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COMPARATIVE PROBABILITY ON VON NEUMANN ALGEBRAS

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ABSTRACT. We continue here the study begun in earlier papers on implementation of comparative probability by states. Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and let $\mathcal{P}(\mathcal{A})$ denote the projections of $\mathcal{A}$. A comparative probability (CP) on $\mathcal{A}$ (or more correctly on $\mathcal{P}(\mathcal{A})$) is a preorder $\preceq$ on $\mathcal{P}(\mathcal{A})$ satisfying:

$0 \preceq P \forall P \in \mathcal{P}(\mathcal{A})$ with $Q \not\preceq 0$ for some $Q \in \mathcal{P}(\mathcal{A})$.
If $P, Q \in \mathcal{P}(\mathcal{A})$, then either $P \preceq Q$ or $Q \preceq P$.
If P, Q and R are all in $\mathcal{P}(\mathcal{A})$ and $P \perp R$, $Q \perp R$, then $P \preceq Q \Leftrightarrow P+R \preceq Q+R$.

A state ω on $\mathcal{A}$ is said to implement a CP $\preceq$ on $\mathcal{A}$ if for $P, Q \in \mathcal{P}(\mathcal{A})$, $P \preceq Q \Leftrightarrow \omega(P) \leq \omega(Q)$. In this paper, we examine the conditions for implementability of a CP on a general von Neumann algebra (as opposed to only type I factors). A crucial tool used here, as well as in earlier results, is the interval topology generated on $\mathcal{P}(\mathcal{A})$ by $\preceq$. A CP $\preceq$ will be termed continuous in a given topology on $\mathcal{A}$ if the interval topology generated by $\preceq$ is weaker than the topology induced on $\mathcal{P}(\mathcal{A})$ by the given topology. We show that uniform continuity of a comparative probability is necessary and sufficient if the von Neumann algebra has no finite direct summand. For implementation by normal states, weak continuity is sufficient and necessary if the von Neumann algebra has no finite direct summand of type I. We arrive at these results by constructing an appropriate additive measure from the CP.

1. INTRODUCTION AND NOTATION

In this paper $\mathcal{A}$ denotes a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{P}(\mathcal{A})$ denotes the (orthogonal) projections of $\mathcal{A}$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The largest projection in $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ (i.e. the identity operator) is denoted by $\mathbf{1}$ and we use 0 for the zero projection. If $P \in \mathcal{P}(\mathcal{A})$, then $P^{\perp}$ denotes $\mathbf{1} - P$ and $\Gamma(P)$ denotes the set of all subprojections of P which are in $\mathcal{P}(\mathcal{A})$. The range projection $\mathcal{R}(B) \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$ of a linear operator $B \in \mathcal{B}(\mathcal{H})$ is defined by $\mathcal{R}(B)\mathcal{H} = \overline{B\mathcal{H}}$. If $B \in \mathcal{A}$, then $\mathcal{R}(B) \in \mathcal{P}(\mathcal{A})$. By $P \sim Q$ we mean the equivalence of $P, Q \in \mathcal{P}(\mathcal{A})$ in the sense of von Neumann's comparison theory, and $P \precsim Q$ means that $P \sim Q'$ for some $Q' \in \Gamma(Q)$.

Definition 1.1. Let $\preceq$ denote a preorder on $\mathcal{P}(\mathcal{A})$. Then $\preceq$ is a comparative probability (CP) on $\mathcal{P}(\mathcal{A})$ if the following conditions are satisfied:

(A1) $0 \preceq P \ \forall P \in \mathcal{P}(\mathcal{A})$ with $Q \not\preceq 0$ for some $Q \in \mathcal{P}(\mathcal{A})$.

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(A2) If $P, Q \in \mathcal{P}(\mathcal{A})$, then either $P \preceq Q$ or $Q \preceq P$.
(A3) If P, Q, R, all in $\mathcal{P}(\mathcal{A})$, are such that $P \perp R$, $Q \perp R$, then $P \preceq Q \Leftrightarrow P+R \preceq Q+R$.

If $P \preceq Q$ and $Q \preceq P$ are simultaneously satisfied, we write $P \cong Q$, and if $Q \not\preceq P$, we write $P \prec Q$. Given a CP $\preceq$ and a real-valued function f on $\mathcal{P}(\mathcal{A})$ (e.g. the restriction of a state on $\mathcal{A}$), we say that f implements $\preceq$ if $P \preceq Q \Leftrightarrow f(P) \leq f(Q)$, or, equivalently, if $P \prec Q \Leftrightarrow f(P) < f(Q)$. Because CP's are defined on projections, the natural domain of study of these objects is the von Neumann algebra. However, an interesting case is that of the $C^\star$ algebra of compact operators on some Hilbert space. This algebra is generated by the finite rank projections. Even though the unit operator is missing when the Hilbert space is infinite dimensional, one can show that, here too, a CP can be satisfactorily defined and conditions for implementability can be demonstrated [12]. For this case (unlike the von Neumann algebra case) the above axioms, as they stand, make $\preceq$ a "comparative weight" in that $\preceq$ is no longer necessarily bounded, where boundedness is defined for an arbitrary $C^\star$ algebra $\mathcal{A}$ as follows:

Definition 1.2. Let $\preceq$ be a preorder on $\mathcal{P}(\mathcal{A})$. Then $\preceq$ is bounded if and only if there exists $P \in \mathcal{P}(\mathcal{A})$ such that $Q \preceq P$ if $Q \in \Gamma(P^\perp)$.

Let μ be a state on $\mathcal{A}$ and $A \in \mathcal{A}$ self-adjoint. Then, in the standard quantum mechanical interpretation, $\mu(A)$ is the mean value of the results (i.e. the expectation value) of measurements of the quantum observable (represented by) A when the system is in the state (represented by) μ. When restricted to $\mathcal{P}(\mathcal{A})$, μ is an "additive probability", i.e. a normalized additive measure, which is completely additive if and only if μ is normal. This probability has the following interpretation: Let $E \subset \sigma(A)$ be a Borel set and let the spectral projection $\chi_E(A)$ of A be defined in the usual functional calculus for self-adjoint operators. Then $\mu(\chi_E(A))$ is the probability that $a \in E$, where a is the result of a single measurement of A, while the system is in the state μ. Clearly, μ induces a CP $\preceq_\mu$ on $\mathcal{P}(\mathcal{A})$ defined by $P \preceq_\mu Q$ if and only if $\mu(P) \leq \mu(Q)$. The idea of a purely comparative probability was first introduced by Ochs [13] who worked mainly on uniqueness rather than existence of implementing states. In fact, Ochs had an additional axiom, namely:

(A4) For $P, Q \in \mathcal{P}(\mathcal{A})$, $P \preceq Q \Leftrightarrow Q^\perp \preceq P^\perp$.

It would appear that inclusion of (A4) offers no additional advantage of any consequence in the search for the conditions of implementability. For example, one can construct, on a finite von Neumann algebra, a continuous CP which satisfies (A4) but which cannot be implemented by a state. One can also show that in general, axioms (A3) and (A4) are mutually independent. But as we shall see later, for many algebras, the addition of a single topological condition on $\preceq$ is sufficient to make (A4) a consequence of (A3).

The question of uniqueness was taken further and completely resolved by Goldstein and Paszkiewicz in [5]. For our investigation of the possibility of implementation by a state we need to consider a continuity concept:

Definition 1.3. The CP $\preceq$ is said to be uniformly (weakly) continuous if whenever the net $Q_j : j \in \mathcal{J}$ in $\mathcal{P}(\mathcal{A})$ uniformly (weakly) converges to Q with $S \preceq Q_j \preceq T$ $\forall j \in \mathcal{J}$ for some $S, T \in \mathcal{P}(\mathcal{A})$, then $S \preceq Q \preceq T$.

In Kelly [8], $\preceq$ is uniformly (weakly) continuous if given $P, Q \in \mathcal{P}(\mathcal{A})$ such that $P \prec Q$, then there exists uniform (weak) neighbourhoods $\mathcal{N}(P)$ and $\mathcal{N}(Q)$ of P

and Q respectively such that $P' \prec Q'$ if $P' \in \mathcal{N}(P)$ and $Q' \in \mathcal{N}(Q)$. One can show that this definition is identical to ours.

As a linear preorder on $\mathcal{P}(\mathcal{A})$, $\preceq$ induces an interval topology on $\mathcal{P}(\mathcal{A})$, which we will call the $\preceq$ topology, which is generated by a neighbourhood base consisting (in obvious terminology) of $\preceq$ intervals $[0, P), (Q, \mathbf{1}]\colon P, Q \in \mathcal{P}(\mathcal{A})$. One easily shows (cf. [10], Proposition 2.2) that $\preceq$ is uniformly (weakly) continuous if and only if the interval topology induced by $\preceq$ is weaker than the topology on $\mathcal{P}(\mathcal{A})$ inherited from the uniform (weak) topology on $\mathcal{B}(\mathcal{H})$. We note here that if a CP $\preceq$ is implemented by some state μ, then the $\preceq$ topology is just that topology on $\mathcal{P}(\mathcal{A})$ for which sets of the form $\{P' \in \mathcal{P}(\mathcal{A})\colon 0 < |\mu(P - P')| < 1/n\}$, $n \in \mathbf{N}$, form a neighbourhood base of P. Unlike the case of states, not every CP is uniformly continuous, and perhaps more surprisingly, not every uniformly continuous CP can be implemented by a state as the following counter-example will show: Let $\mathcal{A}$ be a finite factor of type II and let τ be the restriction to $\mathcal{P}(\mathcal{A})$ of the canonical trace on $\mathcal{A}$. Let ω be a state on $\mathcal{A}$ such that $\omega \neq \tau$. Define the preorder $\trianglelefteq$ on $\mathcal{P}(\mathcal{A})$ as follows: For $P, Q \in \mathcal{P}(\mathcal{A})$, $P \trianglelefteq Q$ if and only if either $\tau(P) < \tau(Q)$ or $\tau(P) = \tau(Q)$ and $\omega(P) \leq \omega(Q)$. It is not difficult to verify that $\trianglelefteq$ is a CP. Now, $\tau^{-1}\{x\}$ is a uniformly connected component of $\mathcal{P}(\mathcal{A})$ for every $x \in [0, 1]$. This follows from the fact that if $P, Q \in \tau^{-1}\{x\}$, then $P \sim Q$. Since $\mathcal{A}$ is finite, we also have $P^{\perp} \sim Q^{\perp}$ so that we actually have unitary equivalence, that is, $P = U^{\star}QU$ for some unitary $U \in \mathcal{A}$ (see, for example, [2], Chapter 4; or [1]). But by the spectral theorem, $U = \exp(iH)$ for some self-adjoint $H \in \mathcal{A}$. Since H is bounded, $t \in [0, 1] \mapsto \exp(-iHt)Q\exp(iHt)$ is a *uniformly* continuous path joining P and Q. On the other hand, if $P \in \tau^{-1}\{x\}$ and $Q \in \tau^{-1}\{y\}$ with $x \neq y$, then $\|P - Q\| = 1$, because $\|P - Q\| < 1$ implies $P \sim Q$ [9], [15]. It easily follows then that $\trianglelefteq$ is, in fact, uniformly continuous. However no state can implement $\trianglelefteq$. To see this let $Q \in \mathcal{P}(\mathcal{A})$ be such that $\tau(Q) = 1/4$ and let the sequence Q_n in $\mathcal{P}(\mathcal{A})$ be such that $Q \perp Q_n$ $\forall n$ and such that $\tau(Q_n) = 2^{-n}$. Clearly Q_n converges to 0 in the $\trianglelefteq$ topology. Since $\omega \neq \tau$, we may assume, by Theorem 2.3 of [5], that $Q \triangleleft P$ for some $P \in \mathcal{P}(\mathcal{A})$ such that $\tau(P) = 1/4$. Thus $Q \triangleleft P \triangleleft Q + Q_n$ $\forall n \in \mathbf{N}$ so that all the $Q + Q_n$ are excluded from the $\trianglelefteq$ neighbourhood $[0, P)$ of Q. Thus addition fails here to be even separately $\trianglelefteq$ continuous on $\mathcal{P}(\mathcal{A})$; consequently no state can implement $\trianglelefteq$.

Let $P \in \mathcal{P}(\mathcal{A})$ be finite and let $\preceq$ be a CP on $\mathcal{P}(\mathcal{A})$. We will say that $\preceq$ is tracial on $\Gamma(P)$ if $\preceq$ is implemented by the canonical trace on the reduced algebra $P\mathcal{A}$. Next, we give some conditions for $\preceq$ to be tracial.

Proposition 1.4. *Let $\mathcal{A}$ be a type II factor and let $P \in \mathcal{P}(\mathcal{A})$ be finite. Let τ be the restriction to $\Gamma(P)$ of the normalized canonical trace on $P\mathcal{A}$. If $\preceq$ is a CP on $\mathcal{P}(\mathcal{A})$ such that $0 \prec P$, then the following are all equivalent*:

(i) *$\preceq$ is tracial on $\Gamma(P)$.*

(ii) *$\tau(R) = \tau(S) \Rightarrow R \cong S$ for $R, S \in \Gamma(P)$.*

(iii) *There exists $x \in (0, 1)$ such that, for $R, S \in \Gamma(P)$, $\tau(R) = \tau(S) = x \Rightarrow R \cong S$.*

Proof. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) are all clear.

(ii) $\rightarrow$ (i) We note that item (ii) implies that $\preceq$ is "faithful" on $\Gamma(P)$ in that $0 \prec P'$ for nonzero $P' \in \Gamma(P)$. To see this suppose there is a $T \in \Gamma(P)$ with $T \neq 0$ and $T \cong 0$. Then, for some $n \in \mathbf{N}$, we can find mutually orthogonal $P_j \in \Gamma(P)\colon 1 \leq j \leq n$ such that $P = \sum_{j=1}^{n} P_j$ and such that $\tau(P_j) < \tau(T)$ for all j. The last condition implies that $P_j \cong 0$ $\forall j$, and hence gives a contradicting $P \cong 0$. Now suppose that $\tau(R) < \tau(S)$ for some $S, T \in \Gamma(P)$. Then there exists $S' \in \Gamma(S)$

such that $\tau(S') = \tau(R)$. Since $\preceq$ is faithful and $S' < S$, we have $R \cong S' \prec S$. We conclude that, for $R, S \in \Gamma(P)$, $\tau(R) \leq \tau(S) \Rightarrow R \preceq S$, which simultaneously gives $R \preceq S \Rightarrow \tau(R) \leq \tau(S)$.

(iii) $\Rightarrow$ (ii) Let $y \in (0,1)$ be such that $y + x < 1$, $y < x$, and let $R, S \in \Gamma(P)$ be such that $\tau(R) = \tau(S) = y$. Now by Proposition 1.6 of Chapter V of [14], $R \vee S - R \sim S - R \wedge S$. Hence $\tau(R \vee S) \leq \tau(R) + \tau(S) = 2y$. As $2y + (x - y) < 1$, there exists $T \in \Gamma(P)$ such that $T \perp (R \vee S)$ and such that $\tau(T) = x - y$. We have $\tau(R+T) = \tau(S+T) = x$ implies, by hypothesis, that $R+T \cong S+T$, which in turn gives, by (A3), $R \cong S$. We conclude that the set $\mathcal{S} = \{t \in (0,1)\colon \tau(R) = \tau(S) < t \Rightarrow R \cong S \text{ for } R, S \in \Gamma(P)\}$ is nonempty. Now let $z \in \mathcal{S}$. Let $z' \in (0,1)$ be such that $z' < z$, $z + z' < 1$, and let z'' be such that $0 < z'' < z'$ and $z + z' + z'' < 1$. Let $R, S \in \Gamma(P)$ be such that $\tau(R) = \tau(S) = z + z''$ and let $R' \in \Gamma(R)$, $S' \in \Gamma(S)$ be such that $\tau(R') = \tau(S') = z'$. Again, we have $\tau(R' \vee S') \leq 2z'$. As $2z' + (z + z'' - z') < 1$, there exists $T \in \Gamma(P)$ such that $T \perp (R' \vee S')$ and such that $\tau(T) = z + z'' - z'$. Further, since $z + z'' - z' < z$, we have $T \cong R - R' \cong S - S'$. Thus, by (A3), $R = R' + R - R' \cong R' + T \cong S' + T \cong S' + S - S' = S$. This shows that $\sup \mathcal{S} = 1$ and the proof is complete. □

Let P_j be a net in $\mathcal{P}(\mathcal{A})$; then convergence to some $P \in \mathcal{P}(\mathcal{A})$ is denoted by $P_j \xrightarrow{w} P (P_j \xrightarrow{u} P)$ in the weak (uniform) topology and by $P_j \overset{\preceq}{\to} P$ in the $\preceq$ topology. It has been shown [10], [11] that if $\preceq$ is a uniformly (weakly) continuous CP on $\mathcal{P}(\mathcal{A})$, where $\mathcal{A}$ is type I_∞ factor, then $\preceq$ is implemented by a state (normal state). The purpose of this paper is to examine the extent to which these results can be extended to the general von Neumann algebra.

2. Topological structure and other results

For the remainder of this paper, unless the contrary is indicated, $\mathcal{A}$ is either a type II_1 factor with $\preceq$ a *weakly* continuous CP on $\mathcal{P}(\mathcal{A})$, or $\mathcal{A}$ is a type II_∞ or a type III factor with $\preceq$ a *uniformly* continuous CP on $\mathcal{P}(\mathcal{A})$. The strategy here, as in [11], will be to construct an additive measure on $\mathcal{P}(\mathcal{A})$ which implements $\preceq$. By a generalization of Gleason's theorem [3], [9], such an additive measure is the restriction of a state. For each $P \in \mathcal{P}(\mathcal{A})$ we define $\mathcal{D}(P)$ to be the set $\{Q \in \Gamma(P)\colon Q \sim P - Q\}$. We will use $\mathcal{D}$ to denote $\mathcal{D}(\mathbf{1})$. Let $P, Q \in \mathcal{D}$. Then by Proposition 6.2.2 of [6], $P \sim Q$ and $P^\perp \sim Q^\perp$ so that P and Q are unitarily equivalent. Consequently, $\mathcal{D}$ is uniformly and hence $\preceq$ connected. This $\preceq$ connectedness implies that all $\preceq$ closed subsets of $\mathcal{D}$ are $\preceq$ order complete and, equivalently, are $\preceq$ compact [8].

We begin with a few technical lemmas. The first one contains a strengthening of Lemma 2.1 (i) in Gunson [4].

Lemma 2.1. *Let $P, Q \in \mathcal{P}(\mathcal{A})$. The following statements are equivalent*:

(i) $\|(P - Q)\phi\| < \|\phi\|$ *if* $\phi \neq 0$.
(ii) $P \wedge Q^\perp = 0 = Q \wedge P^\perp$.
(iii) $\mathcal{R}(PQ) = P$ *and* $\mathcal{R}(QP) = Q$.
(iv) *The map* $f\colon P' \in \Gamma(P) \mapsto \mathcal{R}(QP')$ *is a bijection onto* $\Gamma(Q)$ *satisfying*

$$f(\mathcal{R}(PQ')) = Q' \quad \text{for } Q' \in \Gamma(Q).$$

Proof. (i) $\Rightarrow$ (ii) If $P \wedge Q^\perp \neq 0$, then clearly there exists $\phi \neq 0$ such that $P\phi = \phi$ and $Q\phi = 0$. This immediately gives $\|(P - Q)\phi\| = \|\phi\|$ and the contradiction gives the result.

(ii) $\Rightarrow$ (i) If $P, Q \in \mathcal{P}(\mathcal{A})$ and $\phi \in \mathcal{H}$ is nonzero, then $\|(P-Q)\phi\| = \|\phi\| \Rightarrow \langle\phi|(P-Q)^2\phi\rangle = \|\phi\|^2$. As $\|P-Q\| \leq 1$ we must have $\phi = (P-Q)^2\phi$. Now $Q^\perp(P-Q)^2 = Q^\perp P Q^\perp$ so that $Q^\perp\phi = Q^\perp P(Q^\perp\phi)$. We either have $Q^\perp\phi = 0 \Rightarrow Q\phi = \phi \Rightarrow P\phi = 0 \Rightarrow P^\perp\phi = \phi$, whence $Q \wedge P^\perp > 0$; or else $\psi = Q^\perp\phi \neq 0 \Rightarrow P\psi = \psi = Q^\perp\psi \Rightarrow P \wedge Q^\perp > 0$. The result follows by contraposition.

(ii) $\Rightarrow$ (iv) Let $Q' \in \Gamma(Q)$ be nonzero and let $P' = \mathcal{R}(PQ')$. We wish to show that $Q' = \mathcal{R}(QP')$. Clearly, $\mathcal{R}(Q'PQ') \leq Q'$. Let $Q'\phi = \phi$ for some $\phi \in \mathcal{H}$. Then $\langle Q'PQ'\psi|\phi\rangle = 0\ \forall\psi \in \mathcal{H}$ implies, by choosing $\psi = \phi$, that $PQ'\phi = 0$ which implies, by hypothesis, that $\phi = 0$. We conclude that $\mathcal{R}(Q'PQ') = Q'$ and hence that $\mathcal{R}(QPQ') \geq Q'$. Similarly, if for some $\phi \in \mathcal{H}$ we have $\langle QPQ'\phi|Q'\psi\rangle = 0\ \forall\psi \in \mathcal{H}$, then $QPQ'\phi = 0$. We conclude that $\mathcal{R}(QPQ') = f(\mathcal{R}(PQ')) = Q'$ as required, where we have used the fact that $\mathcal{R}(AB) = \mathcal{R}(A\mathcal{R}(B))$ for $A, B \in \mathcal{A}$. This also shows that f is onto $\Gamma(Q)$. It only remains to show that f is injective. To this end, let $f(P') = f(P'')$ for $P', P'' \in \Gamma(P)$. Interchanging the symmetric roles of P and Q, we have $P' = \mathcal{R}(PQP') = \mathcal{R}(Pf(P')) = \mathcal{R}(Pf(P'')) = \mathcal{R}(PQP'') = P''$, so that f is injective.

(iv) $\Rightarrow$ (iii) Since f is a bijection, we conclude that $\mathcal{R}(QP) = f(P) = Q$ and again, interchanging P and Q, we get $\mathcal{R}(PQ) = P$.

(iii) $\Rightarrow$ (ii) Let $\phi \in \mathcal{H}$. Now $Q^\perp P\phi = \phi \Rightarrow P\phi = \phi$, $QP\phi = 0$. Hence $0 = \langle\psi|QP\phi\rangle = \langle PQ\psi|\phi\rangle\ \forall\psi \in \mathcal{H}$. Since $P\phi = \phi$ and $\overline{PQ\mathcal{H}} = P\mathcal{H}$, we have $\phi = 0$ and hence $Q^\perp \wedge P = 0$. A similar argument gives $P^\perp \wedge Q = 0$. □

Lemma 2.2. *Let $P, Q \in \mathcal{P}(\mathcal{A})$ be such that $\|(P-Q)\phi\| < \|\phi\|$ for $\phi \neq 0$ and let f be as defined in Lemma 2.1. Then the following are true*:

(i) *$f(P') \sim P'$ for $P' \in \Gamma(P)$.*
(ii) *If $P', P'' \in \Gamma(P)$ are such that $f(P') \perp f(P'')$, then $P' \perp f(P'')$ and $P'' \perp f(P')$.*

Proof. (i) By Proposition 6.1.6 of [6], $\mathcal{R}(A) \sim \mathcal{R}(A^*)$, for $A \in \mathcal{A}$. Hence $f(P') = \mathcal{R}(QP') \sim \mathcal{R}(P'Q) = P'$ as required.

(ii) $\mathcal{R}(QP') \perp \mathcal{R}(QP'') \Rightarrow \langle QP'\phi|QP''\psi\rangle = \langle P'\phi|QP''\psi\rangle = 0\ \forall\phi,\ \psi \in \mathcal{H}$. The required results follow at once. □

Lemma 2.3. *Let $P, Q \in \mathcal{P}(\mathcal{A})$ be both nonzero. Then there exists $P' \in \Gamma(P)$ and $Q' \in \Gamma(Q)$, both nonzero, such that $P' \perp Q'$. If, in addition, $0 \prec P$ and $0 \prec Q$, then P' and Q' can be chosen such that $0 \prec P'$ and $0 \prec Q'$.*

Proof. If $R = P \wedge Q^\perp > 0$, then the choice $P' = R$ and $Q' = Q$ will suffice, with a similar result in the case $Q \wedge P^\perp > 0$. Now let $P \wedge Q^\perp = Q \wedge P^\perp = 0$. Let $P'' \in \Gamma(P)$ be such that $0 < P'' < P$. Since the function f in Lemma 2.1 is a bijection, we have $0 < \mathcal{R}(QP'') < Q$ and hence $0 < Q - \mathcal{R}(QP)$. By Lemma 2.2, the choice $P' = P''$ and $Q' = Q - \mathcal{R}(QP'')$ will satisfy our requirements. This completes the proof of the first part. For the second part we now assume that $0 \prec P$ and $0 \prec Q$. As in the first part, if $0 \prec P \wedge Q^\perp$ or $0 \prec Q \wedge P^\perp$, then there is no difficulty in finding the required P' and Q'. So we assume that $0 \cong P \wedge Q^\perp \cong Q \wedge P^\perp$ and set $\widetilde{P} = P - P \wedge Q^\perp$ and $\widetilde{Q} = Q - Q \wedge P^\perp$. Clearly $\widetilde{P} \cong P$ and $\widetilde{Q} \cong Q$. We claim that $\|(\widetilde{P} - \widetilde{Q})\phi\| < \|\phi\|$ for $\phi \neq 0$. To justify this we let $(\widetilde{Q}^\perp \wedge \widetilde{P})\phi = \phi$ for some $\phi \in \mathcal{H}$. Then $\widetilde{P}\phi = \phi = P\phi$ and $\widetilde{Q}\phi = 0$. The latter gives $Q\phi = Q \wedge P^\perp\phi$ and since $P^\perp\phi = 0$, we conclude that $Q\phi = 0$ so that $P \wedge Q^\perp\phi = \phi$. This immediately yields $\phi = 0$ and so $\widetilde{Q}^\perp \wedge \widetilde{P} = 0$. Similarly $\widetilde{P}^\perp \wedge \widetilde{Q} = 0$ and Lemma 2.1 completes the argument. If it is the case that for $Q_1 \in \Gamma(\widetilde{Q})$ we have $\mathcal{R}(\widetilde{P}Q_1) \cong 0 \Rightarrow Q_1 \cong 0$, then we pick any $Q_2 \in \Gamma(\widetilde{Q})$ such that $0 \prec Q_2 \prec \widetilde{Q}$. Clearly, $0 \prec \mathcal{R}(\widetilde{P}Q_2)$. By Lemma 2.2, the choice $P' = \mathcal{R}(\widetilde{P}Q_2)$ and $Q' = \widetilde{Q} - Q_2$ will satisfy our requirements. If, on the other hand, there is a $Q_1 \in \Gamma(\widetilde{Q})$ such that $\mathcal{R}(\widetilde{P}Q_1) \cong 0$ with $0 \prec Q_1$, then by Lemma 2.2 the choice $P' = \widetilde{P} - \mathcal{R}(\widetilde{P}Q_1)$ and $Q' = Q_1$ will suffice. This completes the proof of the proposition. □

Lemma 2.4. *For any $P \in \mathcal{P}(\mathcal{A})$, there exists $P' \in \mathcal{D}(P)$ such that $P' \cong P - P'$.*

Proof. We only need consider the case $0 \prec P$. Let $Q \in \mathcal{D}(P)$. If $Q \cong P - Q$, then there is nothing further to show. So we consider the case $Q \prec P - Q$. Since $\mathcal{D}(P)$ is uniformly path connected, there exists a uniformly, and hence $\preceq$ continuous, path $t \in [0,1] \mapsto P_t \in \mathcal{D}(P)$ such that $P_0 = Q$ and $P_1 = P - Q$. Since the path $t \in [0,1] \mapsto P - P_t$ is also uniformly and hence $\preceq$ continuous, there exists $s \in (0,1)$ such that $0 \le t \le s \Rightarrow P_t \preceq P - P_t$. Let s_0 be the supremum of all such s. We cannot have $P_{s_0} \prec P - P_{s_0}$ since another continuity argument would show that this inequality would also have to be true for a neighbourhood of s_0. Similarly, we cannot have $P - P_{s_0} \prec P_{s_0}$, hence the required result. □

Given any $P \in \mathcal{P}(\mathcal{A})$, repeated application of Lemma 2.4 results in a "partitioning" of P into 2^n subprojections of equal "size". We shall be using this idea several times and the following definition will be useful:

Definition 2.5. Let $n \in \mathbf{N}$ and let $P \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec P$. A set $\{P_j \in \Gamma(P) \colon 1 \le j \le n\}$ is called an equipartition of P of order n if the P_j are all mutually orthogonal, $P_j \cong P_k \ \forall j,k$ and if $\sum_{j=1}^n P_j = P$. □

The following result shows that any sequence of mutually orthogonal projections of $\mathcal{P}(\mathcal{A})$ must be $\preceq$ convergent to 0.

Lemma 2.6. *Let $Q \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec Q$. If $\mathcal{B}$ is a set of mutually orthogonal projections of $\mathcal{P}(\mathcal{A})$ satisfying $Q \preceq P \ \forall P \in \mathcal{B}$, then $\mathcal{B}$ is a finite set.*

Proof. If $\mathcal{A}$ is finite, then any sequence of orthogonal projections must weakly converge to 0 and so the result follows at once. Assume now that $\mathcal{A}$ is infinite. We suppose, for a contradiction, that the contrary is true, that is, $\mathcal{B}$ is an infinite set. We examine first the case where $Q \perp Q' \ \forall Q' \in \mathcal{B}$. By Lemma 2.4, there exists $Q_0 \in \mathcal{D}(Q)$ such that $0 \prec Q_0$. By Proposition 6.2.2 of [6], $\mathcal{D}(Q_0^{\perp}) \subset \mathcal{D}$. We pick sequences S_j and T_j in $\mathcal{B}$, each consisting of distinct projections, such that $S_j \perp T_k \ \forall j,k$ and such that $S = \sum_{j\in\mathbf{N}} S_j \precsim \sum_{j\in\mathbf{N}} T_j$. Again Proposition 6.2.2 of [6] gives $\mathcal{D}(Q_0^{\perp} - S) \subset \mathcal{D}$. Let R_j be a sequence of mutually orthogonal projections of $\mathcal{D}(Q_0^{\perp} - S)$. If we set $Q_j = R_j + S_j$, then $Q_j \in \mathcal{D}$ for all j and hence $\sum_{j=2}^m Q_j \in \mathcal{D}(P)$ for all $m \in [2,\infty]$, where $P = \sum_{j\in\mathbf{N}} Q_j$. Now since $\mathcal{D}(P)$ is uniformly connected with $\sum_{j=2}^n Q_j \le \sum_{j=2}^\infty Q_j \in \mathcal{D}(P)$ for all n, and since $\sum_{j=2}^n Q_j$ is increasing with n, $\sum_{j=2}^n Q_j \overset{\preceq}{\to} \widetilde{Q}$, as $n \to \infty$, for some $\widetilde{Q} \in \mathcal{D}(P)$. Now addition is uniformly continuous on $\mathcal{P}(\mathcal{A})$. Hence $Q_0 + \mathcal{D}(P)$ is also uniformly connected. As $Q_0 + Q_2 \prec \widetilde{Q} \prec Q_0 + \sum_{j=2}^\infty Q_j$, we also have $\widetilde{Q} \cong Q_0 + Q'$ for some $Q' \in \mathcal{D}(P)$. Thus there exists $m \in \mathbf{N}$ for which $Q' \prec \sum_{j=2}^m Q_j$. (A4) yields a contradictory $Q_0 + Q' \preceq Q_{m+1} + Q' \prec \sum_{j=2}^{m+1} Q_j$. Now we look at the case of arbitrary Q. Pick a sequence of distinct projections P_j of $\mathcal{B}$. By Lemma 2.3, there exist mutually orthogonal projections $S \in \Gamma(Q)$ and $T \in \Gamma(P_1)$ such that $0 \prec S$

and $0 \prec T$. By considering equipartitions of T, it is clear from the first case that there exists $T' \in \Gamma(T)$ for which $0 \prec T' \preceq S$. Thus the sequence $P_j : j \geq 2$ is such that $T' \perp P_j$ for all j, and since $0 \prec T'$, this contradicts the result of the first case. The proof of the lemma is now complete. □

Lemma 2.7. *If $\mathcal{A}$ is infinite, then $\mathbf{1}$ is in the $\preceq$ supremum of $\mathcal{D}$.*

Proof. Suppose for a contradiction that $\mathbf{1} \notin_{\preceq} \sup \mathcal{D}$, with $\mathcal{A}$ infinite. Then there exists $R \in \mathcal{P}(\mathcal{A}) \backslash \mathcal{D}$ such that $P \preceq R \prec \mathbf{1}$ for all $P \in \mathcal{D}$. We note that $\mathcal{D}(R) \subset \mathcal{D}$. We can find a sequence of mutually orthogonal nonzero projections of $\mathcal{D}(R)$. Hence, by Lemma 2.6, there exists $P' \in \mathcal{D}$ such that $P' \leq R$ and such that $P' \prec R^\perp$. We conclude that $R = (R - P') + P' \prec R - P' + R^\perp = \mathbf{1} - P'$. Since $\mathbf{1} - P' \in \mathcal{D}$, we have a contradiction, and the proof is complete. □

Lemma 2.8. *Let $P \in \mathcal{P}(\mathcal{A})$ and let P_j be a sequence of projections of $\Gamma(P^\perp)$ such that $P_j \overset{\preceq}{\to} 0$. Then $P + P_j \overset{\preceq}{\to} P$.*

Proof. Let Q_j be a $\preceq$ decreasing sequence of mutually orthogonal projections of $\Gamma(P^\perp)$ such that $0 \prec Q_j \ \forall j$. By Lemma 2.6 and (A3), it is sufficient to show that $P + Q_j \overset{\preceq}{\to} P$. This is trivially true for the case $\mathcal{A}$ is finite, so we assume otherwise. If P is not infinite, we can ensure that $P + Q_1$ is infinite by choosing the sequence Q_j with Q_1 infinite. Now $\mathcal{D}(P + Q_1)$ is uniformly path connected and, by Lemmas 2.6 and 2.7, $\preceq$ dense in $\Gamma(P + Q_1)$. As $P + Q_j \prec P + Q_1$ for all large enough j, we have $P + Q_j \overset{\preceq}{\to} R$, for some $R \in \mathcal{D}(P + Q_1)$. This R must in fact be in the $\preceq$ infimum of the $\preceq$ decreasing $P + Q_j$. Since R is infinite, there exists, by Lemma 2.7, a sequence R_j in $\Gamma(R)$ for which $R_j \prec R$ for all j and $R_j \overset{\preceq}{\to} R$. Pick any $k \in \mathbf{N}$; then there exists $m > 1$ such that $Q_m \prec R - R_k$. By (A3), $R_k + Q_m \preceq R_k + (R - R_k) = R \preceq P + Q_m$, which implies that $R_k \preceq P$. Hence $P \cong R$ as required, since $P \preceq R$. □

Proposition 2.9. *Let $P \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec P$. Then $\Gamma(P) \cap (0, P)$ is a $\preceq$ path connected subset of $\mathcal{P}(\mathcal{A})$ which is $\preceq$ dense in $[0, P]$. Hence $[0, P]$ is $\preceq$ compact for every $P \in \mathcal{P}(\mathcal{A})$.*

Proof. We already know that $\mathcal{D}(P)$ is $\preceq$ path connected. If P is infinite, then by Lemmas 2.6 and 2.7, 0 and P are $\preceq$ limits of $\mathcal{D}(P)$ and the result follows at once.

Now we consider P finite. If $\preceq$ is tracial on $\Gamma(P)$ with τ the implementing normalized trace, then any map $\gamma\colon [0,1] \to \mathcal{P}(\mathcal{A})$ for which $\gamma(x) \in \tau^{-1}\{x\}$ is a $\preceq$ continuous path. Suppose now that $\preceq$ is not tracial on $\Gamma(P)$. Let $x \in (0,1)$ and let $\mathcal{S}$ be the $\preceq$ path connected component of $\Gamma(P)$ containing the uniformly (and hence $\preceq$) path connected set $\tau^{-1}\{x\}$. Set $x_0 = \sup\{t \in [x, 1] \colon \tau^{-1}\{t\} \subset \mathcal{S}\}$. We claim that $x_0 = 1$. To justify this we suppose, for a contradiction, that $x_0 < 1$. By Proposition 1.4, there exists $R, S \in \tau^{-1}\{x_0\}$ such that $R \prec S$. By Lemma 2.8, there exists $R' \in \Gamma(P - R)$ such that $R \prec R + R' \prec S$. Thus $R + R' \in \mathcal{S}$, a contradiction since $\tau(R + R') > x_0$. By Lemma 2.6, 0 is a $\preceq$ limit of nonzero projections of $\Gamma(P)$; this completes the proof. The $\preceq$ density and compactness follow immediately. □

Lemma 2.10. *Let $P \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec P$ and let $P_j \in \Gamma(P)$ be a sequence of mutually orthogonal projections satisfying $0 \prec P_j$. Then $P - P_j \overset{\preceq}{\to} P$. Hence $\mathcal{P}(\mathcal{A})$ is $\preceq$ second countable.*

Proof. By Proposition 2.9, there exists a sequence Q_j in $\Gamma(P)$ such that $Q_j \overset{\preceq}{\Rightarrow} P$ and such that $Q_j \prec P \ \forall j$. Pick any $j \in \mathbf{N}$. By Lemma 2.3, we can find mutually orthogonal $R \in \Gamma(P_1)$ and $S \in \Gamma(P - Q_j)$ such that $0 \prec R \preceq S$. Let $k_0 \in \mathbf{N}$ be such that $P_k \preceq R$ if $k > k_0$. Then, by (A3), $Q_j \leq P - S = (P - S - R) + R \preceq (P - S - R) + S = P - R = (P - R - P_k) + P_k \preceq (P - R - P_k) + R = P - P_k$, for $k > k_0$. The result follows at once. The second countability follows immediately from this result together with Lemma 2.8. □

Proposition 2.11. *Let $T \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec T \prec \mathbf{1}$.*

(i) *Let P, Q, R and S, all in $\Gamma(T)$, be such that $P \perp R$, $Q \perp S$, $P \preceq Q$ and $R \preceq S$. Then $P + R \preceq Q + S$.*
(ii) *Let $P, Q \in \Gamma(T)$ be such that $P \preceq Q$. Then $Q \cong P + R$ for some $R \in \Gamma(P^{\perp})$.*

Proof. (i) We assume that $0 \prec P$, lest the result be trivial. Let $\{P_j \colon 1 \leq j \leq 2^n\}$ and $\{Q_j \colon 1 \leq j \leq 2^n\}$ be equipartitions of P and Q. Assume that n is large enough so that by Lemma 2.6, $P_j \preceq T^{\perp}$ and $Q_j \preceq T^{\perp}$. We claim that $P_j \preceq Q_j \ \forall j$. To see this, suppose for a contradiction that the converse is true, that is $Q_j \prec P_j \ \forall j$. By Proposition 2.9, there exists $T' \in \Gamma(T^{\perp})$ such that $Q_j \prec T' \prec P_j$. Suppose, for an inductive proof, that $Q'_m = \sum_{j=1}^m Q_j \prec \sum_{j=1}^m P_j = P'_m$ for m such that $1 \leq m < 2^n$. Then $Q'_{m+1} = Q'_m + Q_{m+1} \prec Q'_m + T' \prec P'_m + T' \prec P'_m + P_{m+1} = P'_{m+1}$. Since the inequality is true for $m = 1$, we have $Q \prec P$, and the contradiction verifies the claim. A similar inductive argument involving the use of T' shows that $R + P'_m \preceq S + Q'_m$ for $1 \leq m \leq 2^n$. This completes the proof. We note that if the condition $P \preceq Q$ is replaced by $P \cong Q$, then we immediately have $R \preceq S \Leftrightarrow P + R \preceq Q + S$.

(ii) Again, we only need consider the case $0 \prec P$. By Proposition 2.9, there exists $P' \in \Gamma(Q)$ such that $P' \cong P$. Since $(Q - P') + P' = Q \preceq T = (T - P) + P$ and $P' \cong P$, item (i) gives $Q - P' \preceq T - P$. Hence there exists $R \in \Gamma(T - P)$ such that $R \cong Q - P'$. Again, by item (i), we have $Q = P' + (Q - P') \cong P + R$, as required. □

The following result establishes separate $\preceq$ continuity of addition on $\mathcal{P}(\mathcal{A})$.

Proposition 2.12. *Let $P, Q \in \mathcal{P}(\mathcal{A})$ be mutually orthogonal with $0 \prec P$ and let Q_j be a sequence of $\Gamma(P^{\perp})$ such that $Q_j \overset{\preceq}{\Rightarrow} Q$. Then $P + Q_j \overset{\preceq}{\Rightarrow} P + Q$.*

Proof. Proposition 2.9 implies that we can pick a subsequence $P + Q_{j_k}$ of $P + Q_j$ and $\preceq$ converges to T, say. Clearly $P \preceq T$ and by Proposition 2.11 (ii), $T \cong P + Q'$ for some $Q' \in \Gamma(P^{\perp})$. By (A3), $Q_{j_k} \overset{\preceq}{\Rightarrow} Q'$ and so $Q' \cong Q$. Thus every $\preceq$ convergent subsequence of $P + Q_j$ $\preceq$ converges to $P + Q$; hence the result. □

Lemma 2.13. *Let $P, Q \in \mathcal{P}(\mathcal{A})$ be such that $0 \prec P$ and $0 \prec Q$. Let $n \in \mathbf{N}$ and let $\{P_j \colon 1 \leq j \leq 2^n\}$ and $\{Q_j \colon 1 \leq j \leq 2^n\}$ be equipartitions of P and Q respectively. Then we have $\sum_{j=1}^{2^n} P_j \preceq \sum_{j=1}^{2^n} Q_j \Leftrightarrow P_j \preceq Q_j \ \forall j$.*

Proof. It is sufficient to show that $P_j \cong Q_j$ implies that $P \cong Q$. To see this, we suppose for the moment that $P_j \cong Q_j \Rightarrow P \cong Q$ is given. Now, if $P_j \prec Q_j$, then by Proposition 2.9, there exists $Q'_j \in \Gamma(Q_j)$, for each j, such that $P_j \cong Q'_j \prec Q_j$. The given supposition implies that $P \cong \sum_{j=1}^{2^n} Q'_j \prec Q$. Thus we have $P_j \preceq Q_j \Rightarrow P \preceq Q$ and this clearly is equivalent to saying $P_j \preceq Q_j \Leftrightarrow P \preceq Q$. Now let $P_j \cong Q_j$.

Case 1: *$P_j = Q_k$ for some j and k.* We are obviously free to change labels so that $j = k = 2^n$. Suppose inductively that $P'_m = \sum_{j=1}^m P_j \cong \sum_{j=1}^m Q_j = Q'_m$ for some m

such that $1 \leq m < 2^n$. Then $P'_{m+1} = P'_m + P_{m+1} \cong P'_m + P_{2^n} \cong Q'_m + P_{2^n} \cong Q'_{m+1}$, by (A3). Since $P'_1 \cong Q'_1$, we have $P \cong Q$ as required.

Case 2. $P \prec \mathbf{1}$ *and* $Q \prec \mathbf{1}$. We first observe that, because $P \prec \mathbf{1}$, Proposition 2.11 (i) implies that if $\{\widetilde{P}_j : 1 \leq j \leq 2^n\}$ is another equipartition of P, then $\widetilde{P}_j \cong P_j$ and (with the use of Proposition 2.9) also implies that, if $P''_1 \in \Gamma(P)$ is such that $P''_1 \cong P_j$, then P''_1 is always part of some equipartition $\{P''_j : 1 \leq j \leq 2^n\}$ of P. The same observations will, of course, identically hold for Q. Now, by Lemma 2.3 and Proposition 2.9, there exists mutually orthogonal $R \in \Gamma(P)$ and $S \in \Gamma(Q)$ such that $0 \prec R \cong S$. Pick n large enough so that, by Lemma 2.6, $P_j \preceq R$ and $Q_j \preceq R$. Then we may assume from the above observations that $P_{2^n} \perp Q_{2^n}$. As $P_{2^n} + Q_{2^n} \cong Q_{2^n} + Q_1 \prec Q$, there exists, by Proposition 2.11 (ii), $T \in \mathcal{P}(\mathcal{A})$ such that $T \perp (P_{2^n} + Q_{2^n})$ and such that $T + P_{2^n} + Q_{2^n} \cong P$. From earlier arguments, there exists an equipartition $\{T_j : 1 \leq j \leq 2^n\}$ of $T + P_{2^n} + Q_{2^n}$ for which $T_1 = P_{2^n}$ and $T_2 = Q_{2^n}$. We deduce from Case 1 that $Q \cong T + P_{2^n} + Q_{2^n}$ and hence that $P \cong Q$ as required. This only shows that, for n large enough, $\sum_{j=1}^{2^n} P_j \preceq \sum_{j=1}^{2^n} Q_j \Leftrightarrow P_j \preceq Q_j$. We now extend this to arbitrary $n \in \mathbf{N}$. If $m \in \mathbf{N}$, then for each j such that $1 \leq j \leq 2^n$, we pick equipartitions $\{P_{jk} : 1 \leq k \leq 2^m\}$ and $\{Q_{jk} : 1 \leq k \leq 2^m\}$ of P_j and Q_j respectively. For m large enough, the above arguments show that we may assume that $P_{jk} \cong P_{il}$, $P_{jk} \cong Q_{il}$ and $Q_{jk} \cong Q_{il}$ for all i, j, k and l and that we may also assume that $P_{11} \perp Q_{11}$. We also follow these arguments to establish the existence of $T \in \mathcal{P}(\mathcal{A})$ and an equipartition of $\{T_j : 1 \leq j \leq 2^{n+m}\}$ of T for which $T_1 = P_{11}$ and $T_2 = Q_{11}$. This gives the required $P \cong T \cong Q$.

Case 3. $P \cong \mathbf{1}$ *or* $Q \cong \mathbf{1}$. Given the preceding cases, the result will have been established if we can show that $Q \prec \mathbf{1} \Rightarrow P \prec \mathbf{1}$. Now, let $Q \prec \mathbf{1}$. As $P_1 \cong Q_1 \prec Q$, there exists, by Proposition 2.11 (ii), $R \in \Gamma(P_1^{\perp})$ such that $\widetilde{P} = P_1 + R \cong Q$. Case 2 reveals that there exists an equipartition $\{\widetilde{P}_j : 1 \leq j \leq 2^n\}$ of $\widetilde{P}$ for which $\widetilde{P}_1 = P_1$. Case 1 then gives $P \cong \widetilde{P} \cong Q$, completing the proof of the lemma. □

We are now in a position to give what might appear to be only a slight strengthening of axiom (A3). As we shall see, the result, together with the corollary that follows it, is strong enough to place our final implementability theorem well within sight.

Proposition 2.14. *Let P, Q, R and S, all in $\mathcal{P}(\mathcal{A})$, be such that $P \perp R$ and $Q \perp S$. Then $P \preceq Q$, $R \preceq S \Rightarrow P + R \preceq Q + S$.*

Proof. Assume, to avoid triviality, that $0 \prec P$ and $0 \prec R$. We let $\{P_j : 1 \leq j \leq 2^n\}$, $\{Q_j : 1 \leq j \leq 2^n\}$, $\{R_j : 1 \leq j \leq 2^n\}$ and $\{S_j : 1 \leq j \leq 2^n\}$ be equipartitions of P, Q, R and S respectively. By (A3), we have for each j and k, $P_j + R_j \cong P_j + R_k \cong P_k + R_k$ and similarly, $Q_j + S_j \cong Q_k + S_k$.

We assume that n is large enough so that, by a now familiar procedure, P_1, Q_1, R_1 and S_1 may be chosen to be all mutually orthogonal. Lemma 2.13 gives $P_1 \preceq Q_1$ and $R_1 \preceq S_1$. Hence another use of (A3) yields $P_j + R_j \cong P_1 + R_1 \preceq P_1 + S_1 \preceq Q_1 + S_1 \cong Q_j + S_j$ for all j. Since $\{P_j + R_j : 1 \leq j \leq 2^n\}$ and $\{Q_j + S_j : 1 \leq j \leq 2^n\}$ are equipartitions of $P+R$ and $Q+S$ respectively, $P+R \preceq Q+S$ follows immediately from Lemma 2.13. □

Corollary 2.15. *Let $P, Q \in \mathcal{P}(A)$ be mutually orthogonal and let sequences P_j and Q_j in $\mathcal{P}(\mathcal{A})$ be such that $P_j \preceq P$ and $Q_j \preceq Q$, $P_j \perp Q_j$ $\forall j$ and such that $P_j \overset{\preceq}{\Rightarrow} P$ and $Q_j \overset{\preceq}{\Rightarrow} Q$. Then $P_j + Q_j \overset{\preceq}{\Rightarrow} P + Q$.*

Proof. We assume that $0 \prec P$ and $0 \prec Q$, lest the result be trivial. Let P'_j be a sequence of mutually orthogonal projections of $\Gamma(P)$ such that $0 \prec P'_j$ $\forall j$ and let the sequence Q'_j in $\Gamma(Q)$ be similarly defined. Then, by Lemma 2.10, $(P - P'_j) + (Q - Q'_j) \overset{\preceq}{\Rightarrow} P + Q$. For a given $j \in \mathbf{N}$ there exists $k_0 \in \mathbf{N}$ for which $P - P'_j \preceq P_k$ and $Q - Q'_j \preceq Q_k$ if $k > k_0$. Proposition 2.14 gives $(P - P'_j) + (Q - Q'_j) \preceq P_k + Q_k \preceq P + Q$ for all $k > k_0$. This gives the required result. ◻

Proposition 2.16. *$\preceq$ is implemented by an additive measure (= a state) on $\mathcal{P}(\mathcal{A})$.*

Proof. For each $n \in \mathbf{N}$, let $\mathcal{U}_n = \{T_j^{(n)} \colon j \in K(n)\}$, where $K(n) = \{1, 2, 3, \dots, 2^n\}$, be an equipartition of $\mathbf{l}$. For a given n, $T_j^{(n)}$ can also be "halved", by Lemma 2.4. Therefore we can arrange that the $\mathcal{U}_n$ are "nested" in the sense that $T_{2j-1}^{(n+1)} + T_{2j}^{(n+1)} = T_j^{(n)}$ for $n \in \mathbf{N}$ and for $j \in K(n)$. For each $n \in \mathbf{N}$ we define $\mathcal{V}_n$ to be the set $\{\sum_{j \in K} T_j^{(n)} \colon K \subset K(n)\}$. We define $\mathcal{V}$ to be $\bigcup_{n \in \mathbf{N}} \mathcal{V}_n$. We note that if $P \in \mathcal{V}$, then $P^\perp \in \mathcal{V}$ and that, in the case $0 \prec P$, there exists $n \in \mathbf{N}$ such that $P = \sum_{k=1}^m T_{j_k}^{(n)}$, for some $m \leq 2^n$ and $T_{j_k}^{(n)}$ in $\mathcal{U}_n$. The nesting property implies that for each $n \in \mathbf{N}$ and $P \in \mathcal{V}$, there exists an equipartition $\mathcal{U}_n(P) = \{P_j^{(n)} \colon 1 \leq j \leq 2^n\}$ of P such that $\mathcal{U}_n(P) \subset \mathcal{V}$. We define $\mathcal{V}_n(P)$ and $\mathcal{V}(P)$ in analogy with $\mathcal{V}_n(= \mathcal{V}_n(\mathbf{l}))$ and $\mathcal{V}(= \mathcal{V}(\mathbf{l}))$. It is clear that $\mathcal{V}(P) = \Gamma(P) \cap \mathcal{V}$. We claim that $\mathcal{V}(P)$ is $\preceq$ dense in $[0, P]$. To see this, let $S, T \in [0, P]$ be such that $S \prec T$. By Lemma 2.8, there exists $R \in \Gamma(S^\perp)$ such that $S \prec S + R \prec T$. By Lemma 2.6, $P_j^{(n)} \overset{\preceq}{\Rightarrow} 0$ as $n \to \infty$. Choose $n \in \mathbf{N}$ such that $P_j^{(n)} \prec R$ and let m be the largest integer for which $\sum_{j=1}^m P_j^{(n)} \preceq S$. As $S \prec P$, we have $m < 2^n$. Hence, by Proposition 2.14, $\sum_{j=1}^{m+1} P_j^{(n)} \prec S + R \prec T$ and, by the definition of m, we have $S \prec \sum_{j=1}^{m+1} P_j^{(n)}$. Thus we have $S \prec \sum_{j=1}^{m+1} P_j^{(n)} \prec T$, which verifies the claim. We have also demonstrated that there is a sequence P'_j in $\mathcal{V}(P)$ such that $P'_j \overset{\preceq}{\Rightarrow} T$ and such that $P'_j \prec T$.

Now we define the function $\mu \colon \mathcal{V} \to [0, 1]$ by $\mu(\sum_{j \in K} T_j^{(n)}) = 2^{-n} \#(K)$, where $\#(K)$ is the cardinality of the set K. Clearly μ is additive on $\mathcal{V}$ and satisfies $P \preceq Q \Leftrightarrow \mu(P) \leq \mu(Q)$ for P, $Q \in \mathcal{V}$. Now suppose the sequence P_j in $\mathcal{V}$ $\preceq$ converges to $P \in \mathcal{V}$, with $P_j \preceq P (P \preceq P_j)$. Then we may assume, by the nesting property, that $P_j \leq P (P \leq P_j)$ for all j. Hence $P - P_j \overset{\preceq}{\Rightarrow} 0 (P_j - P \overset{\preceq}{\Rightarrow} 0)$. Since for any sequence Q_j in $\mathcal{V}$, we have $Q_j \overset{\preceq}{\Rightarrow} 0 \Leftrightarrow \mu(Q_j) \to 0$, additivity of μ on $\mathcal{V}$ yields $\mu(P_j) \to \mu(P)$. This shows that μ is $\preceq$ continuous on $\mathcal{V}$. For arbitrary $P \in \mathcal{P}(\mathcal{A})$, we define $\mu(P)$ to be $\lim_j \mu(P_j)$, where P_j is any sequence in $\mathcal{V}$ such that $P_j \preceq P$ $\forall j$ and $P_j \overset{\preceq}{\Rightarrow} P$. Henceforth, we will regard μ as being defined on all of $\mathcal{P}(\mathcal{A})$. One also shows very easily that $P \preceq Q \Leftrightarrow \mu(P) \leq \mu(Q)$ for $P, Q \in \mathcal{P}(\mathcal{A})$. Now let S and T in $\mathcal{P}(\mathcal{A})$ be mutually orthogonal with $0 \prec S$ and $0 \prec T$. Let S_j be any sequence in $\mathcal{V}$ such that $S_j \preceq S$ and $S_j \overset{\preceq}{\Rightarrow} S$. If $j \in \mathbf{N}$, then $T \preceq S_j^\perp$ by Proposition 2.14; hence there exists $T_j \in \mathcal{V}(S_j^\perp)$ such that $T_j \preceq T$ and such that T_j is in any desired $\preceq$ neighbourhood of T. This implies that we can choose $T_j \stackrel{\preceq}{\to} T$. Corollary 2.15 and additivity of μ on $\mathcal{V}$ then gives $\mu(S) + \mu(T) = \lim_j \mu(S_j) + \lim_j \mu(T_j) = \lim_j\{\mu(S_j) + \mu(T_j)\} = \lim_j \mu(S_j + T_j) = \mu \lim_j(S_j + T_j) = \mu(S + T)$. This is sufficient to show additivity of μ on $\mathcal{P}(\mathcal{A})$. □

3. Conclusion

If $\preceq$ is a CP on $\mathcal{P}(\mathcal{A})$ for some von Neumann algebra $\mathcal{A}$, then the restriction of $\preceq$ to $\mathcal{P}(\mathcal{B})$, where $\mathcal{B}$ is a subspace of $\mathcal{A}$, is also a CP. Furthermore, weak or uniform continuity of $\preceq$ is automatically inherited on $\mathcal{P}(\mathcal{B})$. Hence type decomposition of von Neumann algebras leads to the final results below (incorporating Theorem 3.14 of [11]):

Theorem 3.1. *(a) Let $\mathcal{A}$ be a von Neumann algebra without a finite direct summand, and let $\preceq$ be a comparative probability on $\mathcal{P}(\mathcal{A})$. Then $\preceq$ is implemented by a state if and only if $\preceq$ is uniformly continuous.*

(b) Let $\mathcal{A}$ be a von Neumann algebra without a direct summary of type I_n, $n < \infty$, and let $\preceq$ be a comparative probability on $\mathcal{P}(\mathcal{A})$. Then $\preceq$ is implemented by a normal state if and only if $\preceq$ is weakly continuous. □

In our search for implementability based on the continuity condition, finiteness of $\mathcal{A}$, whether it be in operator algebraic terms or in terms of linear dimension, seems to be the main obstacle. Thus, where we have the latter form of finiteness, and hence both, we have no result at all. Where we have only the former, i.e., the type II_1 case, we have a result only if we insist on weak continuity as the counterexample in Section 1 shows. From the proofs, it is clear that the difficulty arising from this finiteness is the loss of connectedness of $\mathcal{P}(\mathcal{A})$. Thus in the worst case we have no connectedness of $\mathcal{P}(\mathcal{A})$ in any "reasonable" topology, whereas in the type II_1 case we have weak but not uniform connectedness.

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References

1. H. Araki, M-S. B. Smith, and L. Smith, *On the homotopical significance of the type of von Neumann algebra factors*, Comm. Math. Phys. **22** (1971), 71–88. MR **44:**5783
2. B. Blackadar, *K-theory for operator algebras*, Springer-Verlag, Berlin and New York, 1986. MR **88g:**46082
3. A. M. Gleason, *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–893. MR **20:**2609
4. J. Gunson, *Physical states on quantum logics*. I, Ann. Inst. H. Poincaré, **17** (1972), 295–311. MR **49:**1139
5. S. Goldstein and A. Paszkiewicz, *Comparison of states and Darboux-type properties in von Neumann algebras*, Math. Scand. **63** (1988), 220–232. MR **91d:**46079
6. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Academic Press, New York, 1986. MR **88d:**46106
7. G. Kalmbach, *Measures and Hilbert lattices*, World Scientific, Singapore, 1986. MR **88a:**06013
8. J. L. Kelly, *General topology*, Van Nostrand, Princeton, NJ, 1955. MR **16:**1136C
9. S. Maeda, *Probability measures on projections in von Neumann algebras*, Rev. Math. Phys., **1** (1990), 235–290. MR **92m:**46100

10. S. A. Mutangadura, *Implementation of comparative probability by normal states*, Comm. Math. Phys. **132** (1990), 581–592. MR **91k:**46073
11. ——, *Implementation of comparative probability by states*, Publ. Res. Inst. Math. Sci., **28** (1992), 315–327. MR **93c:**46117
12. ——, *Comparative probability on the C^* algebra of compact operators*, Quaestiones Math. **18** (1995), 155–166.
13. W. Ochs , *Gleason measures and quantum comparative probability*, Quantum Probability and Applications II (Heidelberg Proceedings), Springer, Berlin and Heidelberg, 1985, pp. 388–396. MR **87a:**81020
14. M. Takesaki, *Theory of operator algebras* I, Springer, New York, 1979. MR **81e:**46038
15. J. Zemanek, *Idempotents in Banach spaces*, Bull. London Math. Soc. **11** (1979), 177–183. MR **80h:**46073

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