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FACTORIZATION THEOREMS FOR HARDY SPACES OF THE BIDISC, $0 < p \leq 1$

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ABSTRACT. A factorization theorem is proved in the Hardy spaces H^p of the bi-upper half plane, $0 < p \leq 1$. The proof is based on some fundamental work of Chang-Fefferman on atomic decompositions of H^p.

1. Introduction and preliminaries

We are concerned with a factorization theorem which has been known (for $p = 1$) for the unit disc in $\mathbf{C}$ since the first part of the twentieth century. In 1976, Coifman, Rochberg and Weiss [4] extended it to the unit ball in $\mathbf{C}^n$. In 1992, Krantz and Li [6] proved that it holds on smoothly bounded strongly pseudoconvex domains for $0 < p \leq 1$. In this paper, we prove the analogous factorization theorem for the Hardy spaces H^p of the bi-upper half plane, $0 < p \leq 1$. A good reference for Hardy spaces is the book by Krantz [5, Chapter 8].

The following standard notation will be used: $\mathbf{R}$ denotes the real numbers; $\mathbf{C}$ denotes the complex numbers; $x = (x_1, x_2, \ldots, x_n)$ denotes an element of $\mathbf{R}^n$.

As a consequence of the boundedness of the Hilbert transform on $L^q(\mathbf{R})$ for $1 < q < \infty$ (see [8, p. 38]) it follows that $S\colon L^q(\mathbf{R}) \to H^q(\mathbf{R}^2_+)$ is bounded, where S is the Szegő projection for the upper half plane:

$$Sf(z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(w)}{z - w} dw,$$

and $S(z, w) = \frac{1}{\pi} \frac{1}{z-w}$ is the Szegő kernel on the upper half plane $\mathbf{R}^2_+ := \{z = x + iy\colon x \in \mathbf{R}, y > 0\}$.

It is a simple matter to extend this result to higher dimensions.

Lemma 1.1. *If $q > 1$ and $n \geq 1$, then*

$$S\colon L^q(\mathbf{R}^n) \to H^q((\mathbf{R}^2_+)^n)$$

is bounded.

The following two lemmas will be useful.

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Lemma 1.2. *Let $\psi \in L^2(\mathbf{R})$ with support $\subseteq [-l, l]$ and, for $z \in \mathbf{R}^2_+$ let*

$$\phi(z) = \int_{\mathbf{R}} \psi(w) S(z, w) dw = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\psi(w)}{z - w} dw. \tag{1}$$

Then $\phi \in H^2(\mathbf{R})$ and (identifying ϕ with its boundary values in $L^2(\mathbf{R})$)

(1) $\int |\phi(x)|^2 \, dx \leq \int |\psi(w)|^2 \, dw$.

(2) *If $|x| > 2l$ and $x \in \mathbf{R}$, then*

$$|\phi(x)| \leq c \frac{\|\psi\|_1}{|x|} \quad a.e.$$

(3) *If $\int_{\mathbf{R}} \psi = 0$ and $|x| > 2l$ and $x \in \mathbf{R}$, then*

$$|\phi(x)| \leq \frac{cl\|\psi\|_1}{|x|^2} \quad a.e.$$

Proof. Part (1) of the lemma follows from the fact that ϕ is the image of ψ under the Szegő projection of $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$.

(2) If $|x| > 2l$, $x \in \mathbf{R}$, then for $w \in [-l, l]$,

$$|x| \leq |x - w| + |w| \leq |x - w| + l.$$

Therefore,

$$|x - w| \geq |x| - l \geq |x| - \frac{|x|}{2} = \frac{|x|}{2},$$

and thus

$$|x - w|^{-1} \leq c|x|^{-1}.$$

So for a.e. x, we obtain from (1)

$$|\phi(x)| \leq c|x|^{-1} \int_{-l}^{l} |\psi(w)| dw \leq c|x|^{-1} \|\psi\|_1.$$

(3) If $|x| > 2l$, $x \in \mathbf{R}$, $\int \psi = 0$, then for a.e. x

$$\begin{aligned} \phi(x) &= c \int_{\mathbf{R}} \psi(w) S(x, w) \, dw \\ &= c \int_{\mathbf{R}} \psi(w) [S(x, w) - S(x, 0)] \, dw \\ &= c \int_{-l}^{l} \frac{\psi(w) w}{(x - w) x} \, dw. \end{aligned}$$

As in the argument of part (2), we see that

$$|\phi(x)| \leq cl|x|^{-2} \int_{-l}^{l} |\psi(w)| dw = cl|x|^{-2} \|\psi\|_1.$$

Lemma 1.3. *If $\alpha > 0$, then*

$$\int_{\mathbf{R} \setminus [-2l, 2l]} l^{\alpha} |x|^{-(1+\alpha)} dx = c_{\alpha} = \alpha^{-1} 2^{-(\alpha+1)}.$$

We shall conclude this section with a discussion of the atomic decomposition of Chang and Fefferman for $H^p, 0 < p \leq 1$.

In one variable (see [3], [7]) if $f \in H^1(\mathbf{R}^1)$, then $f(x)$ can be written as

$$f(x) = \sum \lambda_k a_k(x)$$

where $\sum |\lambda_k| \leq C\|f\|_{H^1}$ and $a_k(x)$ are particularly simple functions called "atoms".

An analogous decomposition holds for functions f defined on $\mathbf{R}^2$ which are boundary values of functions in $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, where $\mathbf{R}^2_+$ is the upper half plane.

In what follows, we shall deal exclusively with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ and its Šilov boundary $\mathbf{R}^2$. A point of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted by $z = (z_1, z_2)$ where $z_j = x_j + iy_j$ and $x_j \in \mathbf{R}$, $y_j > 0$, $j = 1, 2$.

Now, we shall introduce some notation: Let $\psi \in C^1(\mathbf{R})$ be supported on $[-1, 1]$ with ψ even and $\int_{-1}^{1} \psi(x)\,dx = 0$. If $y > 0, \psi_y(x) = (1/y)\psi(x/y)$ and if $y = (y_1, y_2)$ and $x = (x_1, x_2) \in \mathbf{R}^2$, then $\psi_y(x) = \psi_{y_1}(x_1)\psi_{y_2}(x_2)$. If f is a function defined on $\mathbf{R}$, then we define $f(x, y) := f * \psi_y(x)$; if $x = (x_1, x_2) \in \mathbf{R}^2$, we denote $\Gamma(x) := \Gamma(x_1) \times \Gamma(x_2)$, where $\Gamma(x_j) := \{(t_j, y_j) \in \mathbf{R}^2 \colon |x_j - t_j| < y_j\}$, $j = 1, 2$.

Given a function f on $\mathbf{R}^2$, we define its double Square function by

$$Q^2(f)(x) := \int\int_{\Gamma(x)} |f(t, y)|^2 \frac{dt\,dy}{y_1^2 y_2^2}.$$

It is a fact that for $1 < p < \infty$

$$\|Q(f)\|_p \leq c_p \|f\|_p.$$

We may also define functions in $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $0 < p < \infty$, as those functions f with $Q(f) \in L^p(\mathbf{R}^2)$ and define $\|f\|_{H^p} = \|Q(f)\|_p$.

The following definition is due to Chang and Fefferman.

Definition 1.4. A *p-atom* $(0 < p \leq 1)$ is a function $a(x), x = (x_1, x_2) \in \mathbf{R}^2$, defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

(1) $\|a\|_{L^2} \leq |\Omega|^{1/2 - 1/p}$.

(2) a can be further decomposed into "elementary particles" a_R as follows:

   (a) $a = \sum_{R \subset \Omega} a_R$, where a_R is supported in a rectangle $R \subseteq \Omega$ (say, $R = I_1 \times I_2$) and the R in the sum have the property that no one R is contained in the triple of any other.

   (b) $\int_{I_1} a_R(x_1, x_2') x_1^k dx_1 = 0 = \int_{I_2} a_R(x_1', x_2) x_2^k dx_2$ for each $x_j' \in I_j$, $j = 1, 2$, and $\forall k = 1, 2, \ldots, k(p)$, where $k(p)$ is an integer depending on $p, k(p) \leq [2/p - 3/2]$.

   (c) a_R is $C^{k(p)+1}$ with

$$\|a_R\|_\infty \leq c_R / |R|^{1/2},$$
$$\|\partial^m a_R / \partial x_j^m\|_\infty \leq c_R / |I_j|^m |R|^{1/2}, \qquad j = 1, 2,$$
$$m \leq k(p) + 1,$$

   and $\sum_R c_R^2 \leq A|\Omega|^{1 - 2/p}$, where A is an absolute constant.

With this definition of p-atom, Chang and Fefferman have shown the following result (see [1], [2]).

Theorem 1.5. *Let* $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, $0 < p \leq 1$. *Then* (*identifying* f *with its boundary values*) f *can be written as* $f = \sum \lambda_k a_k$ *where* a_k *are* p*-atoms and* $\lambda_k \geq 0$ *satisfy*

$$\sum \lambda_k^p \leq A_p \|f\|_{H^p}^p.$$

2. Factorization in Hardy space of bi-upper half plane ($0 < p \leq 1$)

The main purpose of this paper is to study factorization theorems in Hardy spaces H^p, $0 < p \leq 1$, on the bi-upper half plane. The case of the bidisc can then be easily obtained by use of the Cayley transform.

Theorem 2.1 (Factorization Theorem for $\mathbf{R}_+^2 \times \mathbf{R}_+^2$)**.** *Let* $0 < p \leq 1$ *and* $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$. *Then there exist* $g_j, h_j \in H^{2p}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, *such that*

$$f = \sum_{J=1}^{\infty} g_j h_j$$

in the sense of distributions, and $\sum \|g_j\|_{H^{2p}} \|h_j\|_{H^{2p}} \approx \|f\|_{H^p}$.

Similarly to the idea given by Coifman, Rochberg and Weiss [4], with the aid of Theorem 1.5, the proof of Theorem 2.1 can be reduced to the following theorem, that is, it suffices to prove Theorem 2.1 for $f = S(a)$, where a is a p-atom.

Theorem 2.2. *Let* a *be a* p*-atom,* $0 < p \leq 1$. *There exist* $B_j, C_j \in H^{2p}(\mathbf{R}^2)$ *such that* (*with* $A = S(a)$)

$$A = \sum_1^{\infty} B_j C_j$$

(*in the sense of distributions*) *and*

$$\sum_1^{\infty} \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \approx \|A\|_{H^p} \approx 1.$$

By Definition 1.4 a can be written as follows: $a = \sum_{R \in \mathcal{R}_a} a_R$, where a_R is supported in a rectangle $R \subseteq \Omega$ and the R in the sum have the property that no one R is contained in the triple of any other. Moreover,

$$\|a_R\|_\infty \leq c_R |R|^{-1/2}, \qquad \sum_R c_R^2 \leq A|\Omega|^{1-2/p},$$

where A is an absolute constant.

Therefore, the proof of Theorem 2.2 is reduced to Theorem 2.3, as we shall show below.

Theorem 2.3. *Let* a *be a* p*-atom,* $0 < p \leq 1$. *Then for each* $R \in \mathcal{R}_a$, *there exist* $B_j, C_j \in H^{2p}(\mathbf{R}^2)$ *such that*

$$A_R = \sum_1^4 B_j C_j, \quad \text{where } A_R = S(a_R)$$

and

$$\sum_1^4 \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \leq c c_R |R|^{1/p - 1/2}.$$

Proof. By shift, without loss of generality, we may assume that $R = l_1 I \times l_2 I$, where $I = [-1,1], 0 < l_j < \infty$ (in general, $0 < l_j \ll 1$), $j = 1, 2$. Since

$$\begin{aligned} 1 &= \int_R \frac{1}{|R|}\, dw \\ &= \int_{l_1 I} \frac{1}{2l_1} \int_{l_2 I} \frac{1}{2l_2}\, dw_2 dw_1 \\ &= \prod_{j=1}^{2} \int_{l_j I} \frac{1}{2l_j} \frac{1}{(z_j - w_j)} (z_j - w_j)\, dw_j \\ &= g_1(z) + g_2(z) - g_3(z) - g_4(z), \end{aligned}$$

where

$$\begin{aligned} g_1(z) &= z_1 z_2 \int_{l_1 I} \frac{1}{4l_1 l_2} \int_{l_2 I} \frac{dw_2\, dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_2(z) &= \int_{l_1 I} \int_{l_2 I} \frac{w_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_3(z) &= \int_{l_1 I} \int_{l_2 I} \frac{w_1 z_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_4(z) &= \int_{l_1 I} \int_{l_2 I} \frac{z_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}. \end{aligned}$$

Thus

$$A_R(z) = A_R(z) g_1(z) + A_R(z) g_2(z) - A_R(z) g_3(z) - A_R(z) g_4(z).$$

We shall work on each $A_R(z) g_j(z)$ to achieve the desired factorization as claimed in Theorem 2.3.

Step 1. We shall work on $A_R(z) g_1(z)$.

Let η be a number determined by

$$-1 + \frac{1}{2p} < \eta < -1 + \frac{1}{2p} + \varepsilon, \quad \text{with } 0 < \varepsilon \ll 1.$$

Let

$$\begin{aligned} B_1(z) &= z_1^{-\eta} z_2^{-\eta} \int_R \frac{dw_1\, dw_2}{(z_1 - w_1)(z_2 - w_2)}; \\ C_1(z) &= z_1^{1+\eta} z_2^{1+\eta} A_R(z) |R|^{-1}. \end{aligned}$$

Then $g_1(z) A_R(z) = B_1(z) C_1(z)$.

We shall show first

$$\|B_1\|_{H^{2p}} \|C_1\|_{H^{2p}} \le c c_R |R|^{1/p - 1/2}. \tag{2}$$

Lemma 2.4. $\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \le c_p |R|^{1 - 2p\eta}$.

Proof. *Case* 1. $|z_1| > 2l_1, |z_2| > 2l_2$.

Then (see the proof of Lemma 1.2)

$$|B_1(z)| \le c |z_1|^{-(\eta+1)} |z_2|^{-(\eta+1)} |R|.$$

Thus, since $2p(1+\eta) > 1$

$$\begin{aligned}\int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |B_1(z)|^{2p} dz_1\, dz_2 &\le c^{2p}|R|^{2p} \int_{2l_1}^{\infty}\int_{2l_2}^{\infty} t^{-2p(1+\eta)} s^{-2p(1+\eta)} ds\, dt \\ &\le c|R|^{2p} l_1^{-2p(1+\eta)+1} l_2^{-2p(1+\eta)+1} \\ &= c|R|^{1-2p\eta}.\end{aligned}$$

Case 2. $|z|_1 \le 2l_1, |z_2| \le 2l_2$.

Let $q = (1+\eta_0)/\eta_0$ where $\eta_0 > 0$ is chosen so that $\eta < \eta_0$ and $2p\frac{\eta}{\eta_0}(1+\eta_0) < 1$. Then $q' := q/(q-1) = 1+\eta_0$. Since $\frac{\eta}{\eta_0}2p(1+\eta_0) < 1$ if and only if $2p\eta(1+\eta_0) < \eta_0$ if and only if $2p\eta < \eta_0(1-2p\eta)$, and since we know that $0 < 1-2p\eta < 2p$, we have $\eta_0(1-2p\eta) < 2p\eta_0$. So we may choose such η_0. By Hölder's inequality with exponent q, and by Lemma 1.1,

$$\begin{aligned}&\int_{2l_1 I\times 2l_2 I} |B_1(z)|^{2p} dz_1\, dz_2 \\ &\quad = \int_{2R} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw_1\, dw_2}{(z_1-w_1)(z_2-w_2)} \right|^{2p} dz_1 dz_2 \\ &\quad \le \left(\int_{2R} |z_1 z_2|^{-2p\eta\cdot \frac{(1+\eta_0)}{(1+\eta_0)-1}} dz \right)^{\frac{(1+\eta_0)-1}{(1+\eta_0)}} \\ &\qquad \times \left(\int_{2R} \left| \int_R \frac{dw}{(z_1-w_1)(z_2-w_2)} \right|^{2p(1+\eta_0)} dz \right)^{\frac{1}{1+\eta_0}} \\ &\quad \le \left(\int_{2R} |z_1 z_2|^{\frac{-2p\eta(1+\eta_0)}{\eta_0}} dz \right)^{\frac{\eta_0}{1+\eta_0}} \left(\int_{2R} 1 dw \right)^{\frac{1}{1+\eta_0}}.\end{aligned}$$

Thus

$$\begin{aligned}\int_{2R} |B_1(z)|^{2p} dz_1\, dz_2 &\le c\left(|R|^{-\frac{2p\eta(1+\eta_0)}{\eta_0}+1} \right)^{\frac{\eta_0}{1+\eta_0}} |R|^{\frac{1}{1+\eta_0}} \\ &= c|R|^{\frac{-2p\eta(1+\eta_0)+\eta_0+1}{1+\eta_0}} = c|R|^{-2p\eta+1}.\end{aligned}$$

Case 3. $|z_1| \le 2l_1$ and $|z_2| > 2l_2$.

Using part 2 of Lemma 1.2 and Lemma 1.3, we have

$$\begin{aligned}&\int_{2l_1 I\times(\mathbf{R}\setminus 2l_2 I)} |B_1(z)|^{2p} dz_1\, dz_2 \\ &\quad = \int_{2l_1 I\times(\mathbf{R}\setminus 2l_2 I)} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw}{(z_1-w_1)(z_2-w_2)} \right|^{2p} dz \\ &\quad \le \int_{\mathbf{R}\setminus 2l_2 I} |z_2|^{-2p\eta-2p} l_2^{2p} dz_2 \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_R \frac{dw_1}{(z_1-w_1)} \right|^{2p} dz_1 \\ &\quad \le c_p l_2^{2p} l_2^{-2p(1+\eta)+1} \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1-w_1)} \right|^{2p} dz_1.\end{aligned}$$

By using the argument of Case 2 (but in one variable only), we have

$$\int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1 \leq c l_1^{-2p\eta+1}.$$

Thus

$$\int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz \leq c_p l_2^{-2p\eta+1} l_1^{-2p\eta+1} \leq c_p |R|^{-2p\eta+1}.$$

Case 4. $|z_1| > 2l_1$ and $|z_2| \leq 2l_2$. By a similar argument to Case 3, we have

$$\int_{(\mathbf{R} \setminus 2l_1 I) \times 2l_2 I} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Therefore

$$\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Next we shall prove that

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{2-p} |R|^{2p\eta-1} = c_p c_R^{2p} |R|^{1+2p\eta-p},$$

and this will complete the proof of (2). We state this as another lemma.

Lemma 2.5.

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R||^{1+2p\eta-p}.$$

Proof. *Case* 1. $z_1, z_2 \in 2R$.

Then

$$\begin{aligned}
\int_{2R} |C_1(z)|^{2p} dz &= \int_{2R} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1\, dz_2 |R|^{-2p} \\
&\leq c_p |R|^{2p(1+\eta)-2p} \int_{2R} |A_R(z)|^{2p} dz \\
&\leq c_p |R|^{2p\eta} \|a_R\|_2^{2p} |R|^{1-p} \\
&\leq c_p |R|^{2p\eta} c_R^{2p} |R|^{1-p} \\
&= c_p c_R^{2p} |R|^{2p\eta+1-p}.
\end{aligned}$$

Case 2. $|z_1| > 2l_1, |z_2| > 2l_2$.

To prove this case, we need the following sublemmas. First let us introduce some notation. For $z_j, w_j \in \mathbf{R}$ and $j = 1, 2$, let

$$S_j^* = S_j^*(z_j, w_j) := \sum_{m=0}^{k(p)} \frac{1}{m!} \frac{\partial^m S_j(z_j, 0)}{\partial w_j^m} w_j^m.$$

Note that $S_j^*(z_j, w_j)$ is the $k(p)$th partial sum of the Taylor expansion of $S_j^*(z_j, \cdot)$ at 0.

Sublemma 2.6. *For a function ϕ, let*

$$\phi^*(z_j) = \int_{l_j I} [S_j(z_j, w_j) - S_j^*(z_j, w_j)] \phi(w_j)\, dw_j.$$

Then for $|z_j| > 2l_j$, *we have*

$$|\phi^*(z_1, w_2)| \leq cl_1^{k(p)+1}|z_1|^{-(k(p)+2)} \int_{l_1 I} |\phi(w_1)|\, dw_1.$$

In particular, if

$$M(z_2, w_1) = \int_{l_2 I} [S_2(z_2, w_2) - S_2^*(z_2, w_2)] a_R(w)\, dw_2,$$

then

$$|M(z_2, w_1)| \leq cl_2^{k(p)+1}|z_2|^{-(k(p)+2)} \int_{l_2 I} |a_R(w)|\, dw_2.$$

Proof. Since $\forall m \in Z^+$,

$$\frac{\partial^m S_j(z_j, w_j)}{\partial w_j^m} = c(-1)^m m!(z_j - w_j)^{-(m+1)},$$

we have

$$S_j(z_j, w_j) - S_j^* = \frac{\partial^{k(p)+1} S_j(z_j, w_j')}{\partial w_j^{k(p)+1}} z_j^{-(k(p)+2)} w_j^{-(k(p)+1)}$$

where w_j' lies between 0 and w_j.

Then

$$|S_j(z_j, w_j) - S_j^*| \leq c|z_j - w_j'|^{-(k(p)+2)}|w_j|^{-(k(p)+1)}.$$

Claim. $|S_j(z_j, w_j) - S_j^*| \leq c|z_j|^{-(k(po)+2)}|w_j|^{-(k(p)+1)}$.

To prove this we need: $|z_j - w_j'|^{-(k(p)+2)} \leq c|z_j|^{-(k(p)+2)}$, that is, we have to show $|z_j - w_j'|^{-1} \leq c|z_j|^{-1}$.

Since

$$\begin{aligned} |z_j| &= |z_j - w_j' + w_j'| \\ &\leq |z_j - w_j'| + |w_j'| \\ &\leq |z_j - w_j'| + l_j \quad (\text{because } |w_j'| \leq |w_j| \leq l_j), \end{aligned}$$

we obtain

$$\begin{aligned} |z_j - w_j'| &\geq |z_j| - l_j \\ &\geq |z_j| - \frac{1}{2}|z_j| \quad (\text{because } |z_j'| > 2l_j) \\ &= \frac{1}{2}|z_j|. \end{aligned}$$

Hence $|z_j - w_j'|^{-1} \leq 2|z_j|^{-1}$. Therefore the claim is proved.

Thus

$$\begin{aligned} |\phi^*(z_j)| &\leq c \int_{l_j I} |\phi(w_j)| |z_j|^{-(k(p)+2)} |w_j|^{k(p)+1}\, dw_j \\ &\leq cl_j^{k(p)+1}|z_j|^{-(k(p)+2)} \int_{l_j I} |\phi(w_j)|\, dw_j. \end{aligned}$$

Sublemma 2.7. *If* $|z_1| > 2l_1$ *and* $|z_2| > 2l_2$, *then*

$$|A_R(z)| \leq cc_R|R|^{k(p)+3/2}|z_1 z_2|^{-(k(p)+2)}.$$

Proof. By property 2(b) of Definition 1.4,

$$\begin{aligned} A_R(z) &= \int_{l_1 I} S_1(z_1, w_1) \int_{l_2 I} S_2(z_2, w_2) a_R(w) dw_2\, dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] \int_{l_2 I} [S_2(z_2, w_2) - S_2^*] a_R(w) dw_2\, dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] M(z_2, w_1)\, dw_1. \end{aligned}$$

Sublemma 2.6 implies

$$\begin{aligned} |A_R(z)| &\le c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} \int_{l_1 I} |M(z_2, w_1)|\, dw_1 \\ &\le c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} l_2^{k(p)+1} |z_2|^{-(k(p)+2)} \int_{l_1 I} |a_R(w_1, w_2)| dw_1\, dw_2 \\ &\le c |R|^{k(p)+3} |z_1 z_2|^{-(k(p)+2)}. \end{aligned}$$

Now we return to the proof of Case 2.

$$\begin{aligned} &\int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |C_1(z)|^{2p}\, dz \\ &\quad = \int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |R|^{-2p} |z_1 z_2|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1\, dz_2 \\ &\quad \le c c_R^{2p} \int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |R|^{2p(k(p)+1/2)} |z_1 z_2|^{2p(1+\eta) - 2p(k(p)+2)} dz_1\, dz_2 \\ &\quad = c c_R^{2p} |R|^{2p(k(p)+1/2)} \int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1\, dz_2. \end{aligned}$$

Note that

$$\begin{aligned} 2p(k(p)+1-\eta) &= 2pk(p) + 2p - 2p\eta \\ &\ge 2p(2/p - 3/2 - 1) + 2p - 2p\eta \\ &= 4 - 3p + 2p - 2p\eta \\ &> 4 - p - (-2p + 1 + \varepsilon) \\ &= 4 + p - 1 - \varepsilon \\ &= 3 + p - \varepsilon > 1. \end{aligned}$$

Thus, by Lemma 1.3,

$$\int_{\mathbf{R}\setminus 2l_1 I} \int_{\mathbf{R}\setminus 2l_2 I} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1\, dz_2 \le c (l_1 l_2)^{-2p(k(p)+1-\eta)+1}.$$

Hence

$$\begin{aligned} \int_{(\mathbf{R}\setminus 2l_1 I)\times(\mathbf{R}\setminus 2l_2 I)} |C_1(z)|^{2p} dz &\le c_p c_R^{2p} |R|^{2p(k(p)+1/2) - 2p(k(p)+1-\eta)+1} \\ &= c_p c_R^{2p} |R|^{-p+2p\eta+1}. \end{aligned}$$

Case 3. $|z_1| < 2l_1, |z_2| \le 2l_2$.

The key to this case is the following sublemma.

Sublemma 2.8. *If $z_1 \in \mathbf{R}\backslash 2l_1 I$, then*

$$\int_{2l_2 I} |A_R|^{2p} dz_2 \leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}.$$

Proof.

$$\begin{aligned}
&\int_{2l_2 I} \left| \int_R S_1(z_1, w_1) S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\
&\quad = \int_{2l_2 I} \left| \int_{l_1 I} [S_1(z_1, w_1) - S_1^*(z_1)] \int_{l_2 I} S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\
&\quad \leq \int_{2l_2 I} \left[c l_1^{k(p)+3/2} |z_1|^{-(k(p)+2)} \int_{l_1 I} \left| \int_{l_2 I} S_2(z_2, w_2) a(w) dw_2 \right| dw_1 \right]^{2p} \\
&\quad \leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left(\int_R |S_2(a_R)(w_1, z_2)|^2 dz_2 \right)^p \left(\int_{l_2 I} dz_2 \right)^{1-p} \\
&\quad \leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left(\int_R |a_R(w_1, w_2)|^2 \, dw \right)^p |R|^{1-p} \\
&\quad \leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} c_R^{2p} l_2^{1-p} \\
&\quad = c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}.
\end{aligned}$$

With the aid of this sublemma we have

$$\begin{aligned}
&\int_{(\mathbf{R}\backslash 2l_1)\times 2l_2 I} |C_1(z)|^{2p} dz \\
&\quad = \int_{\mathbf{R}\backslash 2l_1} \int_{2l_2 I} |R|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_2 \, dz_1 \\
&\quad \leq \int_{\mathbf{R}\backslash 2l_1} |R|^{-2p} c_p c_R^{2p} l_1^{2p(k(p)+3/2)} (2l_2)^{2p(1+\eta)} l_2^{1-p} |z_1|^{-2p(k(p)+2)+2p(1+\eta)} dz_1 \\
&\quad \leq c_p c_R^{2p} \int_{2l_1}^{\infty} l_1^{2p(k(p)+3/2)} l_2^{2p\eta+1+p} t^{-2p(k(p)+2)+2p(1+\eta)} dt |R|^{-2p} \\
&\quad \leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)-2p(k(p)+\eta+1)+1} l_2^{2p\eta+1+p} |R|^{-2p} \\
&\quad = c_p c_R^{2p} l_1^{p+2p\eta+1} l_2^{2p\eta+1+p} |R|^{-2p} \\
&\quad = c_p c_R^{2p} |R|^{-p+2p\eta+1}.
\end{aligned}$$

Case 4. $|z_1| \leq 2l_1, |z_2| > 2l_2$.

By symmetry with Case 3, we obtain

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{-p+2p\eta+1},$$

and hence the proof of (2) is complete.

Step 2. We now work on $g_2(z) A_R(z)$.

Put

$$g_2(z) A_R(z) = B_2(z) C_2(z),$$

where

$$B_2(z) = (z_1 z_2)^{-\eta} g_2(z) \quad \text{and} \quad C_2(z) = (z_1 z_2)^{\eta} A_R(z).$$

By using an argument similar to that of (2), we obtain

$$\|B_2\|_{H^{2p}} \|C_2\|_{H^{2p}} \le c c_R |R|^{1/p-1/2}. \tag{3}$$

Similarly, if we set

$$g_3(z) A_R(z) = B_3(z) C_3(z),$$

where

$$B_3(z) = z_1^{-\eta} z_2^{-\eta-1} g_3(z) \quad \text{and} \quad C_3(z) = z_1^{\eta} z_2^{\eta+1} A_R(z)$$

and

$$g_4(z) A_R(z) = B_4(z) C_4(z),$$

where

$$B_4(z) = z_1^{-\eta-1} z_2^{-\eta-1} g_4(z) \quad \text{and} \quad C_4(z) = z_1^{\eta+1} z_2^{-\eta} A_R(z),$$

then we can prove

$$\|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \le c c_R |R|^{1/p-1/2}, \qquad j = 3, 4. \tag{4}$$

Combining (2), (3), and (4) completes the proof of Theorem 2.3.

We now prove Theorem 2.2:

$$A = \sum_{R \subset \Omega} S(a_R) = \sum_{R \subset \Omega} \sum_{j=i}^{4} B_j C_j$$

and because $2/p - 1 > 1$,

$$\begin{aligned}
\sum_{R \subset \Omega} \sum_{j=i}^{4} \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} &\le \sum_{R \subset \Omega} c_R |R|^{1/p-1/2} \\
&\le \left(\sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left(\sum_{R \subset \Omega} |R|^{2/p-1} \right)^{1/2} \\
&\le \left(\sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left(\sum_{R \subset \Omega} |R| \right)^{(2/p-1)/2} \\
&\le ((A|\Omega|)^{1-2/p})^{1/2} ((2|\Omega|)^{2/p-1})^{1/2} = c.
\end{aligned}$$

Therefore the proof of Theorem 2.2 is complete.

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