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OPERATOR-VALUED TYPICALLY REAL FUNCTIONS

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ABSTRACT. Generalizing the classical typically real functions in complex analysis, we introduce the operator-valued typically real functions and show how to construct these functions.

1. Notation and definition

Throughout this paper, a Hilbert space is always a complex Hilbert space and an operator is always a bounded linear operator on a Hilbert space. The real and imaginary parts of an operator A are denoted by Re A and Im A respectively:

$$\operatorname{Re} A = \frac{A + A^*}{2}, \qquad \operatorname{Im} A = \frac{A - A^*}{2i}.$$

The capital letter I denotes the identity operator on a Hilbert space. When we have to consider two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ simultaneously, we shall use I and $\tilde{I}$ to denote the identity operator on $\mathcal{H}$ and $\mathcal{K}$ respectively. For a Hermitian operator A, we write $A \geq 0$ to indicate that A is a positive operator, i.e., the inner product $\langle A\xi, \xi\rangle$ is non-negative for every vector ξ in the Hilbert space.

The open unit disk in the complex plane is denoted by $\Delta : \Delta = \{z \in \mathbb{C} : |z| < 1\}$. We recall that a complex-valued function f is said to be *typically real*, if f is analytic on Δ, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with all coefficients a_n real, $a_0 = 0$, $a_1 = 1$ and if Im $f(z) \geq 0$ for all $z \in \Delta$ with Im $z \geq 0$. The complex-valued typically real functions were introduced by W. Rogosinski [4] in 1932.

We shall consider operator-valued analytic functions of a complex variable z : $F(z) = \sum_{n=0}^{\infty} z^n A_n$, where the coefficients A_n are operators on a Hilbert space $\mathcal{H}$. F is said to be *typically real*, if $F(z) = \sum_{n=0}^{\infty} z^n A_n$ is analytic on the open unit disk Δ, if all the coefficients A_n are Hermitian operators on $\mathcal{H}$, $A_0 = 0$, $A_1 = I$ and if Im $F(z) \geq 0$ (i.e., Im $F(z)$ is a positive operator) for all $z \in \Delta$ with Im $z \geq 0$. Notice that all typically real functions are defined and analytic on Δ, but their values are operators on a Hilbert space depending on each function.

2. Construction of all operator-valued typically real functions

Our problem is this: Given a Hilbert space $\mathcal{H}$, how to construct all typically real functions whose values are operators on $\mathcal{H}$. Let us begin with some preparations.

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Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces such that $\mathcal{H}$ is a subspace of $\mathcal{K}$. Let P be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. For an operator D on $\mathcal{K}$, we shall write Pr D to denote the operator on $\mathcal{H}$ defined by

$$(\text{Pr } D)\xi = PD\xi \quad \text{for} \quad \xi \in \mathcal{H}. \tag{1}$$

In other words, Pr D is the restriction to $\mathcal{H}$ of the operator PD on $\mathcal{K}$. The following properties of Pr D are easily verified:

$$(\text{Pr } D)^* = \text{Pr } D^*. \tag{2}$$

(3) If D is a (positive) Hermitian operator on $\mathcal{K}$, then Pr D is a (positive) Hermitian operator on $\mathcal{H}$.

$$\text{Pr Re } D = \text{Re Pr } D, \qquad \text{Pr Im } D = \text{Im Pr } D. \tag{4}$$

$$\|\text{Pr } D\| \le \|D\|. \tag{5}$$

From these simple properties of Pr D, we obtain immediately the following lemma.

Lemma. *Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces such that $\mathcal{H}$ is a subspace of $\mathcal{K}$. If Φ is a typically real function whose values are operators on $\mathcal{K}$, then F defined by*

$$F(z) = \text{Pr } \Phi(z) \quad \textit{for} \quad z \in \Delta$$

is a typically real function with operators on $\mathcal{H}$ as its values.

Theorem 1. (a) *Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces such that $\mathcal{H}$ is a subspace of $\mathcal{K}$, and let P be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. For any contraction C (i.e., $\|C\| \le 1$) on $\mathcal{K}$, the function Φ defined on the open unit disk Δ by*

$$\Phi(z) = z(\tilde{I} - zC)^{-1}(\tilde{I} - zC^*)^{-1} \quad \textit{for} \quad z \in \Delta \tag{6}$$

is a typically real function with operators on $\mathcal{K}$ as its values. If $F(z)$ is the restriction to $\mathcal{H}$ of the operator $P\Phi(z)$:

$$F(z) = \text{Pr } \Phi(z) \quad \textit{for} \quad z \in \Delta, \tag{7}$$

then F is a typically real function whose values are operators on $\mathcal{H}$.

(b) *Conversely, every typically real function F whose values are operators on $\mathcal{H}$ is of the form* (7) *with Φ given by* (6) *for a suitable choice of a contraction C on a suitable larger Hilbert space $\mathcal{K}$. Moreover, C can be required to be a unitary operator on $\mathcal{K}$.*

Proof of part (a). Let C be a contraction on $\mathcal{K}$, and let Φ be defined by (6). As $\|C\| \le 1$, Φ is clearly analytic on Δ. Using $(\tilde{I} - zC)^{-1} = \sum_{n=0}^{\infty} z^n C^n$ and a similar expression for $(\tilde{I} - zC^*)^{-1}$, we find the coefficients D_n in the power series

$$\Phi(z) = \sum_{n=0}^{\infty} z^n D_n \qquad (z \in \Delta). \tag{8}$$

They are $D_0 = 0,\ D_1 = \tilde{I}$ and for $n \geq 2$:

$$D_n = C^{n-1} + C^{n-2}C^* + \cdots + C^{n-k}{C^*}^{k-1} + \cdots + {C^*}^{n-1}. \tag{9}$$

If we rewrite for $n \geq 1$:

$$D_{2n} = 2\,\mathrm{Re}\left(C^{2n-1} + C^{2n-2}C^* + \cdots + C^n {C^*}^{n-1}\right), \tag{10}$$

$$D_{2n+1} = C^n {C^*}^n + 2\,\mathrm{Re}\left(C^{2n} + C^{2n-1}C^* + \cdots + C^{n+1}{C^*}^{n-1}\right), \tag{11}$$

then it is clear that all D_n are Hermitian operators on $\mathcal{K}$. By (6), we verify that for $z \in \Delta$:

$$\mathrm{Im}\ \Phi(z) = N(z)^* M(z) N(z) \tag{12}$$

with

$$M(z) = (\mathrm{Im}\ z)(\tilde{I} - |z|^2 C^* C), \tag{13}$$

$$N(z) = (\tilde{I} - \bar{z}C)^{-1}(\tilde{I} - \bar{z}C^*)^{-1}. \tag{14}$$

For each $z \in \Delta$ with $\mathrm{Im}\ z > 0$, $M(z)$ is an invertible positive operator. Therefore (12) shows that $\mathrm{Im}\ \Phi(z)$ is an invertible positive operator for every $z \in \Delta$ with $\mathrm{Im}\ z > 0$. Thus Φ is a typically real function with operators on $\mathcal{K}$ as its values. By Lemma, it follows that F defined by (7) is a typically real function whose values are operators on $\mathcal{H}$.

Proof of part (b). To prove the converse part of the theorem, we assume that

$$F(z) = \sum_{n=1}^{\infty} z^n A_n \tag{15}$$

is a given typically real function where A_n are operators on $\mathcal{H}$. For each vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, let f_ξ and g_ξ be defined on Δ by

$$f_\xi(z) = \langle F(z)\xi,\ \xi\rangle = \sum_{n=1}^{\infty} z^n \langle A_n \xi,\ \xi\rangle \tag{16}$$

and

$$g_\xi(z) = \frac{1 - z^2}{z} f_\xi(z). \tag{17}$$

As F is an operator-valued typically real function, f_ξ is a complex-valued typically real function for every fixed $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Then by a classical theorem of

Rogosinski [4] (see also [1], p. 56; [3], p. 54), the analytic function g_ξ on Δ defined by (17) has positive real part:

$$\text{Re } g_\xi(z) > 0 \quad \text{for} \quad z \in \Delta. \tag{18}$$

Define the operator-valued analytic function G on Δ by

$$G(z) = \frac{1 - z^2}{z} F(z). \tag{19}$$

For $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and $z \in \Delta$, we have

$$\langle G(z)\xi, \ \xi \rangle = \frac{1 - z^2}{z} f_\xi(z) = g_\xi(z)$$

and therefore

$$\langle [\text{Re } G(z)]\xi, \ \xi \rangle = \text{Re}\langle G(z)\xi, \ \xi \rangle = \text{Re } g_\xi(z) > 0. \tag{20}$$

Thus, for each $z \in \Delta$, Re $G(z)$ is a positive operator on $\mathcal{H}$. Also $G(0) = A_1 = I$. By a theorem of Naimark [2], there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary operator U on $\mathcal{K}$ such that

$$G(z) = \text{Pr } (\tilde{I} + zU)(\tilde{I} - zU)^{-1} \quad \text{for} \quad z \in \Delta. \tag{21}$$

As

$$(\tilde{I} + zU)(\tilde{I} - zU)^{-1} = \tilde{I} + 2\sum_{n=1}^{\infty} z^n U^n,$$

we derive from (15), (19) and (21):

$$(1 - z^2)\sum_{n=1}^{\infty} z^n A_n = (1 - z^2)F(z) = zG(z) = z \ \text{Pr} \left\{ \tilde{I} + 2\sum_{n=1}^{\infty} z^n U^n \right\}$$

and therefore

$$A_1 = I, \quad A_2 = 2\,\text{Pr } U, \quad A_n = A_{n-2} + 2\,\text{Pr } U^{n-1} \quad \text{for} \quad n \geq 3. \tag{22}$$

Since the coefficients A_n in the power series (15) of the typically real function F are Hermitian operators on $\mathcal{H}$, (22) implies that Pr U^n are Hermitian for all $n \geq 1$. Then by (4), we have

$$\text{Pr } U^n = \text{Re } \text{Pr } U^n = \text{Pr } \text{Re } U^n \qquad (n \geq 1). \tag{23}$$

Combining (22) with (23), we have

$$A_1 = I, \quad A_2 = 2\,\text{Pr } \text{Re } U, \ A_n = A_{n-2} + 2\,\text{Pr } \text{Re } U^{n-1} \quad \text{for} \quad n \geq 3$$

or more explicitly:

$$A_{2n} = 2\,\text{Pr } \text{Re}(U + U^3 + U^5 + \cdots + U^{2n-1}) \qquad (n \geq 1), \tag{24}$$

$$A_{2n+1} = I + 2\,\mathrm{Pr}\ \mathrm{Re}(U^2 + U^4 + U^6 + \cdots + U^{2n}) \qquad (n \geq 1). \tag{25}$$

Now, let

$$\Phi(z) = z(\tilde{I} - zU)^{-1}(\tilde{I} - zU^*)^{-1} = \sum_{n=1}^{\infty} z^n D_n \tag{26}$$

for $z \in \Delta$. Observe that (26) is the same as (6) and (8), except that the contraction C on $\mathcal{K}$ in (6) is replaced by the unitary operator U on $\mathcal{K}$ satisfying (21). Since U is unitary, (10) and (11) reduce to

$$D_{2n} = 2\,\mathrm{Re}(U + U^3 + U^5 + \cdots + U^{2n-1}) \qquad (n \geq 1), \tag{27}$$

$$D_{2n+1} = \tilde{I} + 2\,\mathrm{Re}(U^2 + U^4 + \cdots + U^{2n}) \qquad (n \geq 1). \tag{28}$$

From (15), (24), (25), (27) and (28), we obtain (7) with $\Phi(z)$ given by (26). This completes the proof of the converse part of the theorem.

Theorem 1 just proved may be reformulated in the following form.

Theorem 2. *Let $F(z) = \sum_{n=1}^{\infty} z^n A_n$, $(A_1 = I)$ be an operator-valued analytic function on the open unit disk Δ, with operators on a Hilbert space $\mathcal{H}$ as its values. The following are equivalent*:

(a) *F is typically real.*

(b) *There exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary operator U on $\mathcal{K}$ such that* (24), (25) *hold.*

(c) *There exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a resolution of the identity $\{E(\theta) : 0 \leq \theta \leq 2\pi\}$ formed by orthogonal projections acting on $\mathcal{K}$ such that*

$$A_n = \mathrm{Pr} \int_0^{2\pi} \frac{\sin n\theta}{\sin \theta}\, dE(\theta) \qquad (n \geq 1). \tag{29}$$

(d) *There exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a resolution of the identity $\{E(\theta) : 0 \leq \theta \leq 2\pi\}$ formed by orthogonal projections acting on $\mathcal{K}$ such that*

$$F(z) = \mathrm{Pr} \int_0^{2\pi} \frac{z}{1 - 2z\cos\theta + z^2}\, dE(\theta) \quad \text{for} \quad z \in \Delta. \tag{30}$$

Proof. The equivalence of (a) and (b) is already established in the proof of Theorem 1. To prove the equivalence of (b) and (c), we use the spectral decomposition of the unitary operator U:

$$U^n = \int_0^{2\pi} e^{in\theta}\, dE(\theta) \qquad (n \geq 0).$$

Then (24), (25) become

$$A_{2n} = 2\,\mathrm{Pr}\int_0^{2\pi}[\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta]\,dE(\theta)$$
$$= \mathrm{Pr}\int_0^{2\pi}\frac{\sin 2n\theta}{\sin\theta}\,dE(\theta)$$

and

$$A_{2n+1} = \mathrm{Pr}\int_0^{2\pi}[1 + 2(\cos 2\theta + \cos 4\theta + \cdots + \cos 2n\theta)]\,dE(\theta)$$
$$= \mathrm{Pr}\int_0^{2\pi}\frac{\sin(2n+1)\theta}{\sin\theta}\,dE(\theta).$$

The equivalence of (c) and (d) follows from

(31) $$\frac{z}{1 - 2z\cos\theta + z^2} = \sum_{n=1}^{\infty}\frac{\sin n\theta}{\sin\theta}z^n \quad \text{for} \quad z \in \Delta.$$

3. The case of 1-dimensional Hilbert space

When the complex plane $\mathbb{C}$ is regarded as a 1-dimensional Hilbert space, a complex-valued typically real function f on Δ is actually an operator-valued function. For each $z \in \Delta$, $f(z)$ is that operator on $\mathbb{C}$ which sends each vector $w \in \mathbb{C}$ to the vector $f(z)w$. Therefore, in the case of a 1-dimensional Hilbert space $\mathcal{H} = \mathbb{C}$, we have the following corollary of Theorem 2.

Corollary. *Let U be an $m \times m$ unitary matrix, and let $U^n = \left(u_{ij}^{(n)}\right)$ $(n = 1, 2, 3, \ldots; 1 \le i \le m, 1 \le j \le m)$. If*

(32) $$a_{2n} = 2\,\mathrm{Re}\left(u_{11}^{(1)} + u_{11}^{(3)} + \cdots + u_{11}^{(2n-1)}\right) \qquad (n \ge 1),$$

(33) $$a_{2n+1} = 1 + 2\,\mathrm{Re}\left(u_{11}^{(2)} + u_{11}^{(4)} + \cdots + u_{11}^{(2n)}\right) \qquad (n \ge 1),$$

then the function

(34) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta)$$

is typically real.

Example. For the unitary matrix

$$U = \begin{pmatrix} 0 & 1 \\ e^{i2\theta} & 0 \end{pmatrix}$$

where θ is a fixed real number, (32) and (33) become

$$a_{2n} = 0, \quad a_{2n+1} = 1 + 2(\cos 2\theta + \cos 4\theta + \cdots + \cos 2n\theta)$$
$$= \frac{\sin(2n+1)\theta}{\sin\theta}.$$

Hence, for any fixed real number θ, the function

(35) $$f(z) = \sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{\sin\theta}z^{2n+1}$$

is typically real.

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