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CHARACTERISATION OF COMPACTNESS THROUGH CONVERGENCE IN TYCHONOFF SPACES

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Abstract. In this article we first give a characterisation of compact spaces among $T_{3.5}$ spaces by improving a theorem of J. Ewert. Then, with the aid of a new type of convergence, we give a characterisation of the pseudocompact and of the Lindelöf $T_{3.5}$ spaces.

1. Introduction

Let X be a topological space, $(Y,\mathcal{U})$ a uniform space and $\mathcal{P}_{\mathcal{U}}$ a collection of pseudometrics on Y defining $\mathcal{U}$. As in [2], we denote by $\mathcal{F}(X,Y)$ $(\mathcal{C}(X,Y))$ the collection of all functions (all continuous functions) from X into Y.

A net $\{f_s : s \in S\} \subseteq \mathcal{F}(X,Y)$ is called almost uniformly convergent to some $f \in \mathcal{F}(X,Y)$, at a point $x_0 \in X$, if for each $\varepsilon > 0$, $\rho \in \mathcal{P}_{\mathcal{U}}$ there exist $s_0 \in S$ and a neighborhood U of x_0 such that $\rho(f_s(x), f(x)) < \varepsilon$ for each $s \geq s_0$, $x \in U$.

The net is called almost uniformly convergent to some f, if it is almost uniformly convergent to f at each point $x \in X$ [2].

Introducing a new type of convergence, we define:

A net $\{f_s : s \in S\} \subseteq \mathcal{F}(X,Y)$ is called countably uniformly convergent to some $f \in \mathcal{F}(X,Y)$, if for every $\varepsilon > 0$, $\rho \in \mathcal{P}_{\mathcal{U}}$ there exist a locally finite countable open covering $\{U_n\}$ of X and a sequence $\{s_n\}$, $s_n \in S$, $n \in \mathbf{N}$, such that

$$\rho(f_s(x), f(x)) < \varepsilon$$

for every $s \geq s_n$ and for all $x \in U_n$.

It is easily seen that uniform convergence implies countable uniform convergence implies almost uniform convergence.

2. Results

The following theorem is due to J. Ewert [2, Theorem 2.2].

Theorem 1. *Let X be a $T_{3.5}$ space and let us consider the following properties:*

(a) *X is a compact space.*

(b) *For any uniform space $(Y,\mathcal{U})$ the uniform convergence and the almost uniform convergence coincide on $\mathcal{F}(X,Y)$.*

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(c) *Almost uniform convergence and uniform convergence coincide on* $\mathcal{C}(X,[0,1])$.

(d) X *is a pseudocompact space.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

We will now improve this theorem, showing that (c) implies (a), and give in this way a characterisation of the compact $T_{3.5}$ spaces.

Theorem 2. *In Theorem* 1, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. (c) $\Rightarrow$ (a) From (c) we have that X is a pseudocompact space. Recall that a $T_{3.5}$ space X is pseudocompact iff every compatible uniformity is totally bounded; thus X is compact iff X is complete relative to any compatible uniformity.

Suppose on the contrary that X is not complete relative to a compatible uniformity, and let $\widehat{X}$ be the completion of X relative to this uniformity. Let $x_0 \in \widehat{X} \backslash X$, and let $\widehat{U}_s$, $s \in S$, denote the open neighborhoods of x_0. It is clear that the sets $U_s \coloneqq \widehat{U}_s \cap X$, $s \in S$, are open nonempty subsets of X.

We define on S the direction: $s_1 < s_2$ iff $U^c_{s_1} \subseteq U^c_{s_2}$. We choose a point $x_s \in U_s$ for every $s \in S$, and since X is a $T_{3.5}$ space, we define continuous functions f_s as follows:

$$f_s(x) = 0 \quad \text{if } x \in U^c_s, \qquad f_s(x_s) = 1.$$

We claim that the net of continuous functions $\{f_s : s \in S\}$ converges almost uniformly but, obviously, not uniformly to the function $f = 0$ on X.

We will show that for every $x_1 \in X$ there exist an open set W which contains x_1, and an $s_0 \in S$ such that $|f_s(x)| = 0$, for every $x \in W$ and every $s > s_0$.

Indeed, since $\widehat{X}$ is $T_{3.5}$, there exists an open set A in $\widehat{X}$ which contains x_0 and does not contain x_1. Now, because X is $T_{3.5}$ there exist open sets U_{s_0} and W such that U_{s_0} contains x_0, W contains A^c and

$$U_{s_0} \cap W = \varnothing, \quad \text{i.e. } W \subseteq U^c_{s_0}.$$

By construction of the net $\{f_s, s \in S\}$, we have that $f_s(x) = 0$ for every $s > s_0$ and for every $x \in U^c_s \supseteq U^c_{s_0} \supseteq W$.

Theorem 3. *Let X be a $T_{3.5}$ topological space. Then the following are equivalent*:

(a) X *is pseudocompact.*

(b) *For any uniform space* $(Y,\mathcal{U})$ *uniform convergence and countable uniform convergence coincide on* $\mathcal{F}(X,Y)$.

(c) *Uniform convergence and countable uniform convergence coincide on* $\mathcal{C}(X,[0,1])$.

Proof. (a) $\Rightarrow$ (b) Let $\{f_s : s \in S\} \subseteq \mathcal{F}(X,Y)$ be countably uniformly convergent to $f \in \mathcal{F}(X,Y)$ and $\varepsilon > 0$, $\rho \in \mathcal{P}_{\mathcal{U}}$ be given. Then, from the definition of this convergence, there exists a locally finite countable open covering $\{U_n\}$ of X. Since X is pseudocompact, $\{U_n\}$ must be finite [1, Theorem 3.10.22], from which it follows that $\{f_s : s \in S\}$ converges uniformly to f.

(b) $\Rightarrow$ (c) This follows from the fact that the uniform limit of continuous functions is a continuous function.

(c) $\Rightarrow$ (a) Suppose X is not pseudocompact. Then, by [5, Theorem 10.2.5], there exists a locally finite disjoint family of open sets which is not finite.

Let $\{U_n, n \in \mathbb{N}\}$ be a countable subcollection of this family.

Since X is $T_{3.5}$, for every U_n, $n \in \mathbb{N}$, there exists a nonempty open set V_n such that $V_n \subseteq \overline{V}_n \subseteq U_n$.

As the set $U_0 = [\bigcup_{n=1}^{+\infty} \overline{V}_n]^c$ is open [5, Theorem 10.2.2], the family

$$\{U_n,\ n = 0, 1, 2, \dots\}$$

is a locally finite open cover of X.

Choosing $x_n \in V_n$, $n \in \mathbb{N}$, we observe that $x_n \notin \bigcup_{i=0}^{n-1} \overline{V}_i$, where $V_0 = [\bigcup_{n=1}^{+\infty} V_n]^c$.

Since X is $T_{3.5}$, there exist continuous functions $f_n : X \to [0, 1]$ such that $f_1(x) = 0$ for every $x \in V_0$, $f_1(x_1) = 1$ and $f_n(x) = 0$ for every $x \in \bigcup_{i=0}^{n-1} \overline{V}_i$, $f_n(x_n) = 1$, $n = 2, 3, \dots$.

It is not difficult to see that the net $\{f_n,\ n \in \mathbb{N}\}$ converges countably uniformly, but not uniformly to the zero function.

This contradicts (c) and the theorem is proved.

So far it is known that for a $T_{3.5}$ space X we have: X is compact, if and only if, the almost uniform convergence and the uniform convergence coincide on $\mathcal{C}(X, [0, 1])$, and X is pseudocompact, if and only if, the countable uniform convergence and the uniform convergence coincide on $\mathcal{C}(X, [0, 1])$. It is also well known that a $T_{3.5}$ space is compact, if and only if, it is pseudocompact and Lindelöf.

So it is natural to ask:

If for a $T_{3.5}$ space X, the almost uniform convergence and the countable uniform convergence coincide on $\mathcal{C}(X, [0, 1])$, must then X be a Lindelöf space?

A positive answer to this question is given by the next theorem. We give first a Lemma.

Lemma. *Let (X, T) be a topological space which is not Lindelöf. Then there exists a well-ordered strictly increasing, with respect to inclusion, open cover of X, which has no countable subcover.*

Proof. We can easily prove this Lemma, by taking any open cover of X with no countable subcover and using transfinite induction.

Theorem 4. *Let X be a $T_{3.5}$ topological space. Then the following are equivalent*:

(a) *X is a Lindelöf space.*

(b) *For any uniform space $(Y, \mathcal{U})$ countable uniform convergence and almost uniform convergence coincide on $\mathcal{F}(X, Y)$.*

(c) *Countable uniform convergence and almost uniform convergence coincide on $\mathcal{C}(X, [0, 1])$.*

Proof. (a) $\Rightarrow$ (b) Let $\{f_s : s \in S\}$ be a net of functions which is almost uniformly convergent to $f \in \mathcal{F}(X, Y)$ and let $\varepsilon > 0$, $\rho \in \mathcal{P}_{\mathcal{U}}$ be given. Then, for every $x \in X$, there exist a neighborhood $\mathcal{U}(x)$ of x and an $s_x \in S$ such that

$$\rho(f_s(x), f(x)) < \varepsilon$$

for every $s > s_x$ and for every $x \in \mathcal{U}(x)$.

The collection of neighborhoods $\mathcal{U}(x)$, $x \in X$, forms a covering of X, which obviously has a countable subcovering.

Since every Lindelöf $T_{3.5}$ space is paracompact [3, III.4.C], there exists a locally finite refinement of this countable subcovering which, without loss of generality, can be taken countable.

So the net $\{f_s : s \in S\}$ converges countably uniformly to f.

(b) $\Rightarrow$ (c) This follows from the fact that the countable uniform limit of a net of continuous functions is a continuous function [2], [4].

(c) $\Rightarrow$ (a) Suppose that X is not Lindelöf.

Then, by the Lemma, there exists a well-ordered strictly increasing, with respect to inclusion, open cover $\mathcal{V} = \{V_i, 0 \leq i < k\}$, which has no countable subcover.

For every, i, $0 \leq i < k$, we fix a point $x_{i+1} \in V_{i+2} \setminus V_{i+1}$.

Let $\mathcal{K} = \{K_l, l \in L\}$ be the collection of all closed subsets of X which are subsets of some V_i, $0 \leq i < k$.

Since X is a $T_{3.5}$ space, we can define the net of continuous functions $\{f_l, l \in L\}$ with:

$$f_l(x) = 0 \text{ for every } x \in K_l \text{ and } f_l(x_i) = 1,$$

where i is the least index j, $0 \leq j < k$, such that $K_l \subseteq V_j$ and L is equipped with the direction $l_1 < l_2$ iff $K_{l_1} \subseteq K_{l_2}$.

The net $\{f_l, l \in L\}$ converges almost uniformly to the function $f(x) = 0$, $x \in X$.

In fact, for every $p \in X$ we have that $p \in V_{i_0}$ for some i_0 and since X is regular, there exists a closed neighborhood of p, say K_{l_0}, such that $K_{l_0} \subseteq V_{i_0}$.

It follows that $f_l(x) = 0$ for all $l > l_0$ and for all $x \in K_{l_0}$.

Suppose now that the net $\{f_l, l \in L\}$ converges countably uniformly to the function $f(x) = 0$, $x \in X$.

Then for $\varepsilon = \frac{1}{2}$ there exist an open cover $\{U_n,\ n \in \mathbb{N}\}$ and a sequence $\{l_n,\ n \in \mathbb{N}\}$ such that $|f_l(x)| < \varepsilon = \frac{1}{2}$, for every $l > l_n$ and every $x \in U_n$.

Since $\bigcup_{n=1}^{+\infty} U_n = X$, it follows that there exists a U_n, say U_{n_0}, which contains a subfamily of the family $\{x_{i+1},\ 0 \leq i < k\}$ of cardinality k.

To U_{n_0}, by definition of the countable uniform convergence, corresponds an index l_{n_0}, which gives us a set $K_{l_{n_0}}$. Let i_{n_0} be the least j, $0 \leq j < k$, such that $K_{l_{n_0}} \subseteq V_j$.

Consider an $x_{m+1} \in U_{n_0}$, with $m + 1 > i_{n_0}$.

Then there exists a closed neighborhood $K(x_m)$ of x_m, which is a subset of V_{m+1}.

Now obviously we have that

$$K_{l'} := K_{l_{n_0}} \cup K(x_m) \supseteq K_{l_{n_0}}$$

and since $K_{l_{n_0}} \subseteq V_{i_{n_0}} \subseteq V_{m+1}$, it holds that the least j, $0 \leq j < k$, with $K_{l'} \subseteq V_j$, is the $m + 1$.

So we will have:

$$l' > l_{n_0} \quad \text{and} \quad f_{l'}(x_{m+1}) = 1,$$

although $x_{m+1} \in U_{n_0}$, a contradiction.

Remarks. As it is known that a $T_{3.5}$ space is compact iff it is pseudocompact and Lindelöf, we can get another indirect proof of Theorem 2 combining Theorems 3 and 4.

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