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# CHAINS OF STRONGLY NON-REFLEXIVE DUAL GROUPS OF INTEGER-VALUED CONTINUOUS FUNCTIONS

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*Dedicated to Professor Shōzō Sasada on his* 60*th birthday*

Abstract. Answering a question of Eklof-Mekler (*Almost free modules, set-theoretic methods*, North-Holland, Amsterdam, 1990), we prove: (1) If there exists a non-reflecting stationary set of $\omega_i$ consisting of ordinals of cofinality $\omega$ for each $1 < i < \omega$, then there exist abelian groups $A_n (n \in \mathbb{Z})$ such that $A_n^* \cong A_{n+1}$ and $A_n \ncong A_{n+2}$ for each $n \in \mathbb{Z}$. (2) There exist abelian groups $A_n (n \in \mathbb{Z})$ such that $A_n^* \cong A_{n+1}$ for each $n \in \mathbb{Z}$ and $A_n \ncong A_{n+2}$ for each $n < 0$. The groups $A_n$ are the groups of $\mathbb{Z}$-valued continuous functions on a topological space and their dual groups.

## 1. Introduction

For an abelian group $A$, let $A^*$ denote the $\mathbb{Z}$-dual group of $A$, i.e., $A^* = \operatorname{Hom}(A, \mathbb{Z})$, where $\mathbb{Z}$ is the group of integers. An abelian group $A$ is said to be *reflexive* if the canonical homomorphism $A \to A^{**}$ is an isomorphism, and $A$ is said to be *strongly non-reflexive* if $A$ is not isomorphic to $A^{**}$. In [4, p.455], Eklof and Mekler asked: Is there a $\mathbb{Z}$-chain of strongly non-reflexive dual groups, i.e., abelian groups $A_n (n \in \mathbb{Z})$ such that for all $n$, $A_n^* = A_{n+1}$ and $A_n$ is not isomorphic to $A_{n+2}$? The purpose of this note is to give a consistency answer and a partial answer to this question by using the abelian groups $C(X, \mathbb{Z})$ of continuous functions from a topological space $X$ to the discrete group $\mathbb{Z}$. For a regular, uncountable cardinal $\kappa$, let $E(\kappa)$ be the following statement: There exists a non-reflecting stationary set of $\kappa$ consisting of ordinals of cofinality $\omega$. Here, a subset $E$ of $\kappa$ is *non-reflecting* if $E \cap \lambda$ is not stationary in $\lambda$ for each ordinal $\lambda < \kappa$ with cofinality$> \omega$. Obviously, $E(\omega_1)$ is true. It is known (cf. [4]) that under $V = L$, $E(\kappa)$ holds for any regular, not weakly compact cardinal $\kappa$. In section 3, we prove:

**Theorem 1.** *If $E(\omega_i)$ holds for each $1 < i < \omega$, then there exist topological spaces $X_n (n \in \mathbb{Z})$ such that, if $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$ for each $n \in \mathbb{Z}$, then $A_n^* \cong A_{n+1}$ and $A_n \ncong A_{n+2}$ for each $n \in \mathbb{Z}$.*

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**Theorem 2.** *There exist topological spaces $X_n (n \in \mathbb{Z})$ such that, if $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A^*_{2n}$ for each $n \in \mathbb{Z}$, then $A^*_n \cong A_{n+1}$ for each $n \in \mathbb{Z}$ and $A_n \ncong A_{n+2}$ for each $n < 0$.*

Strongly non-reflexive dual groups were constructed by several authors ([3], [4], [5] and [6]) and, in particular, Sageev-Shelah and Eklof-Mekler-Shelah [5] constructed an $\omega$-chain of strongly non-reflexive dual groups assuming $\diamondsuit_{\omega_1}$. The history of non-reflexive dual groups can be found in [4] and [5].

Throughout the paper, by a group we mean an abelian group and by a space we mean a 0-dimensional $T_0$-space, i.e., a topological space with a base $\mathcal{B}$, consisting of clopen (= closed and open) sets, such that $\bigcap\{U : x \in U \in \mathcal{B}\} = \{x\}$ for each $x \in X$. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of spaces. Take a copy $X'_\alpha$ of $X_\alpha$ for each $\alpha$ so that $X'_\alpha \cap X'_\beta = \emptyset$ if $\alpha \neq \beta$. Let $X = \bigcup_{\alpha \in A} X'_\alpha$ and define a topology for $X$ by declaring a subset $U \subset X$ to be open if and only if $U \cap X'_\alpha$ is open in $X'_\alpha$ for each $\alpha \in A$. The space $X$ with this topology is called the *discrete sum* of $\{X_\alpha\}_{\alpha \in A}$, and is denoted by $\bigsqcup_{\alpha \in A} X_\alpha$, or by $X_1 \sqcup X_2$ if $A = \{1, 2\}$. The cardinality of a set $A$ is denoted by $|A|$. For ordinals $\lambda$ and $\mu$ with $\lambda < \mu$, we write $(\lambda, \mu] = \{\alpha : \lambda < \alpha \leq \mu\}$ and $[\lambda, \mu) = \{\alpha : \lambda \leq \alpha < \mu\}$; as usual, we also write $\mu$ for $[0, \mu)$. Other terminology and notation will be used as in [4].

## 2. Preliminaries

We state two lemmas, which play key roles in our proof of Theorems 1 and 2, together with some definitions. Except for Lemma 2 below, each definition and result stated here appears in [2] or [4]. A space $X$ is called *$\mathbb{Z}$-compact* if it is homeomorphic to a closed subspace of a product $\mathbb{Z}^\kappa$ for some cardinal $\kappa$. It is known that each space $X$ can be densely embedded in a unique $\mathbb{Z}$-compact space $\beta_{\mathbb{Z}} X$ to which each $\mathbb{Z}$-valued continuous function on $X$ admits a continuous extension. By the property of $\beta_{\mathbb{Z}} X$, the groups $C(X, \mathbb{Z})$ and $C(\beta_{\mathbb{Z}} X, \mathbb{Z})$ are isomorphic. A $\mathbb{Z}$-valued function on a space $X$ is said to be *$k$-continuous* if the restriction $f|K$ is continuous for each compact set $K$ of $X$. Let $k_{\mathbb{Z}} X$ be the set $X$ equipped with the smallest topology so that every $\mathbb{Z}$-valued $k$-continuous function on $X$ is continuous.

**Lemma 1** ([1, Lemma 4.4]). *If $X$ is a $\mathbb{Z}$-compact space and no compact set of $X$ is of measurable cardinality, then $C(X, \mathbb{Z})^{**}$ is isomorphic to $C(k_{\mathbb{Z}} X, \mathbb{Z})$.*

For a space $X$, let $w(X) = \omega \cdot \min\{|\mathcal{B}| : \mathcal{B}$ is a base of $X\}$.

**Lemma 2** ([3, Theorem 3.1]). *Let $X$ be a space and $\kappa$ an infinite cardinal. Then there exists a subgroup of $C(X, \mathbb{Z})^*$ isomorphic to $\mathbb{Z}^\kappa$ if and only if there exists a compact set $K$ of $\beta_{\mathbb{Z}} X$ with $w(K) \geq \kappa$.*

A *clopen ultrafilter* on a space $X$ is a collection $p$ of clopen sets in $X$ which has the finite intersection property and has the property that for every clopen set $U$ in $X$, exactly one of $U$ and $X \setminus U$ is in $p$. A clopen ultrafilter $p$ is *$\omega_1$-complete* if $p$ intersects with each countable cover of $X$ consisting of clopen sets, and *$p$ converges* to a point $x$ if $\bigcap\{V : V \in p\} = \{x\}$. A clopen ultrafilter which converges is also called principal. It is known (cf. [2] or [4]) that a space $X$ is $\mathbb{Z}$-compact if and only if every $\omega_1$-complete clopen ultrafilter on $X$ converges. It follows that the discrete sum $\bigsqcup_{\alpha \in A} X_\alpha$ of $\mathbb{Z}$-compact spaces $X_\alpha$ is $\mathbb{Z}$-compact when $|A|$ is non-measurable.

The notion of $\mathbb{Z}$-compact spaces originated with Mrówka (see the references of [2]); many topologists use the term '$N$-compact' instead of '$\mathbb{Z}$-compact'.

## 3. Proof of Theorems 1 and 2

Throughout this section, let $\kappa$ be a regular, uncountable, non-measurable, successor cardinal and $\Omega$ the set of all ordinals less than $\kappa$ which are successors or limits of cofinality $\omega$. For a subset $X$ of $\kappa+1$, we call the subspace topology on $X$ induced from the order topology the *relative order topology.* For a set $X$ with $\kappa \in X \subset \kappa+1$ and a topology $\tau$ on $X$, we call the space $(X,\tau)$ an *adequate space* if it satisfies the following conditions (1)-(4):

(1) $\tau$ is finer than the relative order topology on $X$.
(2) $\{(\lambda,\kappa]\cap X : \lambda<\kappa\}$ is a neighborhood base of the point $\kappa$ in $(X,\tau)$.
(3) Each point of $X\cap\Omega$ has a countable neighborhood base in $(X,\tau)$.
(4) $\beta_{\mathbb{Z}}((X\cap\Omega)\cup\{\kappa\}) = (X,\tau)$, where $(X\cap\Omega)\cup\{\kappa\}$ is a subspace of $(X,\tau)$.

Note that (4) implies that an adequate space is $\mathbb{Z}$-compact. For a space $Y$, define $kw(Y)=\sup\{w(K) : \text{compact } K\subset Y\}$. For spaces $X$ and $Y$, we write $X\approx Y$ if $X$ is homeomorphic to $Y$.

**Lemma 3.** *Assume that $E(\kappa)$ holds and let $E$ be a non-reflecting, stationary and co-stationary set of $\kappa$ with $E\subset\Omega$. Let $X=(X,\tau)$ be an adequate space. Then there exists a topology $\upsilon$ on $E\cup\{\kappa\}$ such that the following holds for the space $Y=(E\cup\{\kappa\},\upsilon)$:*

(a) *$Y$ is an adequate space.*
(b) *$kw(Y)=\omega$.*
(c) *$kw(\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y)=kw(X)$.*
(d) *$\beta_{\mathbb{Z}}k_{\mathbb{Z}}\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \approx \beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}X$.*

*Proof.* Since $E$ can be decomposed into $\kappa$ many disjoint stationary sets, we can write $E=\bigcup\{E_\lambda : \lambda\in(X\cap\Omega)\cup\{\kappa\}\}$, where each $E_\lambda$ is stationary and $E_\lambda\cap E_\mu=\emptyset$ for $\lambda\neq\mu$. Moreover, we can assume that

$$\lambda\notin E_\lambda \text{ for each } \lambda\in X\cap\Omega. \tag{5}$$

Define a map $\varphi : E\to\Omega$ by $\varphi(\alpha)=\lambda$ if $\alpha\in E_\lambda$ for $\lambda\in X\cap\Omega$ and $\varphi(\alpha)=\alpha$ if $\alpha\in E_\kappa$. Put $G=\{\langle\alpha,\varphi(\alpha)\rangle : \alpha\in E\}$, $H=\{\langle\kappa,\lambda\rangle : \lambda\in X\}$ and $S=G\cup H$ $(\subset (E\cup\{\kappa\})\times(\kappa+1))$. Topologize $S$ as follows: A basic neighborhood of $\langle\alpha,\lambda\rangle\in S$ for $\alpha\notin E_\kappa$ is $((\beta,\alpha]\times U)\cap S$ for $\beta<\alpha$ and an open neighborhood $U$ of $\lambda$ in $X$ such that $(\beta,\alpha]\cap U=\emptyset$, and a basic neighborhood of $\langle\alpha,\alpha\rangle\in S$ for $\alpha\in E_\kappa$ is $((\beta,\alpha]\times(\beta,\alpha])\cap S$ for $\beta<\alpha$. By (1), (2) and (5), this defines a topology on $S$. Let $T$ be the quotient space obtained from $S$ by collapsing the set $H$ to the point $\langle\kappa,\kappa\rangle$ and $\psi : S\to T$ the quotient map. Then there is a bijection $h : T\to E\cup\{\kappa\}$ such that $h\circ\psi=\pi$, where $\pi : S\to E\cup\{\kappa\}$ is the projection. Define $\upsilon$ to be the topology on $E\cup\{\kappa\}$ which makes the map $h$ a homeomorphism and put $Y=(E\cup\{\kappa\},\upsilon)$.

To prove (a)-(d), let $\gamma$ denote the relative order topology on $E\cup\{\kappa\}$. We begin by proving (a). By the definition, $\gamma\subset\upsilon$, and hence $Y$ satisfies (1). By using (2) for $X$ and the regularity of $\kappa$, it is easily checked that every open set in $S$ including $H$ includes an open set of the form $((\mu,\kappa]\times(\kappa+1))\cap S$ for some $\mu<\kappa$. This implies that Y satisfies (2). By the fact that $E\subset\Omega$ and (3) for $X$, each point of $G$ has a countable neighborhood base. Hence, $Y$ satisfies (3). Since $E\subset\Omega$, $Y=(Y\cap\Omega)\cup\{\kappa\}$. Thus, if we show that $Y$ is $\mathbb{Z}$-compact, then (4) would be satisfied. To do this and to prove Claim 4 below, let $Y_\mu=Y\cap\mu$ for each $\mu<\kappa$.

*Claim* 1. For each $\mu<\kappa$, $Y_\mu$ is $\mathbb{Z}$-compact.

*Proof.* In case $\kappa = \omega_1$, this is obvious since each $Y_\mu$ is then countable. So we prove the case $\kappa > \omega_1$ by induction. Assume that $\mu < \kappa$ and $Y_\lambda$ is $\mathbb{Z}$-compact for each $\lambda < \mu$. Since a union of a $\mathbb{Z}$-compact space and a singleton is $\mathbb{Z}$-compact, $Y_\mu$ is $\mathbb{Z}$-compact if $\mu$ is a successor. If $\mu$ is a limit, then either cofinality $\mu = \omega$ or $Y_\mu$ is not stationary in $\mu$, because $E$ is non-reflecting. In each case, since $\gamma \subset \upsilon$, we can write $Y_\mu = \bigsqcup_{\xi \in \Xi} A_\xi$, where each $A_\xi$ is bounded in $\mu$ and clopen in $Y$. By our assumption, each $A_\xi$ is $\mathbb{Z}$-compact. Since $\kappa$ is non-measurable, $Y_\mu$ is $\mathbb{Z}$-compact. □

To prove that $Y$ is $\mathbb{Z}$-compact, let $p$ be an $\omega_1$-complete clopen ultrafilter on $Y$. If $p$ does not converge to $\kappa$, then $\kappa \notin U$ for some $U \in p$. Since $Y$ satisfies (2), there is $\mu < \kappa$ such that $U \subset Y_\mu$. By Claim 1, $p$ converges to a point of $U$. Thus $Y$ is $\mathbb{Z}$-compact and hence satisfies (4).

*Claim* 2. Every compact set of $Y$ is countable.

*Proof.* Suppose that $K \subset Y$ is compact and $|K| > \omega$. Then there is a strictly increasing sequence $\{\lambda_\alpha : \alpha < \omega_1\}$ in $K$. Put $\lambda = \sup \lambda_\alpha$. Since $\gamma \subset \upsilon$, $K$ is compact in $\gamma$, and hence, $K \cap \lambda$ is closed and unbounded in $\lambda$ with respect to its relative order topology. If $\kappa = \omega_1$, then $\lambda = \omega_1$ and this contradicts the fact that $E$ is co-stationary. If $\kappa > \omega_1$, this contradicts the fact that $E$ is non-reflecting. □

The statement (b) follows from Claim 2 and (2), (3) for $Y$.

*Claim* 3. $k_{\mathbb{Z}} Y \approx G$ and $k_{\mathbb{Z}} S \approx G \sqcup k_{\mathbb{Z}} X$.

*Proof.* By Claim 2 and (2) for $Y$, the characteristic function of $E$ on $Y$ is $k$-continuous, which implies that $k_{\mathbb{Z}} Y \approx k_{\mathbb{Z}} E \sqcup \{\kappa\}$. Since each point of $E$ has a countable neighborhood base, $k_{\mathbb{Z}} E = E$. Since $E \sqcup \{\kappa\} \approx E \approx G$, $k_{\mathbb{Z}} Y \approx G$. Next, let $K$ be a compact set of $S$. Since the projection $\pi : S \to Y$ is continuous, it follows from Claim 2 that $|\pi[K]| \leq \omega$. This implies that the characteristic function of $G$ on $S$ is $k$-continuous. Hence, $k_{\mathbb{Z}} S \approx k_{\mathbb{Z}} G \sqcup k_{\mathbb{Z}} H$. Since $k_{\mathbb{Z}} G = G$ and $H \approx X$, $k_{\mathbb{Z}} S \approx G \sqcup k_{\mathbb{Z}} X$. □

*Claim* 4. $\beta_{\mathbb{Z}} G = S$.

*Proof.* First, we show that $S$ is $\mathbb{Z}$-compact. Let $p$ be an $\omega_1$-complete clopen ultrafilter on $S$. If there is $U \in p$ with $U \cap H = \emptyset$, then $U \subset \pi^{-1}[Y_\mu]$ for some $\mu < \kappa$. Since $\pi^{-1}[Y_\mu] \approx Y_\mu$, $p$ converges by Claim 1. If $U \cap H \neq \emptyset$ for each $U \in p$, put $q = \{U \cap H : U \in p\}$. Observe that (i) for each clopen set $V$ in $H$, there is a clopen set $W$ in $S$ such that $W \cap H = V$ and that (ii) if $U$ is a clopen set in $S$ with $U \cap H \neq \emptyset$, then there is $\nu < \kappa$ such that $((\nu, \kappa] \times \{\lambda \in X : \langle \kappa, \lambda \rangle \in U\}) \cap S \subset U$. (i) implies that $q$ is a clopen ultrafilter on $H$ and (ii) implies that $q$ is $\omega_1$-complete. Since $H \approx X$ and $X$ is $\mathbb{Z}$-compact, $q$ converges, and hence, $p$ converges to the same point. Thus, $S$ is $\mathbb{Z}$-compact. Next, we show that each $\mathbb{Z}$-valued continuous function $f$ on $G$ can be continuously extended over $S$. For each $\lambda \in (X \cap \Omega) \cup \{\kappa\}$, put $F_\lambda = \{\langle \alpha, \varphi(\alpha) \rangle : \alpha \in E_\lambda\}$. Then the subspace $F_\lambda$ of $G$ is homeomorphic to $E_\lambda$ with the relative order topology. Since $E_\lambda$ is stationary, there is $\beta(\lambda) < \kappa$ such that $f$ takes a constant value $g(\lambda)$ on $\{\langle \alpha, \varphi(\alpha) \rangle \in F_\lambda : \alpha > \beta(\lambda)\}$. We show that $g$ is a continuous function on the subspace $(X \cap \Omega) \cup \{\kappa\}$ of $X$. Applying Forder's lemma to $F_\kappa$, we can find $\nu_0 < \kappa$ such that $f[((\nu_0, \kappa] \times (\nu_0, \kappa]) \cap G] = \{g(\kappa)\}$. Thus, $g(\xi) = g(\kappa)$ for each $\xi \in X \cap \Omega \cap (\nu_0, \kappa]$, which shows that $g$ is continuous at $\kappa$. For $\lambda \in X \cap \Omega$, put $\beta_0 = \sup_{\xi \leq \lambda} \beta(\xi)$ and $F = \bigcap_{\xi < \lambda} \mathrm{cl} E_\xi$, where the closure is taken in $\kappa$ with the order topology. Then $F$ is closed and unbounded, and hence, there is $\beta_1 \in F \cap E_\lambda \cap (\beta_0, \kappa]$. Since $f(\langle \beta_1, \lambda\rangle) = g(\lambda)$, there is a neighborhood $U$ of $\lambda$ in $X$ such that $f[((\beta_0, \beta_1] \times U) \cap G] = \{g(\lambda)\}$. By the condition (1) for $X$, we may assume that $U \subset \lambda + 1$. Since $(\beta_0, \beta_1] \cap E_\xi \neq \emptyset$ for each $\xi \in U \cap \Omega$, $g[U \cap \Omega] = \{g(\lambda)\}$. Hence, $g$ is continuous at $\lambda$. By the condition (4) for $X$, $g$ can be extended to a continuous function $\overline{g}$ on $X$. Extend $f$ over $S$ by letting $f(\langle \kappa, \lambda\rangle) = \overline{g}(\lambda)$ for each $\lambda \in X$. Then, it is easily checked that the extension is continuous. Hence, $\beta_{\mathbb{Z}} G \approx S$. ☐

By Claims 3 and 4, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y \approx \beta_{\mathbb{Z}} G \approx S$. To prove (c), let $K$ be a compact set of $S$. Then, by Claim 2, $|\pi[K]| \leq \omega$, which implies that $K \cap G$ is countable and clopen in $K$. Since each point of $G$ has a countable neighborhood base, $w(K \cap G) \leq \omega$. Since $H \approx X$, $w(K) \leq kw(X)$, and hence, $kw(S) \leq kw(X)$. Conversely, since $X \approx H \subset S$, $kw(X) \leq kw(S)$. Thus, we have (c). Finally, by Claim 3, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} S \approx \beta_{\mathbb{Z}} G \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} X \approx \beta_{\mathbb{Z}} k_{\mathbb{Z}} Y \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} X$. Since $\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y \approx S$, we have (d), which completes the proof of Lemma 3. ☐

**Lemma 4.** *Assume that $E(\kappa)$ holds. Then there exists a collection $\{Y_{\kappa,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$-compact spaces such that*:
(a′) *$\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y_{\kappa,n} \approx Y_{\kappa,n+1}$ for each $n \in \mathbb{Z}$,*
(b′) *$Y_{\kappa,n} = Y_{\kappa,1}$ for each $n \geq 1$,*
(c′) *$kw(Y_{\kappa,n}) = \omega$ for each $n < 0$, and*
(d′) *$kw(Y_{\kappa,n}) = \kappa$ for each $n \geq 0$.*

Before proving Lemma 4, we define some symbols. For a space $X$, let $\bigsqcup_\omega X$ denote the discrete sum of $\omega$ many copies of $X$; in other words, $\bigsqcup_\omega X = X \times \mathbb{Z}$. We denote the space $Y$ constructed from an adequate space $X$ in Lemma 3 by $\Delta(X)$. Since $\Delta(X)$ is an adequate space again, we can define $\Delta(\Delta(X))$. Inductively, we write $\Delta^{i+1}(X) = \Delta(\Delta^i(X))$ for each $i \geq 0$, where $\Delta^0(X) = X$.

*Proof of Lemma* 4. Let $X = \kappa + 1$ with the order topology. Then $X$ is compact, and hence, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} X = X$ and $kw(X) = \kappa$. Since $X$ is an adequate space, we can define the space $\Delta^i(X)$ for each $i \geq 0$. For each $n \in \mathbb{Z}$, define

$$Y_{\kappa,n} = \bigsqcup_{i \geq m(n)} \left( \bigsqcup_\omega \beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta^i(X) \right), \text{ where } m(n) = \max\{1 - n, 0\}.$$

Then $Y_{\kappa,n}$ is $\mathbb{Z}$-compact and $Y_{\kappa,n} = Y_{\kappa,1}$ for each $n \geq 1$; thus (b′) holds. For $n \geq 1$, by (d) in Lemma 3,

$$\begin{aligned}\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y_{\kappa,n} &\approx \bigsqcup_\omega \beta_{\mathbb{Z}} k_{\mathbb{Z}} \beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta^0(X) \\ &\sqcup \left( \bigsqcup_{i \geq 1} \left( \bigsqcup_\omega \left( \beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta^i(X) \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta^{i-1}(X) \right) \right) \right).\end{aligned}$$

Since $\beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta^0(X) = \Delta^0(X)$, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y_{\kappa,n}$ is homeomorphic to $Y_{\kappa,n+1}$. For $n \leq 0$,

$m(n) - 1 = m(n+1)$ and by (d) in Lemma 3 again,

$$\begin{aligned}\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y_{\kappa,n} &\approx \bigsqcup_{i \geq m(n)} \left( \bigsqcup_{\omega} \left( \beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta^i(X) \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta^{i-1}(X) \right) \right) \\ &\approx \bigsqcup_{i \geq m(n+1)} \left( \bigsqcup_{\omega} \beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta^i(X) \right) \approx Y_{\kappa,n+1}.\end{aligned}$$

Hence, we have (a$'$). By (b) and (c) in Lemma 3, $kw(\beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta^i(X)) = kw(\varDelta^{i-1}(X)) = \omega$ for each $i \geq 2$ and $kw(\beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta^i(X)) = kw(X) = \kappa$ for $i = 0, 1$. These imply (c$'$) and (d$'$). ☐

*Proof of Theorem* 1. Assume that $E(\omega_i)$ holds for each $1 < i < \omega$. Note that $E(\omega_1)$ is always true. By Lemma 4, for each $i \geq 1$, there exists a collection $\{Y_{\omega_i,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$-compact spaces satisfying (a$'$)-(d$'$). For each $n \in \mathbb{Z}$, define $X_n = \bigsqcup_{i \geq 1} Y_{\omega_i, n-i}$. Then $X_n$ is $\mathbb{Z}$-compact and by (a$'$),

$$\beta_{\mathbb{Z}} k_{\mathbb{Z}} X_n \approx X_{n+1} \text{ for each } n \in \mathbb{Z}. \tag{6}$$

Moreover, by (c$'$) and (d$'$),

$$kw(X_n) = \omega_n \text{ for each } n \geq 0, \tag{7}$$

where $\omega_0 = \omega$. Put $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$ for each $n \in \mathbb{Z}$. Then, it follows from (6) and Lemma 1 that

$$\begin{aligned}A_{2n+1}^* &= C(X_n, \mathbb{Z})^{**} \\ &\cong C(k_{\mathbb{Z}} X_n, \mathbb{Z}) \\ &\cong C(\beta_{\mathbb{Z}} k_{\mathbb{Z}} X_n, \mathbb{Z}) \\ &\cong C(X_{n+1}, \mathbb{Z}) = A_{2n+2}.\end{aligned}$$

If $n \geq 0$, then by (7) and Lemma 2, no subgroup of $A_{2n+1}$ is isomorphic to $\mathbb{Z}^{\omega_{n+1}}$ but $A_{2n+3}$ has a subgroup isomorphic to $\mathbb{Z}^{\omega_{n+1}}$. Hence, $A_{2n+1} \not\cong A_{2n+3}$ for $n \geq 0$. This means that $A_n$ is strongly non-reflexive for each $n \in \mathbb{Z}$. ☐

*Proof of Theorem* 2. Since $E(\omega_1)$ is always true, there exists a collection $\{Y_{\omega_1,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$-compact spaces satisfying (a$'$)-(d$'$) by Lemma 4. For each $n \in \mathbb{Z}$, define $X_n = Y_{\omega_1,n}$, and put $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$. Then, similarly to the proof of Theorem 1, $A_n^* \cong A_{n+1}$ for each $n \in \mathbb{Z}$. By (c$'$), $kw(X_{-1}) = \omega$ and by (d$'$), $kw(X_0) = \omega_1$. Hence, it follows from Lemma 2 that $A_{-1} \not\cong A_1$. This means that $A_n$ is strongly non-reflexive for each $n < 0$. ☐

*Remark* 1. Let $X_n$ and $A_n (n \in \mathbb{Z})$ be the same as in the proof of Theorem 2. By the proof of Lemma 3, $k_{\mathbb{Z}} X_n \neq X_n$ for each $n \in \mathbb{Z}$. Hence, it follows from [1, Theorem 4.1] that $A_n$ is non-reflexive for all $n \geq 0$ too. However, by (b$'$) in Lemma 4, $A_n \cong A_{n+2}$ for each $n \geq 2$. With some effort, it can also be proved that $A_n \cong A_{n+2}$ for $n = 0, 1$.

*Remark* 2. Our construction yields another strongly non-reflexive dual group $A = C(Y, \mathbb{Z})$. Let $X = \{\omega_1\} \cup \{\alpha < \omega_1 : \alpha \text{ is a successor}\}$ with the relative order topology. Then $X$ is an adequate space. Define $Y = \beta_{\mathbb{Z}} k_{\mathbb{Z}} \varDelta(X)$ and $A = C(Y, \mathbb{Z})$; then $A$ is a dual group by Lemma 1. Since $k_{\mathbb{Z}} \varDelta(X)$ has no partition by $\omega_1$ many nonempty clopen sets, it follows from [3, Theorem 3.3] that $A$ ($\cong C(k_{\mathbb{Z}} \varDelta(X), \mathbb{Z})$) has no subgroup isomorphic to $\mathbb{Z}^{\omega_1}$. On the other hand, by Lemma 1, $A^{**} \cong C(\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y, \mathbb{Z})$ and by (d) in Lemma 3, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y \approx Y \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} X$. Since every compact set of $X$ is finite, $k_{\mathbb{Z}} X$ is a discrete space, and hence, so is $\beta_{\mathbb{Z}} k_{\mathbb{Z}} X$. These mean that there is a summand of $A^{**}$ isomorphic to $\mathbb{Z}^{\omega_1}$. Hence, $A$ is strongly non-reflexive. Assuming $E(\kappa)$, $\omega_1$ can be replaced by $\kappa$.

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