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ON FREE GROUP ALGEBRAS IN DIVISION RINGS WITH UNCOUNTABLE CENTER

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Abstract. Let D be a division algebra with uncountable center k. If D contains a noncommutative free k-algebra, then D also contains the k-group algebra of the free group of rank 2.

1. Introduction

Let D be a division ring with center k. In [7] Makar-Limanov states the following conjecture:
(ML) *If D is finitely-generated and infinite dimensional over k, then D contains a 2-generator free algebra.*

An earlier claim about the existence of the free subgroups in division rings was stated by Lichtman [2]:
(L) *The multiplicative group of a noncommutative division ring contains a free group of rank* 2.

Evidence for (ML) can be found in [1], [5], [11], [12], and for (L) in [2], [3]. Recently, Makar-Limanov [11] surprisingly proved that the ring of fractions of the first Weyl algebra A_1, over the complex field $\mathbb{C}$, contains a $\mathbb{C}$-free group algebra (= the group algebra of a free group over $\mathbb{C}$).

In this note we show that this result is not accidental, but a general phenomenon. We will use a reasoning similar to the one in [9] to prove:

Theorem. *If k is uncountable and D contains a k-free algebra of rank* 2*, then D contains a k-free group algebra.*

Thus, we obtain new examples of division rings containing free group algebras. In particular, if k is uncountable, then (ML) implies (L).

We observe that A_1 can be regarded as a skew polynomial ring over the field of rational functions $\mathbb{C}(t)$. Therefore the existence of free group algebras in the ring of fractions of A_1 follows from the existence of free algebras in the ring of fractions of skew polynomial rings. (See [1], [5] for more on this.)

2. Proof of the theorem

The lemma below is proved in [12], Lemma 1, for algebras inside domains. It is also true for free group algebras.

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Lemma 2.1. *Let A be a domain with prime subfield k_0, and let k be any central subfield of A. If $x_1, x_2 \in A$, then x_1 and x_2 are free over k_0 if and only if they are free over k.* □

We also need the following:

Lemma 2.2. *Let $D[x]$ be the polynomial ring in the central indeterminate x, and let $D(x)$ be its ring of fractions. If $f \in D(x)$ and $f(z) = 0$ for infinitely many $z \in k$, then f is identically zero.*

Proof. See [9], Lemma 1. □

Let L be a field, and let us denote by $L\{x_1, x_2\}$ the free L-group algebra on the generators x_1 and x_2. Let $L\langle y_1, y_2\rangle$ be the free L-algebra in the generators y_1 and y_2, and let $L\langle y_1, y_2\rangle[[\lambda_1, \lambda_2]]$ be the ring of formal power series in the central indeterminates λ_1 and λ_2 over $L\langle y_1, y_2\rangle$. With this notation we have:

Lemma 2.3. *The homomorphism ψ from $L\{x_1, x_2\}$ to $L\langle y_1, y_2\rangle[[\lambda_1, \lambda_2]]$ defined by*

$$\psi(x_i) = 1 + \lambda_i y_i, \quad \psi(x_i^{-1}) = \sum_{j=0}^{\infty} (-\lambda_i y_i)^j, \qquad i = 1, 2,$$

is injective.

Proof. See [10], Lemma 1. □

Proof of the theorem. Let a and b be elements of D that generate a free k-algebra. We claim that there exist σ and τ in k such that $1+\sigma a$ and $1+\tau b$ generate a free k-group algebra. In view of Lemma 2.1 it is enough to show that they generate a free k_0-group algebra. Fix $\sigma \neq -a^{-1}$. Since $k_0\{x_1, x_2\}$ is countable but k is not, there exists $0 \neq f_\sigma(x_1, x_2) \in k_0\{x_1, x_2\}$ such that for uncountably many $\tau \in k \backslash \{-b^{-1}\}$ we have

$$f_\sigma(1 + \sigma a, 1 + \tau b) = 0.$$

By Lemma 2.2 we may replace τ by the central variable λ_2, obtaining the identity

$$f_\sigma(1 + \sigma a, 1 + \lambda_2 b) = 0 \quad \text{in } D(\lambda_2).$$

Arguing analogously, we may find a $0 \neq f(x_1, x_2) \in k_0\{x_1, x_2\}$ among all f_σ, such that

$$(*) \qquad f(1 + \lambda_1 a, 1 + \lambda_2 b) = 0 \quad \text{in } D(\lambda_2)(\lambda_1).$$

Now we embed $D(\lambda_1)(\lambda_2)$ in the Laurent series division ring $D((\lambda_2))((\lambda_1))$. We obtain

$$0 \neq \psi(f(x_1, x_2)) = \lambda_1^{m_1} \lambda_2^{m_2} f_{m_1, m_2}(y_1, y_2) + R,$$

with $f_{m_1, m_2} \neq 0$, and where all the terms in R possess higher (λ_1, λ_2)-degree (in the lexicographic order, say). But from $(*)$ $f_{m_1, m_2}(a, b) = 0$. This contradicts the hypothesis that a and b generate a free k_0-algebra. The result follows. □

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