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WEIGHTED INEQUALITIES FOR SOME ONE-SIDED OPERATORS

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Abstract. We give a characterization of the pairs of weights (u, v) such that the Weyl fractional integral operator maps $L^p(vdx)$ into weak $L^q(udx)$, $1 < p \le q < \infty$ or $p = 1 < q < \infty$. For the case $p < q$ we give necessary and sufficient conditions for the weak type of a maximal operator that includes as particular cases the Weyl fractional integral, the dual of the Hardy operator and the fractional one-sided maximal operator. As a consequence we give a new characterization of the pairs of weights for which the fractional one-sided maximal operator is bounded.

1. Introduction

Let f be a locally integrable function on $\mathbb{R}$ and $0 < \alpha < 1$. The Weyl fractional integral is defined as

$$(W_\alpha f)(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy.$$

It is known that $p > 1$, $1/p > \alpha$ and $1/q = 1/p - \alpha$ imply W_α maps L^p into L^q.

In 1988 Andersen and Sawyer [AS] characterized those u for which it is true that

$$\left(\int_{\mathbb{R}} |W_\alpha f|^q u^q\right)^{1/q} \le C \left(\int_{\mathbb{R}} |f|^p u^p\right)^{1/p}$$

with the above restrictions on p, q, α. In this note we study the weak type inequalities for this operator in the case $p = q$ and in the case $p < q$ we obtain necessary and sufficient conditions for the weak type of the operator

$$N^+ f(x) = \sup_{x<c} h(x,c) \int_x^c f(y) k(x,y)\, dy,$$

where $h(s,t)$ and $k(s,t)$ are positive measurable functions defined on $s \le t$. This operator includes as particular cases the Weyl fractional integral ($h \equiv 1$ and $k(s,t) = (t-s)^{\alpha-1}$), the fractional one-sided Hardy-Littlewood maximal operator ($k \equiv 1$ and $h(s,t) = (t-s)^{-\alpha}$) and the dual of the Hardy operator ($h \equiv 1$

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and $k \equiv 1$). Finally we apply this result together with those in [MT1] to obtain a new characterization of the pairs of weights for which the fractional one-sided Hardy-Littlewood maximal operator is bounded. By weights we understand locally integrable positive functions.

Throughout the paper, C will be a constant that may change from line to line. If A is any measurable set and g a positive function, $g(A)$ will stand for the integral of g over A and if $1 < p < \infty$, then p' will denote its conjugate exponent. M_u will denote the maximal operator defined by

$$M_u g(x) = \sup_{x \in I} \frac{1}{\int_I u} \int_I |gu|,$$

which is known to be of weak type $(1,1)$ with respect to the measure udx. Our main results are the following theorems.

Theorem 1. *Let $1 \leq p < q < \infty$ and consider the following two conditions:*

(1) *There exists C such that*

$$u(\{x : N^+ f(x) > \lambda\}) \leq C \left(\frac{1}{\lambda^p} \int f^p v \right)^{q/p}. \tag{1.1}$$

(2) *There exists C such that for any $a < b < c$*

$$h(a,c) \left(\int_a^b u \right)^{1/q} \left(\int_b^c v^{1-p'}(y) k^{p'}(a,y)\, dy \right)^{1/p'} \leq C, \qquad p > 1, \tag{1.2}$$

or there exists C such that for any $a < b$ and almost all $c > b$

$$h(a,c) \left(\int_a^b u \right)^{1/q} \leq C\ v(c) k^{-1}(a,c), \qquad p = 1.$$

If h is non-increasing on its second variable, then (1.2) $\Longrightarrow$ (1.1). If h and k are non-decreasing on the first variable, then (1.1) $\Longrightarrow$ (1.2).

Theorem 2. *Let $R_\alpha f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} dy$ be the formal adjoint of W_α. Let $1 < p \leq q < \infty$ or $p = 1 < q < \infty$. Then (1.1) for W_α holds iff: There exists C such that for any bounded interval I*

$$\int_I (R_\alpha(\chi_I u))^{p'} v^{1-p'} \leq C \left(\int_I u \right)^{p'/q'}, \qquad \text{if } 1 < p \leq q < \infty, \tag{1.3}$$

or

$$||R_\alpha(\chi_I u) v^{-1}||_{L^\infty(v)} \leq C \left(\int_I u \right)^{1/q'}, \qquad \text{if } p = 1 < q < \infty.$$

Theorem 3. *Let* $M_\alpha^+ f(x) = \sup_{h>0} h^{\alpha-1} \int_x^{x+h} |f|$. *Then* M_α^+ *is bounded from* $L^p(v)$ *to* $L^q(u)$, $1 < p < q < \infty$, *if, and only if, there exists* C *such that for any* $a < b$

$$\left(\int_{-\infty}^{a} \frac{u(y)}{(b-y)^{(1-\alpha)q}}\,dy\right)^{\frac{1}{q}} \left(\int_a^b v^{1-p'}\right)^{\frac{1}{p'}} \le C.$$

Remarks. (1) One can, of course, change the orientation of the real line and obtain theorems for R_α and for the operator

$$N^- f(x) = \sup_{c<x} h(c,x) \int_c^x f(y)k(y,x)\,dy.$$

(2)Sufficient conditions for the weak type of the Weyl fractional integral were obtained by Kokilashvili and Gabidzashvili in [GK].

(3) Theorem 2 for the two-sided fractional integral was obtained first by E. Sawyer [S].

2. Proofs of the theorems

Proof of Theorem 1. (1.1) $\implies$ (1.2). Assume first $p > 1$. We fix $a < b < c$, and consider the function

$$f(y) = v^{1-p'}(y)k^{(p'-1)}(a,y)\chi_{(b,c)}(y).$$

If $a < x < b$, we have

$$\begin{aligned} N^+ f(x) &\ge h(x,c) \int_b^c v^{1-p'}(y)k^{(p'-1)}(a,y)k(x,y)\,dy \\ &\ge h(a,c) \int_b^c v^{1-p'}(y)k^{p'}(a,y)\,dy = \lambda. \end{aligned}$$

This means that $(a,b) \subset \{x : N^+ f(x) > \lambda\}$ and then

$$\lambda^q \int_a^b u \le C \left(\int_b^c v^{1-p'}(y)k^{p'}(a,y)\,dy\right)^{\frac{q}{p}}.$$

If $\lambda < \infty$ this is equivalent to (1.2). If $\lambda = \infty$, then $f(y) = v^{-1}(y)k(a,y) \notin L^{p'}(v\chi_{(b,c)})$ and therefore there exists a non-negative function $g \in L^p(v)$ such that $\infty = \int fgv = \int_b^c g(y)k(a,y)\,dy$. But then for any $x \in (a,b)$

$$N^+ g(x) \ge h(a,c) \int_b^c g(y)k(x,y)\,dy \ge h(a,c) \int_b^c g(y)k(a,y)\,dy = \infty,$$

which contradicts (1.1).

The case $p = 1$ is obtained taking $a < b < c-t < c$, considering $f = \chi_{(c-t,c)}$ and observing that then $x \in (a,b) \Rightarrow N^+ f(x) \ge h(x,c) \int_{c-t}^c k(x,y)\,dy \ge h(a,c) \int_{c-t}^c k(a,y)\,dy = \lambda$ and therefore

$$h(a,c) \int_{c-t}^c k(a,y)\,dy \left(\int_a^b u\right)^{1/q} \le C \int_{c-t}^c v,$$

which implies $h(a,c)\left(u(a,b)\right)^{1/q} \leq Cv(c)k^{-1}(a,c)$ by Lebesgue's Differentiation Theorem.

Let us assume now that (1.2) holds and that h is non-increasing on the second variable. We fix $\lambda > 0$ and $f \geq 0$ and x such that $N^+f(x) > \lambda$. Then there exists $c > x$ such that $h(x,c)\int_x^c f(y)k(x,y)\,dy > \lambda$. Assume $p > 1$. For any $x < b < c$ we may write

$$\begin{aligned}
\lambda <& h(x,c)\int_x^b f(y)k(x,y)\,dy + h(x,c)\int_b^c f(y)k(x,y)\,dy \\
\leq& h(x,b)\int_x^b f(y)k(x,y)\,dy + h(x,c)\left(\int_b^c f^p v\right)^{\frac{1}{p}}\left(\int_b^c v^{1-p'}(y)k^{p'}(x,y)\,dy\right)^{\frac{1}{p'}} \\
\leq& h(x,b)\int_x^b f(y)k(x,y)\,dy + C\left(\int_b^c f^p v\right)^{\frac{1}{p}}\left(\int_x^b u\right)^{-\frac{1}{q}}.
\end{aligned}$$

By continuity of the integral we can choose b so that

$$C\left(\int_b^c f^p v\right)^{\frac{1}{p}}\left(\int_x^b u\right)^{-\frac{1}{q}} = \lambda/2,$$

which implies $\lambda/2 < h(x,b)\int_x^b f(y)k(x,y)\,dy$.

We define a sequence of points in (x,b) as follows: $x_0 = b$, $u(x,x_{k+1}) = \frac{1}{2}u(x,x_k)$. From the definition it follows easily that $x_0 = b > x_1 > \cdots > x_k > \cdots > x$, $\lim x_k = x$. We may therefore write

$$\begin{aligned}
\lambda/2 < h(x,b)\int_x^b f(y)k(x,y)\,dy &= h(x,b)\sum_{k=0}^{\infty}\int_{x_{k+1}}^{x_k} f(y)k(x,y)\,dy \\
&= h(x,b)\sum_{k=0}^{\infty}\int_{x_{k+1}}^{x_k} f(y)v^{1/p}(y)v^{-1/p}(y)k(x,y)\,dy.
\end{aligned}$$

Hölder's inequality gives

$$\begin{aligned}
\frac{\lambda}{2} &< h(x,b)\sum_{k=0}^{\infty}\left(\int_{x_{k+1}}^{x_k} f^p v\right)^{1/p}\left(\int_{x_{k+1}}^{x_k} v^{1-p'}(y)k^{p'}(x,y)\,dy\right)^{1/p'} \\
&\leq C\sum_{k=0}^{\infty}\frac{h(x,b)}{h(x,x_k)}\left(\int_{x_{k+1}}^{x_k} f^p v\right)^{1/p}\left(\int_x^{x_{k+1}} u\right)^{-1/q} \\
&\leq C\sum_{k=0}^{\infty}\left(\int_x^{x_{k+1}} u\right)^{1/p-1/q}\left(\frac{\int_x^{x_k} f^p v u^{-1} u}{\int_x^{x_k} u}\right)^{1/p} \\
&\leq C\sum_{k=0}^{\infty}\left(\int_x^{x_{k+1}} u\right)^{1/p-1/q}\left(M_u f^p v u^{-1}\right)^{1/p}(x) \\
&\leq Cu(x,b)^{1/p-1/q}\left(M_u f^p v u^{-1}\right)^{1/p}(x),
\end{aligned}$$

where the last inequality follows from the facts that $1/p - 1/q > 0$ and $\int_x^{x_k} u = 2^{-k}u(x,b)$. Since $\int_x^b u = C\left(\int_b^c f^p v\right)^{\frac{q}{p}} \lambda^{-q}$, this last inequality gives

$$\lambda^q < C\left(\int_b^c f^p v\right)^{\frac{q}{p}-1} \left(M_u f^p v u^{-1}\right)(x) \leq C\left(\int_{\mathbb{R}} f^p v\right)^{\frac{q}{p}-1} \left(M_u f^p v u^{-1}\right)(x).$$

We have therefore proved that

$$\{x:\ N^+ f(x) > \lambda\} \subset \{x:\ \left(M_u f^p v u^{-1}\right)(x) > C\lambda^q \left(\int_{\mathbb{R}} f^p v\right)^{1-\frac{q}{p}}\},$$

and (1.1) follows from the fact that the operator M_u is of weak type one-to-one with respect to the measure udx.

The case $p = 1$ is proved in a similar way.

Proof of Theorem 2. Let us assume that (1.1) for W_α holds. If $p > 1$ we have

$$\begin{aligned}\left(\int_I (R_\alpha(\chi_I u))^{p'} v^{1-p'}\right)^{1/p'} &\leq ||R_\alpha(\chi_I u)v^{-1}||_{L^{p'}(v)} \\ &= \sup_{\{f\geq 0:||f||_{L^p(v)}=1\}} \int v^{-1} R_\alpha(\chi_I u) f v \\ &= \sup_{\{f\geq 0:||f||_{L^p(v)}=1\}} \int_I (W_\alpha f) u \\ &= \sup_{\{f\geq 0:||f||_{L^p(v)}=1\}} \int_0^\infty u(\{x\in I: W_\alpha f(x) > \lambda\})d\lambda \\ &\leq \int_0^\infty \min(u(I), C\lambda^{-q})d\lambda = Cu(I)^{1/q'}.\end{aligned}$$

The case $p = 1$ is treated in the same way.

To prove that the condition is sufficient we fix $f \geq 0$ and $\lambda > 0$. Let $O_\lambda = \{x : W_\alpha f(x) > \lambda\} = \bigcup I_k$ where $I_k = (a_k, b_k)$ are disjoint open intervals. We fix $0 < \beta < 1$ and call F the set of k's such that $\frac{1}{u(I_k)} \int_{I_k} W_\alpha(f\chi_{I_k})u > \beta\lambda$ and G all the other k's.

If $k \in F$, then

$$\lambda^q u(I_k) < u(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} W_\alpha(f\chi_{I_k})u\right)^q = u(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} fR_\alpha(u\chi_{I_k})\right)^q.$$

If $p > 1$ we use Hölder's inequality and obtain

$$\begin{aligned}\lambda^q u(I_k) &\leq u(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} f^p v\right)^{q/p}\left(\int_{I_k} (R_\alpha(u\chi_{I_k}))^{p'} v^{1-p'}\right)^{q/p'} \\ &\leq Cu(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} f^p v\right)^{q/p} u(I_k)^{q/q'} = C\beta^{-q}\left(\int_{I_k} f^p v\right)^{q/p}.\end{aligned}$$

If $p=1$ we have

$$\begin{aligned}\lambda^q u(I_k) &\le u(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} fR_\alpha(u\chi_{I_k})vv^{-1}\right)^q\\ &\le Cu(I_k)^{1-q}\beta^{-q}\left(\int_{I_k} fv\right)^q u(I_k)^{q/q'} = C\beta^{-q}\left(\int_{I_k} fv\right)^q.\end{aligned}$$

Let now $x\in(a_k,b_k)$ be such that $W_\alpha f(x)>2\lambda$. Then

$$\begin{aligned}2\lambda &< \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}dy = \int_x^{b_k}\frac{f(y)}{(y-x)^{1-\alpha}}dy + \int_{b_k}^\infty \frac{f(y)}{(y-x)^{1-\alpha}}dy\\ &\le (W_\alpha f\chi_{I_k})(x) + \int_{b_k}^\infty \frac{f(y)}{(y-b_k)^{1-\alpha}}dy \le (W_\alpha f\chi_{I_k})(x)+\lambda.\end{aligned}$$

Therefore $\{x\in(a_k,b_k): W_\alpha f(x)>2\lambda\}\subset\{x\in I_k: W_\alpha(f\chi_{I_k})(x)>\lambda\}$.

Now if $k\in G$ we may write

$$\begin{aligned}u(\{x\in I_k : W_\alpha f(x)>2\lambda\}) &\le u(\{x\in I_k: W_\alpha(f\chi_{I_k})(x)>\lambda\})\\ &= \int_{\{x\in I_k: W_\alpha(f\chi_{I_k})(x)>\lambda\}} u(x)dx \le \frac{1}{\lambda}\int_{I_k} W_\alpha(f\chi_{I_k})u \le \beta u(I_k).\end{aligned}$$

We have thus proved that $\lambda^q u(I_k)\le C\beta^{-q}\left(\int_{I_k} f^p v\right)^{q/p}$ if $k\in F$ and $u(\{x\in I_k : W_\alpha f(x)>2\lambda\})\le\beta u(I_k)$ if $k\in G$. Therefore

$$\begin{aligned}&(2\lambda)^q u(\{x: W_\alpha f(x)>2\lambda\})\\ &\quad= \sum_k (2\lambda)^q u(\{x\in I_k: W_\alpha f(x)>2\lambda\})\\ &\quad\le 2^q\sum_{k\in F}\lambda^q u(I_k) + 2^q\sum_{k\in G}\lambda^q u(\{x\in I_k: W_\alpha f(x)>2\lambda\})\\ &\quad\le C2^q\beta^{-q}\sum_{k\in F}\left(\int_{I_k} f^p v\right)^{q/p} + 2^q\lambda^q\beta\sum_{k\in G}u(I_k)\\ &\quad\le C2^q\beta^{-q}\left(\int f^p v\right)^{q/p} + 2^q\lambda^q\beta u(\{x: W_\alpha f(x)>\lambda\}).\end{aligned}$$

Choosing $\beta=\frac{1}{2^{q+1}}$ we have

$$(2\lambda)^q u(\{x: W_\alpha f(x)>2\lambda\}) \le C\left(\int f^p v\right)^{q/p} + \frac{1}{2}\lambda^q u(\{x: W_\alpha f(x)>\lambda\}).$$

For any positive t we have

$$\begin{aligned}A_t &= \sup_{0<\lambda<t}\lambda^q u(\{x: W_\alpha f(x)>\lambda\})\\ &\le C\left(\int f^p v\right)^{q/p} + \frac{1}{2}\sup_{0<\lambda<t/2}\lambda^q u(\{x: W_\alpha f(x)>\lambda\})\\ &\le C\left(\int f^p v\right)^{q/p} + \frac{A_t}{2},\end{aligned}$$

and it is enough to prove that A_t is finite. Let us consider the following condition:

there exists C such that for any $a < b$

$$\left(\int_a^b u\right)^{1/q}\left(\int_b^\infty \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy\right)^{1/p'} \le C, \qquad p>1, \tag{2.1}$$

or

$$\left(\int_a^b u\right)^{1/q} \le C \text{ ess inf } \{v(y)(y-a)^{1-\alpha} : y > b\}, \qquad p = 1.$$

Claim. (1.3) $\Rightarrow$ (2.1). If $p > 1$ we reason as follows: Let $a < b < c$ be such that $u(a,c) \le 3u(a,b)$. Then

$$\begin{aligned}\int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy &= \int_b^c \left(\int_a^b u\right)^{p'} \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy \left(\int_a^b u\right)^{-p'} \\ &\le \int_b^c \left(\int_{-\infty}^y \frac{u(s)\chi_{(a,c)}(s)}{(y-s)^{(1-\alpha)}}ds\right)^{p'} v^{1-p'}(y)dy \left(\int_a^b u\right)^{-p'} \\ &\le \int_a^c (R_\alpha u\chi_{(a,c)})^{p'} v^{1-p'} \left(\int_a^b u\right)^{-p'} \\ &\le C\left(\int_a^c u\right)^{p'/q'} \left(\int_a^b u\right)^{-p'} \le C\left(\int_a^b u\right)^{-p'/q}.\end{aligned}$$

We have thus seen that if $u(a,c) \le 3u(a,b)$, then (1.3) implies

$$\left(\int_a^b u\right)^{1/q}\left(\int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy\right)^{1/p'} \le C.$$

Let now $a < b < c$. Let us choose $x_0 = a$, $x_1 = b$, x_k such that $2^k u(a,b) = u(x_k, x_{k+1})$ and assume $x_N < c \le x_{N+1}$. Then

$$\begin{aligned}&\left(\int_a^b u\right)^{p'/q} \int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy \\ &= \left(\int_a^b u\right)^{p'/q} \sum_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy + \left(\int_a^b u\right)^{p'/q} \int_{x_N}^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy \\ &\le \sum_{k=1}^{N-1} (2^{-(k-1)})^{p'/q} \left(\int_{x_{k-1}}^{x_k} u\right)^{p'/q} \int_{x_k}^{x_{k+1}} \frac{v^{1-p'}(y)}{(y-x_{k-1})^{(1-\alpha)p'}}dy \\ &\quad + (2^{-(N-1)})^{p'/q} \left(\int_{x_{N-1}}^{x_N} u\right)^{p'/q} \int_{x_N}^c \frac{v^{1-p'}(y)}{(y-x_{N-1})^{(1-\alpha)p'}}dy \\ &\le C\sum_{k=1}^{N} 2^{-(k-1)p'/q} \le C.\end{aligned}$$

Letting c go to infinity we obtain (2.1). If $p=1$ we fix $a<b$ and observe that for $y>b$ one has:

$$\begin{aligned}\frac{1}{v(y)(y-a)^{1-\alpha}} &= \int_a^b u(s)ds\frac{v^{-1}(y)}{(y-a)^{1-\alpha}}\left(\int_a^b u\right)^{-1}\\ &\le \int_a^b \frac{u(s)\chi_{(a,b)}(s)}{(y-s)^{1-\alpha}}dsv^{-1}(y)\left(\int_a^b u\right)^{-1}\\ &\le \int_{-\infty}^y \frac{u(s)\chi_{(a,b)}(s)}{(y-s)^{1-\alpha}}dsv^{-1}(y)\left(\int_a^b u\right)^{-1}\\ &= R_\alpha(u\chi_{(a,b)})(y)v^{-1}(y)\left(\int_a^b u\right)^{-1} \le C\left(\int_a^b u\right)^{-1/q};\end{aligned}$$

therefore,

$$\left(\int_a^b u\right)^{1/q} \le C \text{ ess inf } \{v(y)(y-a)^{1-\alpha} : y>b\}.$$

Finally we will prove (2.1) implies that A_t is finite. It is enough to consider the case of small t, since

$$\begin{aligned}\sup_{t_1<\lambda<t_2} \lambda^q u(\{x: W_\alpha f(x)>\lambda\}) &\le t_2^q u(\{x: W_\alpha f(x)>t_1\})\\ &\le \left(\frac{t_2}{t_1}\right)^q \sup_{0<\lambda<t_1+\varepsilon} \lambda^q u(\{x: W_\alpha f(x)>\lambda\}).\end{aligned}$$

We may assume that f is bounded and with compact support. Let a and b be real numbers such that support of f is contained in (a,b). Let us suppose then that t is so small that $\lambda<t \Rightarrow \left(\frac{1}{\lambda}\int_a^b f\right)^{\frac{1}{1-\alpha}} = s > b-a$. Then

$$\begin{aligned}&\lambda^q u(\{x: W_\alpha f(x)>\lambda\})\\ &\quad= \lambda^q u(\{x<a: W_\alpha f(x)>\lambda\}) + \lambda^q u(\{a<x<b: W_\alpha f(x)>\lambda\})\\ &\quad\le \lambda^q u(\{x<a: W_\alpha f(x)>\lambda\}) + t^q u(a,b).\end{aligned}$$

We must then prove that

$$\sup_{0<\lambda<t} \lambda^q u(\{x<a: W_\alpha f(x)>\lambda\}) < \infty.$$

But $x<a; W_\alpha f(x)>\lambda \Rightarrow \lambda < \int_a^b \frac{f(y)}{(y-x)^{1-\alpha}}dy \le \frac{1}{(a-x)^{1-\alpha}}\int_a^b f \Rightarrow a-x < \left(\frac{1}{\lambda}\int_a^b f\right)^{\frac{1}{1-\alpha}} = s$, i.e. $\{x<a: W_\alpha f(x)>\lambda\} \subset (a-s,a)$, and thus

$$\lambda^q u(\{x<a: W_\alpha f>\lambda\}) \le \lambda^q \int_{a-s}^a u = \left(\int_a^b f\right)^q s^{(\alpha-1)q}\int_{a-s}^a u.$$

If $p>1$ we may write $\int_a^b f = \int_a^b f v^{1/p} v^{-1/p}$, use Hölder's inequality and get

$$\lambda^q u(\{x<a : W_\alpha f > \lambda\}) \le \left(\int_a^b f^p v\right)^{q/p} \int_{a-s}^a u \left(\int_a^b \frac{v^{1-p'}(y)}{s^{(1-\alpha)p'}} dy\right)^{q/p'}$$
$$\le C\left(\int_a^b f^p v\right)^{q/p} \int_{a-s}^a u \left(\int_a^b \frac{v^{1-p'}(y)}{(y-a+s)^{(1-\alpha)p'}} dy\right)^{q/p'} \le C\left(\int_a^b f^p v\right)^{q/p}.$$

We have used (2.1) and the fact that $s > b-a$ implies $y-a+s<2s$. If $p=1$ we have

$$\left(\int_a^b f\right)^q s^{(\alpha-1)q} \int_{a-s}^a u \le \left(\int_a^b f(y) \frac{v(y)v^{-1}(y)}{(y-a+s)^{(1-\alpha)}} dy\right)^q \int_{a-s}^a u$$
$$\le C\left(\int_a^b fv\right)^q.$$

Proof of Theorem 3. The pairs of weights (u,v) for which the operator M_α^+ is bounded from $L^p(v)$ to $L^q(u)$ were characterized in [MT1] by the condition $S^+_{p,q,\alpha}$: $\left(\int_I (M_\alpha^+(\sigma\chi_I))^q u\right)^{\frac{1}{q}} \le C\left(\int_I \sigma\right)^{\frac{1}{p}}$, where $\sigma = v^{1-p'}$. The following duality argument shows that this condition is implied by the weak type of the fractional integral R_α from $L^{q'}$ to $L^{p'}$ with respect to the weights $(u^{1-q'}, v^{1-p'})$. If we observe that $M_\alpha^+ f(x) \le W_\alpha f(x)$ and define $B = \{g \ge 0 : \|g\|_{L^{q'}(u^{1-q'})} = 1\}$, we have:

$$\left(\int_I (M_\alpha^+(\sigma\chi_I))^q u\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}} (M_\alpha^+(\sigma\chi_I) u^{\frac{1}{q}} \chi_I)^q\right)^{\frac{1}{q}} = \sup_B \int_{\mathbb{R}} M_\alpha^+(\sigma\chi_I)\chi_I g$$
$$\le \sup_B \int_{\mathbb{R}} W_\alpha(\sigma\chi_I)\chi_I g = \sup_B \int_{\mathbb{R}} \sigma\chi_I R_\alpha(\chi_I g)$$
$$= \sup_B \int_0^\infty \sigma(\{x\in I : R_\alpha(\chi_I g) > \lambda\})\, d\lambda$$
$$\le C\int_0^\infty \min\{\sigma(I), \lambda^{-p'}\}\, d\lambda$$
$$\le C\sigma(I)^{\frac{1}{p}}.$$

But Theorem 1 for the operator N^- in the particular case $h\equiv 1$ and $k(y,x) = (x-y)^{\alpha-1}$ tells us that the weak type of R_α from $L^{q'}$ to $L^{p'}$ with respect to the weights $(u^{1-q'}, v^{1-p'})$ is equivalent to the condition

$$\left(\int_{-\infty}^a \frac{u(y)}{(b-y)^{(1-\alpha)q}}\, dy\right)^{\frac{1}{q}} \left(\int_a^b \sigma\right)^{\frac{1}{p'}} \le C, \qquad 1<p<q.$$

Since it is an easy exercise to check that this condition is also necessary (even if $p=q$), the proof is finished.

Final remarks. (1) Condition (2.1) characterizes the weak type of the Weyl fractional integral if $1 \le p < q < \infty$. We have proved it is actually equivalent to the apparently weaker condition: There exists C such that for any $a < b < c$ for which $\int_a^c u \le 3 \int_a^b u$

$$\left(\int_a^b u\right)^{1/q} \left(\int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy\right)^{1/p'} \le C.$$

(2) Even though we have proved that (1.3) $\implies$ (2.1), the proof of Theorem 2 is needed because Theorem 1 does not include the case $p = q$.

(3) The case $1 = p = q$ remains open even in the case of the two-sided fractional integral.

References

[AS] K. F. Andersen and E. T. Sawyer, *Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators*, Trans. Amer. Math. Soc. **308** (1988), 547–558. MR **89h:**26006

[GK] M. Gabidzashvili and V. Kokilashvili, *Two weight weak type inequalities for fractional-type integrals*, Math. Inst. Czech. Acad. Sci. Prague **45** (1989), 547–558.

[MT1] F. J. Martín-Reyes and A. de la Torre, *Two weight norm inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. **117** (1993), 483–489. MR **94b:**42010

[MT2] ______, *Weights for general one-sided maximal operators*, preprint.

[S] E. T. Sawyer, *A two weight weak type inequality for fractional integrals*, Trans. Amer. Math. Soc. **281** (1984), 339–345. MR **85j:**26010

[SW] E. T. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874. MR **94i:**42024

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