E/_{Z(E)} \to F \to 1$ is essential too.*

Proof. Let us first show that $Z(N)/_{Z(E)}$ is torsion-free . Assume $x \in Z(N)$ and $x^m \in Z(E)$, for $m \ge 2$. Then, for each $y \in E$,

$$1 = [x^m, y] \overset{(1)}{=} \prod_{j=1}^{m} [x, y]^{x^{m-j}} = [x, y]^m$$

and, as N is torsion-free , $x \in Z(E)$.

Now, $1 \to Z(N)/_{Z(E)} \to N/_{Z(E)} \to N/_{Z(N)} \to 1$ shows that $N/_{Z(E)}$ is a central extension of a torsion-free , abelian group by a torsion-free nilpotent group. Therefore $N/_{Z(E)}$ itself is torsion-free and, of course, nilpotent of class $\le c$ (if N is of class $\le c$).

$N/_{Z(E)}$ is quite easily seen to be maximal nilpotent in $E/_{Z(E)}$. Indeed, suppose N' is a subgroup of E containing N such that $N'/_{Z(E)}$ is nilpotent. Then $1 \to Z(E) \to N' \to N'/_{Z(E)} \to 1$ is a central extension by a nilpotent group and consequently N' is nilpotent. This contradicts the maximal nilpotency of N. ☐

This allows the introduction of the following

Definition 3.2. For an AC-group E given by an essential extension $1 \to N \to E \to F \to 1$, we define the associated **AC-series** $(E_i)_{i\in\mathbb{N}}$ inductively as follows:

$$E_0 = E, E_{i+1} = E_i/_{Z(E_i)} \cong \mathrm{Inn}(E_i).$$

Remark 3.3. By defining, in a similar way, $N_0 = N$, $N_{i+1} = N_i/_{Z(E_i)}$, it is clear that each extension $1 \to N_i \to E_i \to F \to 1$ is essential and hence the groups E_i are AC-groups. This motivates the term AC-series.

As could be expected from the definition of the AC-series, there is an interesting and nice relation of this concept and the upper central series terms of E and N. We summarise as follows:

Lemma 3.4. *Assume $1 \to N \to E \to F \to 1$ is an essential extension, and $(E_i)_i$ is the associated AC-series of E.*

1. $\forall i : E_i \cong E/_{Z_i(E)}$ *and* $N_i \cong N/_{Z_i(E)}$.
2. $\forall i : Z_i(E) \subset Z_i(N)$.
3. $\forall i : (Z_{i+1}(E) \setminus Z_i(E)) \subset (Z_{i+1}(N) \setminus Z_i(N))$.
4. *If c is the nilpotency class of N, then, after at most c steps, the AC-series $(E_i)_i$ becomes constant.*

Proof. (1) The first part is obvious from the definition of E_i. Since $E_i/_{N_i} \cong E/_N \cong F$, we conclude that $N_i \cong N/_{Z_i(E)}$.

(2) Use induction on i. If $x \in Z_{i+1}(E)$, then $x \in N$ and $xZ_i(E) \in Z(E/_{Z_i(E)} \cong E_i)$. Hence, $[x, E] \subset Z_i(E) \subset Z_i(N)$ and consequently also $[x, N] \subset Z_i(N)$, which means that $x \in Z_{i+1}(N)$.

(3) Again we proceed by induction on i. For $x \in Z_{i+2}(E) \setminus Z_{i+1}(E)$, there exists an element y_0 in E such that $[x, y_0] \in Z_{i+1}(E)$ and $[x, y_0] \notin Z_i(E)$. By the induction hypothesis, $[x, y_0] \in Z_{i+1}(N) \setminus Z_i(N)$.

We claim that $x \notin Z_{i+1}(N)$. Write k for the index of N in E. Then, $y_0^k \in N$. Assume $[x, y_0^k] \in Z_i(N)$. Remark that, as $[x, y_0] \in Z_{i+1}(E)$, for each $z \in E$, $[[x, y_0], z] \in Z_i(E) \subset Z_i(N)$. Therefore, in $N/_{Z_i(N)}$, we obtain

$$\begin{aligned} 1 = [x, y_0^k]Z_i(N) &\overset{(1)}{=} \Big(\prod_{j=0}^{k-1} [x, y_0]^{y_0^j}\Big)\, Z_i(N) \\ &= [x, y_0]^k Z_i(N) = ([x, y_0]Z_i(N))^k. \end{aligned}$$

As $N/_{Z_i(N)}$ is torsion-free, $[x, y_0] \in Z_i(N)$ which is a contradiction. Hence, $[x, y_0^k] \notin Z_i(N)$ and $x \notin Z_{i+1}(N)$.

(4) If N is of class c, then $Z_c(N) = N$ and consequently $Z_{c+1}(E) \setminus Z_c(E) \subset N \setminus N = \{1\}$ or $Z_{c+1}(E) = Z_c(E)$. Then $E_{c+1} = E_c$. □

For an essential extension $1 \to N \to E \to F \to 1$, defining the AC-group E, we now know that there exists a minimal $\ell \in \mathbb{N}$ such that $E_{\ell+i} = E_\ell$, for all $i \in \mathbb{N}$. Moreover, according to Lemma 3.4, $0 \le \ell \le c$ (c the nilpotency class of N). Let us call this ℓ the *length of the AC-series of E*.

As a direct consequence of Lemma 1.2, we have

Corollary 3.5. *Let E be an AC-group and ℓ the length of the associated AC-series of E. If $b_1(E_i) = 0$ for some $1 \le i < c$, then $\ell \le i$.*

4. Automorphisms of almost crystallographic groups II

Write L_1 for the Mal′cev completion of $N_1 = N/_{Z(E)}$, and also assume that $Z(E)$, which is torsion-free abelian, is of rank k, i.e. $Z(E) \cong \mathbb{Z}^k$. We can now state

Theorem 4.1. *Let $1 \to N \to E \to F \to 1$ be an essential extension and L the Mal′cev completion of N. Then,*

(1) *if* $\ell = 0$ *and* $L^F = \{1\}$*, there exists a faithful representation* $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aff}(L)$;
(2) *if* $\ell \neq 0$, $b_1(E_1) = 0$ *and* $L_1^F = \{1\}$*, there exists a faithful representation of* $\mathrm{Aut}(E) \hookrightarrow \mathrm{Gl}(k, \mathbb{Z}) \times \mathrm{Aff}(L_1)$.

Proof. The case $\ell = 0$ (E is centerless) was treated already before.

Assume $\ell > 0$. First remark that if $b_1(E_1) = 0$, then, because of Corollary 3.5, the AC-series of E has length $\ell = 1$ (or $Z(E_1) = \{1\}$). An automorphism σ of E restricts to an automorphism $\varphi(\sigma)$ of $Z(E)$, and consequently induces an automorphism $\bar{\sigma}$ of E_1. So, σ gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & Z(E) & \to & E & \to & E_1 & \to & 1 \\ & & \downarrow \varphi(\sigma) & & \downarrow \sigma & & \downarrow \bar{\sigma} & & \\ 1 & \to & Z(E) & \to & E & \to & E_1 & \to & 1 \end{array}$$

Since L_1^F is trivial, there exists a faithful representation $\rho(\sigma)$ of $\bar{\sigma}$ in $\mathrm{Aff}(L_1)$. Clearly φ and ρ are homomorphisms. It remains to show that $\varphi \times \rho : \mathrm{Aut}(E) \to \mathrm{Gl}(k_0, \mathbb{Z}) \times \mathrm{Aff}(L_1)$ is injective. An automorphism $\sigma \in \mathrm{Aut}(E)$ lies in the kernel of $\varphi \times \rho$ if and only if σ induces the identity on both $Z(E)$ and E_1. The subgroup of all such automorphisms in $\mathrm{Aut}(E)$ is isomorphic to $Z^1(E_1, Z(E))$ (e.g. see [IM94]). Since $Z(E)$ is a trivial E_1-module and $b_1(E_1) = 0$ (use (2)), we conclude that this $Z^1(E_1, Z(E))$ is trivial and, hence, that $\varphi \times \rho$ is faithful. □

Remark 4.2. If E is crystallographic, Theorem 4.1 reduces to the result found in [Lee82, Lemma 1].

AB-groups E are precisely the fundamental groups of the infra-nilmanifolds. These manifolds are aspherical. In view of this, the study of $\mathrm{Out}(E)$ can be considered of special interest (e.g. [CR77], [IM94]). In a search to represent and to control $\mathrm{Out}(E)$, Theorem 4.1 can be most useful. This is certainly true in a situation where E_ℓ is crystallographic. A good algebraic source of examples of AB-groups is found in [DIKL93] (all isomorphism types in dimension 3) and in [Dek93] (dimension ≤ 4). The following example uses an AB-group of type **27** in [Dek93].

Example 4.3. Consider the group E presented as:

$$E: \ \langle a, b, c, d, \alpha, \beta \parallel \begin{array}{lll} [b,a] = d^4 & [c,a] = 1 & [c,b] = 1 \\ [d,a] = 1 & [d,b] = 1 & [d,c] = 1 \\ \alpha a = a^{-1}\alpha & \alpha b = b^{-1}\alpha & \alpha c = c\alpha \\ \alpha d = d\alpha & \alpha^2 = d & \\ \beta a = a\beta & \beta b = b^{-1}\beta & \beta c = c\beta \\ \beta d = d^{-1}\beta & \beta^2 = c & \alpha\beta = \beta\alpha d^{-1} \end{array} \rangle$$

E fits into an essential extension

$$1 \to N \to E \to F \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1,$$

where N is a 2-step nilpotent group of rank 4 given by

$$N: \ \langle a, b, c, d \parallel [b,a] = d^4, \ [c,a] = [c,b] = [d,a] = [d,b] = [d,c] = 1 \rangle.$$

The center $Z(E)$ of E is the subgroup generated by c, and hence:

$$E_1: \ \langle a, b, d, \alpha, \beta \parallel \begin{array}{lll} [b,a] = d^4 & [d,a] = 1 & [d,b] = 1 \\ \alpha a = a^{-1}\alpha & \alpha b = b^{-1}\alpha & \alpha d = d\alpha \\ \beta a = a\beta & \beta b = b^{-1}\beta & \beta d = d^{-1}\beta \\ \alpha^2 = d & \beta^2 = 1 & \alpha\beta = \beta\alpha d^{-1} \end{array} \rangle$$

E_1 is a centerless, 3-dimensional AC-group (type number **6** in [DIKL93]). The Mal'cev completion of its Fitting subgroup is the Heisenberg group H (see also 2.2). We leave it to the reader to verify that $b_1(E_1) = 0$ and that $H^F = \{1\}$ (use 2.3).

We know that E_1 can be embedded in $\mathrm{Aff}(H) = H \rtimes \mathrm{Aut}(H)$. We leave it to the reader to verify that there is a faithful representation $\mathrm{Aff}(H) \hookrightarrow \mathrm{Aff}(\mathbb{R}^3)$ defined as follows: an affine transformation $(h, \alpha) \in H \rtimes \mathrm{Aut}(H)$ s.t.

$$h = A^x B^y C^z \in H \text{ and } \alpha : H \to H : \begin{cases} A \mapsto A^{\alpha_{11}} B^{\alpha_{21}} C^{p_1}, \\ B \mapsto A^{\alpha_{12}} B^{\alpha_{22}} C^{p_2} \end{cases}$$

is represented in $\mathrm{Aff}(\mathbb{R}^3)$ as:

$$\begin{pmatrix} -\alpha_{12}\alpha_{21} + \alpha_{11}\alpha_{22} & -\frac{\alpha_{11}\alpha_{21} + \alpha_{21}x - \alpha_{11}y}{2} + p_1 & -\frac{\alpha_{12}\alpha_{22} + \alpha_{22}x - \alpha_{12}y}{2} + p_2 & -\frac{xy}{2} + z \\ 0 & \alpha_{11} & \alpha_{12} & x \\ 0 & \alpha_{21} & \alpha_{22} & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that the following is a faithful representation of E_1 into $\mathrm{Aff}(H)$ (thus, realising E_1 as a genuine AC-group):

$$a = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{8} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Given an automorphism σ of E, we can compute, using this representation of E_1, the (unique) element of $\mathrm{Aff}(H)$ which realises, through conjugation, the induced E_1-automorphism. The image of c under σ then completes the representation of σ into $\mathrm{Gl}(1, \mathbb{Z}) \times \mathrm{Aff}(\mathbb{R}^3)$.

In [IM94] we present a systematic method to study $\mathrm{Aut}(E)$ and $\mathrm{Out}(E)$ in terms of commutative diagrams. Using the information there, we obtain a short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Out}(E) \to Q \to 1$$

where Q fits in

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to Q \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1.$$

In fact, $\mathrm{Out}(E)$ is generated by the images of the following E-automorphisms:

$$\sigma_1:\left\{\begin{array}{ccc} a & \mapsto & a\\ b & \mapsto & b\\ c & \mapsto & c\\ d & \mapsto & d\\ \alpha & \mapsto & \alpha\\ \beta & \mapsto & d\beta \end{array}\right.,\ \sigma_2:\left\{\begin{array}{ccc} a & \mapsto & ad^2\\ b & \mapsto & b\\ c & \mapsto & c\\ d & \mapsto & d\\ \alpha & \mapsto & b\alpha\\ \beta & \mapsto & b\beta \end{array}\right.,\ \sigma_3:\left\{\begin{array}{ccc} a & \mapsto & a\\ b & \mapsto & bd^2\\ c & \mapsto & c\\ d & \mapsto & d\\ \alpha & \mapsto & a^{-1}\alpha\\ \beta & \mapsto & \beta \end{array}\right.,$$

$$\sigma_4:\left\{\begin{array}{ccc} a & \mapsto & a\\ b & \mapsto & b\\ c & \mapsto & c^{-1}\\ d & \mapsto & d\\ \alpha & \mapsto & \alpha\\ \beta & \mapsto & c^{-1}\beta \end{array}\right.,\ \sigma_5:\left\{\begin{array}{ccc} a & \mapsto & b\\ b & \mapsto & a\\ c & \mapsto & c\\ d & \mapsto & d^{-1}\\ \alpha & \mapsto & d^{-1}\alpha\\ \beta & \mapsto & \alpha\beta \end{array}\right..$$

These automorphisms are represented in $\mathrm{Gl}(1,\mathbb{Z})\times \mathrm{Aff}(\mathbb{R}^3)$ as

$$\sigma_1=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&\frac18\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\ \sigma_2=\begin{pmatrix}1&0&0&0&0\\0&1&\frac14&0&0\\0&0&1&0&0\\0&0&0&1&\frac12\\0&0&0&0&1\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0&0&0&0\\0&1&0&\frac14&0\\0&0&1&0&-\frac12\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$\sigma_4=\begin{pmatrix}-1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\ \sigma_5=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&\frac1{16}\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}.$$

Observe that a conjugation $\mu(a^{\alpha_1}b^{\alpha_2}d^{\alpha_3})$ ($\alpha_i\in\mathbb{Z}$) in E is represented as

$$\mu(a^{\alpha_1}b^{\alpha_2}d^{\alpha_3})=\begin{pmatrix}1&0&0&0&0\\0&1&\frac{\alpha_2}{2}&-\frac{\alpha_1}{2}&-\frac{\alpha_1\alpha_2}{2}+\frac{\alpha_3}{4}\\0&0&1&0&\alpha_1\\0&0&0&1&\alpha_2\\0&0&0&0&1\end{pmatrix}.$$

Now it is not too hard to verify that $\sigma_1^2=\mu(d)$, $\sigma_2^2=\mu(b)$, $\sigma_3^2=\mu(a^{-1})$, $\sigma_4^2=\sigma_5^2=1$, $[\sigma_5,\sigma_1]=\mu(d)$, $\sigma_5\sigma_2\sigma_5^{-1}=\mu(a)\sigma_3$, $\sigma_5\sigma_3\sigma_5^{-1}=\mu(b^{-1})\sigma_2$, $[\sigma_3,\sigma_2]=\mu(d)$, $[\sigma_5,\sigma_4]=[\sigma_4,\sigma_1]=[\sigma_4,\sigma_2]=[\sigma_4,\sigma_3]=[\sigma_3,\sigma_1]=[\sigma_2,\sigma_1]=1$. From this, finally, it can be deduced that $\mathrm{Out}(E)$ is generated by the projections of σ_1, σ_4, $\sigma_2\sigma_5$ and σ_2 and is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathcal{D}_4$ ($\mathcal{D}_4$ is the dihedral group of order 8).

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