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NON-NORMAL, STANDARD SUBGROUPS OF THE BIANCHI GROUPS

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ABSTRACT. Let S be a subgroup of $SL_n(K)$, where K is a Dedekind ring, and let $\mathbf{q}$ be the K-ideal generated by $x_{ij}, x_{ii} - x_{jj}$ $(i \neq j)$, where $(x_{ij}) \in S$. The subgroup S is called *standard* iff S contains the normal subgroup of $SL_n(K)$ generated by the $\mathbf{q}$-elementary matrices. It is known that, when $n \geq 3$, S is standard iff S is normal in $SL_n(K)$. It is also known that every standard subgroup of $SL_2(K)$ is normal in $SL_2(K)$ when K is an arithmetic Dedekind domain with infinitely many units.

The ring of integers of an imaginary quadratic number field, $\mathcal{O}$, is one example (of three) of such an arithmetic domain with finitely many units. In this paper it is proved that every *Bianchi group* $SL_2(\mathcal{O})$ has uncountably many non-normal, standard subgroups. This result is already known for related groups like $SL_2(\mathbb{Z})$.

INTRODUCTION

Let R be a commutative ring, and let $\mathbf{q}$ be an R-ideal. We put $SL_n(R, \mathbf{q}) = \operatorname{Ker}(SL_n(R) \to SL_n(R/\mathbf{q}))$, where $n \geq 2$. Let $U_n(R, \mathbf{q})$ (resp. $NE_n(R, \mathbf{q})$) be the subgroup (resp. normal subgroup) of $SL_n(R)$ generated by the unipotent (resp. elementary) matrices in $SL_n(R, \mathbf{q})$. It is clear that $NE_n(R, \mathbf{q})$ is a subgroup of $U_n(R, \mathbf{q})$. We put $U_n(R, R) = U_n(R)$ and $NE_n(R, R) = NE_n(R)$. By definition we have $SL_n(R, R) = SL_n(R)$.

Let $\mathcal{O}$ $(= \mathcal{O}_d)$ be the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, where $\mathbb{Q}$ is the set of rational numbers and d is a square-free positive integer. In this paper we prove the following result for the *Bianchi groups*, $SL_2(\mathcal{O})$.

Theorem. *For infinitely many $\mathcal{O}$-ideals $\mathbf{q}$, $SL_2(\mathcal{O}, \mathbf{q})/U_2(\mathcal{O}, \mathbf{q})$ (and hence $SL_2(\mathcal{O}, \mathbf{q})/NE_2(\mathcal{O}, \mathbf{q})$) has a free, non-cyclic quotient.*

Our proof is based on the fundamental paper [16] of Zimmert. Central to Zimmert's approach is a special finite subset of $\mathbb{N}$, the set of positive integers, determined by $\mathcal{O}$, which is now usually referred to as the *Zimmert set*. In this paper we apply some elementary analytic number theory to the Zimmert set and then make use of a previous result [11, Theorem 4.9] of the first author.

Within the context of the class of linear groups $SL_n(A)$, where A is a *Dedekind ring of arithmetic type* [1, p. 83], our theorem represents a two-dimensional anomaly. In a celebrated paper Bass, Milnor and Serre [1, Corollary 4.3, p. 95] have proved

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that, when $n \geq 3$, the quotient group $SL_n(A, \mathbf{q})/NE_n(A, \mathbf{q})$ is always finite, cyclic. Liehl [4] and Vaserstein [15] have proved that this is also true for $n = 2$, provided A^*, the set of units of A, is infinite. By a classical theorem of Dirichlet, A^* is finite if and only if (i) $A = \mathcal{O}$, (ii) $A = \mathbb{Z}$, the ring of rational integers, or (iii) $A = \mathcal{C} = \mathcal{C}(C, P, k)$, the coordinate ring of the affine curve obtained by remaining a closed point P from a (suitable) projective curve C over a *finite* field k. (The simplest case of type (iii) is the polynomial ring $k[t]$.) When $A = \mathbb{Z}$ or $\mathcal{C}$, it is known ([9, Lemma 3.5], [7, Corollary 2.6]) that $SL_2(A, \mathbf{q})/U_2(A, \mathbf{q})$ has a free, non-cyclic quotient, for all but finitely many $\mathbf{q}$. (Note that, as $\mathbb{Z}$ is a principal ideal domain, $NE_2(\mathbb{Z}, \mathbf{q}) = U_2(\mathbb{Z}, \mathbf{q})$, for all $\mathbf{q}$.)

An immediate consequence of our theorem represents another two-dimensional anomaly. A subgroup S of $SL_n(R)$ is called *standard* if and only if $NE_n(R, \mathbf{q}_0) \leq S$, where $\mathbf{q}_0$ is the R-ideal generated by $x_{ij}, x_{ii} - x_{jj}$ $(i \neq j)$, for all $(x_{ij}) \in S$. Let K be a Dedekind ring (or, more generally, a Noetherian domain of Krull dimension 1). Then, by [1, Theorems 7.4, 7.5(e), p. 106] and [5, Theorem 3.2], when $n \geq 3$, a subgroup S of $SL_n(K)$ is standard if and only if S is normal in $SL_n(K)$. It is also known [6, Corollary 1.3] that every standard subgroup of $SL_2(A)$ is normal, provided A^* is infinite. The following result shows that, when A^* is finite, this situation is completely different.

Corollary. *$SL_2(\mathcal{O})$ has uncountably many non-normal, standard subgroups.*

This result is already known for $A = \mathcal{C}$ [7, Theorem 3.2] and for $A = \mathbb{Z}$ [8, Theorem 3].

A weaker version (in the other direction) of the above classification theorem of normal subgroups of $SL_n(K)$ says that, when $n \geq 3$, every non-central normal subgroup of $SL_n(K)$ contains $NE_n(K, \mathbf{q}')$, for some *non-zero* K-ideal, $\mathbf{q}'$. A non-central, normal subgroup N of $SL_2(K)$ is called a *normal subgroup of level zero* if and only if $NE_2(K, \mathbf{q}) \leq N$ *only* when $\mathbf{q} = \{0\}$. Serre [14, Proposition 2, p. 492] has proved that, when A^* is infinite, $SL_2(A)$ has no normal subgroups of level zero. Again the situation is completely different when A^* is finite. When $A = \mathbb{Z}, \mathcal{O}$ or $\mathcal{C}$, it is known that $SL_2(A)$ has uncountably many normal subgroups of level zero. (See [8, Theorem 1], [10, Theorem 4], [7, Theorem 3.1].)

1. Zimmert sets

Let D be the discriminant of $\mathbb{Q}(\sqrt{-d})$. It is well known that $D = -4d$, unless $d \equiv 3 \pmod 4$, in which case $D = -d$. Let

$$\omega = \begin{cases} \sqrt{-d}, & d \equiv 1, 2 \pmod 4, \\ (1 + \sqrt{-d})/2, & d \equiv 3 \pmod 4. \end{cases}$$

For each $m \in \mathbb{N}$ let $\mathcal{O}_m$ be the order of index m in $\mathcal{O}$. It is known that

$$\mathcal{O}_m = \mathbb{Z} + m\omega\mathbb{Z}.$$

(By definition, $\mathcal{O}_1 = \mathcal{O}$.)

Definition. For each d, m the *Zimmert set* $Z(d, m)$ is the set of all $n \in \mathbb{N}$ such that

(i) $4n^2 \leq m^2|D| - 3$.
(ii) $(|a + m\omega|^2, n) = 1$, for all $a \in \mathbb{Z}$.
(iii) $n \neq 2$.

(It is clear that $Z(d,m) \neq \varnothing$ iff $m^2|D| \geq 7$.) The original definition (for the case $m = 1$ only) is due to Zimmert [16]. The extended definition is due to Grunewald and Schwermer [3]. The principal purpose of Zimmert's paper [16, Satz 1(i)] is to prove that $SL_2(\mathcal{O})$ has a free quotient of rank $|Z(d,1)|$ and this result is extended [3] (without too much difficulty) to $SL_2(\mathcal{O}_m)$ by Grunewald and Schwermer.

Notation. Let $r(d,m) = |Z(d,m)|$.

Theorem 1. *For each d there exists $m_0 = m_0(d)$ such that, when $m > m_0$,*

$$r(d,m) > \frac{9}{20}\frac{M}{\ln M} - \frac{\ln m}{\ln 2},$$

where $M = \frac{1}{2}\sqrt{(m^2|D| - 3)}$.

Proof. For each odd prime p and integer a, let

$$\left(\frac{a}{p}\right)$$

denote the Legendre symbol.

Let $Z^*(d,m)$ be the set of odd primes p such that

(i) $4p^2 \leq m^2|D| - 3$.
(ii) $(p, dm) = 1$.
(iii) $(\frac{-d}{p}) = -1$.

It is clear that $Z^*(d,m) \subseteq Z(d,m)$. Let $r^* = |Z^*(d,m)|$. Now

$$d = 2^\varepsilon q_1 \cdots q_r,$$

where $\varepsilon = 0$ or 1 and $q_1, \ldots, q_r$ are distinct odd primes. Let

$$N = 8q_1 \cdots q_r.$$

For any odd prime p, by quadratic reciprocity,

$$\begin{aligned}\left(\frac{-d}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^\varepsilon \prod_{i=1}^{r}\left(\frac{q_i}{p}\right) \\ &= (-1)^{(\frac{p-1}{4})}(-1)^{\varepsilon(\frac{p^2-1}{8})} \prod_{i=1}^{r}\left(\frac{p}{q_i}\right)^{(\frac{p-1}{2})(\frac{q_i-1}{2})}.\end{aligned}$$

It follows that, if p' is any prime for which $p \equiv p' \pmod{N}$, then

$$\left(\frac{-d}{p'}\right) = \left(\frac{-d}{p}\right).$$

The condition $(\frac{-d}{p}) = -1$ is therefore equivalent to p lying in one of $\frac{1}{2}\phi(N)$ arithmetic progressions $(\operatorname{mod} N)$, where ϕ is the Euler function.

For each $y \in \mathbb{Z}$, where $(y, N) = 1$, let $\pi(y, N; M)$ be the number of primes p such that $p \leq M$ and $p \equiv y \pmod{M}$. By the prime number theorem for arithmetic progressions,

$$\pi(y, N; M) = \frac{1}{\phi(N)} \cdot \frac{M}{\ln M} + o\left(\frac{M}{\ln M}\right).$$

(For this formulation of the theorem see, for example, [2, Theorem 3.5.1, p. 72].)

Now let $\pi(N;M)$ be the number of primes p such that $p \leq M$ and $(p, N) = 1$. It follows that there exists $m_0 = m_0(d)$ such that, when $m > m_0$,

$$\pi(N;M) > \frac{9}{10} \cdot \frac{1}{2} \cdot \frac{M}{\ln M}.$$

We deduce that, when $m > m_0$,

$$r^* > \frac{9}{20} \frac{M}{\ln M} - \omega(m),$$

where $\omega(m)$ is the number of primes dividing m. It is clear that

$$2^{\omega(m)} \leq m.$$

The result follows. □

The following conclusion is immediate.

Corollary 2. *For each d,*

$$r(d,m) \to \infty, \quad \text{as } m \to \infty.$$

In [12, Corollary 5] it is proved that $r(d,1) \to \infty$, as $d \to \infty$. (This is a more difficult problem.)

2. Free quotients

Let F_t be the free group of rank t, where $t \geq 0$.

Theorem 3. *Given t, d, there exists an epimorphism*

$$\rho : SL_2(\mathcal{O}_m)/U_2(\mathcal{O}_m) \to F_t,$$

for all but finitely many m.

Proof. It is known [11, Theorem 4.1] that there exists an epimorphism

$$\rho : SL_2(\mathcal{O}_m)/U_2(\mathcal{O}_m) \to F_s,$$

where $s = r(d,m) - 1$. The result follows from Corollary 2. □

When $r(d,1) \geq 3$, it follows that $SL_2(\mathcal{O})/U_2(\mathcal{O})$ and hence $SL_2(\mathcal{O}_m)/U_2(\mathcal{O}_m)$, for all m, have a free, non-cyclic quotient. By [12, Corollary 5] it is known that $r(d,1) \geq 3$, for all but finitely many d. (The smallest such value is $d = 67$.)

This does not (in general) happen when $r(d,1) < 3$. For example, when $d = 1, 2, 3, 7, 11$, it is a classical result that $\mathcal{O}$ is a Euclidean ring. For these cases $SL_2(\mathcal{O}) = U_2(\mathcal{O})$ $(= E_2(\mathcal{O}))$, i.e. $SL_2(\mathcal{O})$ is generated by elementary matrices.

Corollary 4. *Given t, d, there exists an epimorphism*

$$\sigma : SL_2(\mathcal{O}, \mathbf{q})/U_2(\mathcal{O}, \mathbf{q}) \to F_t$$

for infinitely many $\mathcal{O}$-ideals $\mathbf{q}$.

Proof. Fix m as in Theorem 3 and let $\mathbf{q}_0$ be the *conductor* of $\mathcal{O}_m$ in $\mathcal{O}$, i.e. the largest (non-zero) $\mathcal{O}$-ideal contained in $\mathcal{O}_m$. Let $\mathbf{q}$ be any non-zero $\mathcal{O}$-ideal contained in $\mathbf{q}_0$. From Theorem 3 $SL_2(\mathcal{O}_m)$ has a normal subgroup N, containing $U_2(\mathcal{O}_m)$, such that

$$SL_2(\mathcal{O}_m)/N \cong F_t.$$

Now $\mathcal{O}_m$ is a Noetherian domain of Krull dimension 1 and so

$$NE_2(\mathcal{O}_m) \cdot SL_2(\mathcal{O}_m, \mathbf{q}) = SL_2(\mathcal{O}_m),$$

by [5, Theorem 3.1]. It follows that $N \cdot SL_2(\mathcal{O}_m, \mathbf{q}) = SL_2(\mathcal{O}_m)$ and hence that

$$SL_2(\mathcal{O}_m, \mathbf{q})/SL_2(\mathcal{O}_m, \mathbf{q}) \cap N \cong F_t.$$

We note that $U_2(\mathcal{O}_m, \mathbf{q}) \leq N, SL_2(\mathcal{O}_m, \mathbf{q}) = SL_2(\mathcal{O}, \mathbf{q})$ and $U_2(\mathcal{O}_m, \mathbf{q}) = U_2(\mathcal{O}, \mathbf{q})$. □

Corollary 5. *$SL_2(\mathcal{O})$ has $2^{\aleph_0}$ non-normal, standard subgroups.*

Proof. Fix $\mathcal{O}, \mathbf{q}$ as in Corollary 4 with $t \geq 2$. Let S be any subgroup of $SL_2(\mathcal{O})$ such that

$$U_2(\mathcal{O}, \mathbf{q}) \leq S \leq SL_2(\mathcal{O}, \mathbf{q}).$$

It is clear that the $\mathcal{O}$-ideal generated by $x_{ij}, x_{ii} - x_{jj}$ $(i \neq j)$, where $(x_{ij}) \in S$, is $\mathbf{q}$, i.e. S is standard. In the proof of [7, Theorem 3.2] it is shown that a free group of rank at least 2 contains $2^{\aleph_0}$ non-normal subgroups. The result follows. □

3. Concluding remarks

By [12, Corollary 5] a stronger version of Corollary 4 holds for all but finitely many $\mathcal{O}$. When $r(d, 1) \geq 3$, it is known [11, Theorem 4.1] that $SL_2(\mathcal{O})/U_2(\mathcal{O})$ has a free, non-cyclic quotient. This is also then true for $SL_2(\mathcal{O}, \mathbf{q})/U_2(\mathcal{O}, \mathbf{q})$, where $\mathbf{q}$ is *any non-zero* $\mathcal{O}$-ideal.

For the remaining $\mathcal{O}$ however infinitely many $\mathbf{q}$ in general are excluded from the statement of Corollary 4. Consider for example the *Picard group*, $SL_2(\mathbb{Z}[i])$, where $i^2 = -1$. (In this case $d = 1$ and $\omega = i$.) Let p be any rational prime, where $p \equiv 1 \pmod 4$, and let α be a quadratic residue $\pmod p$.

The $\mathbb{Z}$-module,

$$\mathbb{Z}(1 + \alpha i) + \mathbb{Z}pi,$$

is a prime $\mathbb{Z}[i]$-ideal $\mathbf{p}$, say, and it is clear that the *only* order in $\mathbb{Z}[i]$ containing $\mathbf{p}$ is $\mathbb{Z}[i]$ itself. The proof of Corollary 4 tells us nothing about the structure of $SL_2(\mathbb{Z}[i], \mathbf{p})/U_2(\mathbb{Z}[i], \mathbf{p})$, since $SL_2(\mathbb{Z}[i]) = U_2(\mathbb{Z}[i])$. (We recall that $\mathbb{Z}[i]$ is a Euclidean ring.)

It is likely however that restrictions of this type are merely a consequence of the method of proof. Accordingly we conclude with the following.

Conjecture. *$SL_2(\mathcal{O}, \mathbf{q})/U_2(\mathcal{O}, \mathbf{q})$ has a free, non-cyclic quotient, for all but finitely many $\mathcal{O}$-ideals $\mathbf{q}$.*

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