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A PROBLEM ON PRODUCTS OF TOEPLITZ OPERATORS

KUN YU GUO

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Abstract. A natural and interesting problem on classical Hardy space of one complex variable is the following:

Problem: If $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n}=0$, then there exist some i such that $\varphi_i=0$.

In this note, we establish the kernel inclusion theorem for the products of Toeplitz operators. Using this fact, in case $n=5$, we give the above question an affirmative answer.

Let D be the open unit disk in the complex plane and T its boundary. For the definitions and related conceptions of Hardy space $H^2(T,d\theta/2\pi)$ and Toeplitz operators on $H^2(T,d\theta/2\pi)$, refer to [1, Chapters 6 and 7]; for convenience, write $H^2(T,d\theta/2\pi)$ as H^2. To prove the main results in this note, we establish the following lemmas.

Lemma 1. *For* $g\in L^\infty$ *and* $g\neq 0$, *let* E *be a measurable subset of* T *with* $0<\operatorname{meas}E<2\pi, g|_E=0$. *Then* $\ker T_g=\{0\}$.

The proof of Lemma 1 is trivial.

Lemma 2. *For* $f,g\in L^\infty$ *and* $f,g\neq 0$, *let* E *be a measurable subset of* T *with* $0<\operatorname{meas}E<2\pi$ *and* $g|_E=0$. *If* $\ker T_fT_g\neq\{0\}$, *then there exists a function* $\psi(\neq 0)$ *such that*

$$T_gh=\psi h \quad \forall h\in\ker T_fT_g.$$

Proof. For $h_0\in\ker T_fT_g, h_0\neq 0$, by Lemma 1, there exist $h_1\in H^2$ and $h_1\neq 0$ such that $fT_gh_0=\overline{h_1}$. So, $T_gh_0=\frac{\overline{h_1}}{f}$. It follows that there is $h_2\in H^2$ such that $gh_0=\frac{\overline{h_1}}{f}-\overline{h_2}$. Hence $\frac{1}{f}|_E=(\frac{\overline{h_2}}{\overline{h_1}})|_E$. Similarly for $h_0'\in\ker T_fT_g$ and $h_0'\neq h_0$, there exist $h_1',h_2'\in H^2$ such that $\frac{1}{f}|_E=(\frac{\overline{h_2'}}{\overline{h_1'}})|_E$. Therefore

$$\left(\overline{\frac{h_2}{h_1}}\right)\bigg|_E=\left(\overline{\frac{h_2'}{h_1'}}\right)\bigg|_E$$

i.e., $(h_2h_1'-h_2'h_1)|_E=0$. Since $h_2h_1'-h_2'h_1\in H^1$ and $0<\operatorname{meas}E<2\pi$, we get that $h_2h_1'=h_2'h_1$. Put $h'=\frac{h_2}{h_1}=\frac{h_2'}{h_1'}$; according to the preceding reasoning, we

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obtain

$$\begin{aligned} gh_0 &= \frac{\overline{h_1}}{f} - \overline{h_2} = \frac{\overline{h_1}}{f} - \overline{h_1 h'} = \overline{h_1}\left(\frac{1}{f} - \overline{h'}\right) \\ &= f\left(\frac{1}{f} - \overline{h'}\right) T_g h_0 = (1 - f\overline{h'}) T_g h_0. \end{aligned}$$

Similarly, we have

$$gh_0' = (1 - f\overline{h'}) T_g h_0'.$$

Since $\frac{gh_0}{gh_0'}|_{T\setminus E} = \frac{h_0}{h_0'}|_{T\setminus E}$, the above argument leads to the following:

$$\frac{h_0}{h_0'} = \frac{T_g h_0}{T_g h_0'},$$

i.e., $\frac{T_g h_0}{h_0} = \frac{T_g h_0'}{h_0'}$. Set $\psi = \frac{T_g h_0}{h_0}$; clearly such a ψ is independent on the choice of h_0 in the $\ker T_f T_g$. Thus the following is true:

$$T_g h = \psi h \quad \forall h \in \ker T_f T_g.$$

This completes the proof of Lemma 1. $\square$

Next, we establish the kernel inclusion theorem, which is interesting in itself.

Theorem 1. *Under the same assumption as in Lemma* 2*, there exist* $\varphi \in L^\infty$ *and* $\varphi \neq 0$ *such that*

$$\ker T_f T_g \subset \ker T_\varphi.$$

Proof. First claim $\ker T_f T_g \otimes \overline{H^2} \neq L^1$, where $\ker T_f T_g \otimes \overline{H^2}$ denotes the L^1-closure of linear manifold $\{$finite sum $\sum \lambda_i h_i \overline{h_i'} \colon \forall h_i \in \ker T_f T_g, \forall h_i' \in H^2\}$. If not, by Lemma 2 there exists a function $\psi (\neq 0)$ such that for any $h \in \ker T_f T_g$ we have $T_g h = \psi h$. Since

$$\begin{aligned} \int g\left(\sum \lambda_i h_i \overline{h_i'}\right) d\theta &= \sum \lambda_i \langle T_g h_i, h_i' \rangle = \sum \lambda_i \langle \psi h_i, h_i' \rangle \\ &= \int \psi \left(\sum \lambda_i h_i \overline{h_i'}\right) d\theta, \end{aligned}$$

it follows that

$$\left| \int \psi \left(\sum \lambda_i h_i \overline{h_i'}\right) d\theta \right| \leq \|g\|_\infty \left\| \sum \lambda_i h_i \overline{h_i'} \right\|_{L^1}.$$

Because the above inequality is true on the dense subset $\ker T_f T_g \otimes \overline{H}^2$ of L^1, M_ψ, determines a bounded linear functional on L^1. From this, we obtain a $\psi \in L^\infty$ and $\psi = g$. By Lemma 1, $|\psi| > 0$ a.e. This contradicts $g|_E = 0$. It follows that $\ker T_f T_g \otimes \overline{H^2} \neq L^1$. We prove the claim.

According to the claim there exists a bounded linear functional which annihilates $\ker T_f T_g \otimes \overline{H^2}$, i.e., there exist $\varphi \in L^\infty$ and $\varphi \neq 0$ such that for any $h_0 \in \ker T_f T_g$, $h \in H^2$, we have

$$\int \varphi h_0 \overline{h} d\theta = \langle T_\varphi h_0, h \rangle = 0.$$

Hence $h_0 \in \ker T_\varphi$, and it follows that $\ker T_f T_g \subset \ker T_\varphi$. This completes the proof of Theorem 1. $\square$

To prove Theorem 2, we need the following lemma.

Lemma 3 ([1, p. 179])**.** *If* $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n}=0$, *then* $\varphi_1\varphi_2\cdots\varphi_n=0$.

Theorem 2. *If* $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}=0$, *then there exist some* i *such that* $\varphi_i=0$.

Proof. Suppose $\ker T_{\varphi_1}\neq 0$ and $\ker T_{\overline{\varphi_5}}\neq 0$ without loss of generality. Then clearly $|\varphi_1\varphi_5|>0$ a.e. By Lemma 3 we have

$$\varphi_2\varphi_3\varphi_4=0.$$

Assume $\varphi_3\neq 0$. It follows that there exists a measurable subset E of T with $0<\operatorname{meas}E<2\pi$ such that

$$\varphi_2|_E=0 \quad\text{or}\quad \varphi_4|_E=0.$$

In case $\varphi_2|_E=0$, suppose $T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}\neq 0$; hence $\ker T_{\varphi_1}T_{\varphi_2}\neq\{0\}$. From Theorem 1 and Rang $T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}\subset\ker T_{\varphi_1}T_{\varphi_2}$, we know that there exists a $\varphi\in L^\infty$, $\varphi\neq 0$, such that

$$(*)\qquad T_\varphi T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}=0.$$

Since $|\varphi|>0$, $|\varphi_5|>0$ a.e., by Lemma 3 $\varphi_3\varphi_4=0$. Thus there exists a measurable subset E' of T with $0<\operatorname{meas}E'<2\pi$ such that $\varphi_4|_{E'}=0$. By $(*)$, we have

$$T_{\overline{\varphi_5}}T_{\overline{\varphi_4}}T_{\overline{\varphi_3}}T_{\overline{\varphi}}=0.$$

Since $T_{\overline{\varphi_3}}T_{\overline{\varphi}}\neq 0$, by Theorem 1 and Rang $T_{\overline{\varphi_3}}T_{\overline{\varphi}}\subset\ker T_{\overline{\varphi_5}}T_{\overline{\varphi_4}}$, there exists a $\varphi'\in L^\infty, |\varphi'|>0$ a.e., such that

$$T_{\varphi'}T_{\overline{\varphi_3}}T_{\overline{\varphi}}=0.$$

Again by Lemma 3, $\varphi_3=0$. This contradicts the assumption; thus we obtain

$$\varphi_2=0 \quad\text{or}\quad \varphi_4=0.$$

Let $\varphi_4|_E=0$, and assume $\varphi_3\neq 0$. Similarly, we have

$$\varphi_2=0 \quad\text{or}\quad \varphi_4=0.$$

Therefore if $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}=0$, then there exists some i such that $\varphi_i=0$. This completes the proof of Theorem 2. □

Using the approach in this note, we may prove the following

Theorem 3. *If* $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}$ *has finite rank, then there exists some* i *such that* $\varphi_i=0$.

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DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENG DU, 610064, PEOPLE'S REPUBLIC OF CHINA

Current address: Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China