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A FIXED-POINT THEOREM FOR UV^n USCO MAPS

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ABSTRACT. The familiar fixed-point theorem of Kakutani is strengthened by weakening the hypotheses on the set-valued mapping. Applications are made for UV^n and UV^ω decompositions of compact metric spaces.

1. INTRODUCTION

A central position in this paper occupies the following result.

Theorem 1.1. *Let X be a metric space with $\dim(X) \leq n+1$, and Y a compact metric AR, and let $\varphi : X \to \mathcal{F}(Y)$ be u.s.c. such that $\varphi(x)$ is UV^n for all $x \in X$. Then for every continuous $g : Y \to X$ there exists a point $y_0 \in Y$ such that $y_0 \in \varphi(g(y_0))$.*

Here, $\mathcal{F}(Y)$ denotes $\{S \subset Y : S \neq \varnothing,\ S \text{ closed in } Y\}$. A set-valued mapping $\varphi : X \to \mathcal{F}(Y)$ is *upper semi-continuous*, or *u.s.c.*, if $\varphi^{\#}(W) = \{x \in X : \varphi(x) \subset W\}$ is open in X for every open W in Y. A metric space Y is called an AR if it is a retract of every metric space Z containing it as a closed subset. Let $n \geq -1$; a compact metric space A is UV^n provided it embeds in the Hilbert cube Q so that for each neighbourhood U of A in Q there is a smaller one V such that every continuous image of a k-sphere ($k \leq n$) in V is contractible in U.

There are several interesting consequences of the above result. Among them, let us first especially mention the following "dimension type-restriction" version of Kakutani's fixed-point theorem [3], which is so simple that we shall prove it right here.

Theorem 1.2. *Let X be a compact metric AR with $\dim(X) \leq n+1$, and let $\varphi : X \to \mathcal{F}(X)$ be u.s.c. such that $\varphi(x)$ is UV^n for all $x \in X$. Then there is a point $x_0 \in X$ such that $x_0 \in \varphi(x_0)$.*

Proof. Immediately from Theorem 1.1 by taking $Y = X$ and g to be the identity of X. ☐

Before stating our next consequence, there is a little to be said about the UV^n requirement in Theorems 1.1 and 1.2. Denote $\mathbb{B}^{n+1}$ to be the $(n+1)$-ball and $\mathbb{S}^n$

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to be the n-sphere. In [2], Dranishnikov constructed a u.s.c. retraction

$$\varphi : \mathbb{B}^{n+1} \to \mathcal{F}(\mathbb{S}^n) \subset \mathcal{F}(\mathbb{B}^{n+1})$$

($\varphi(x) = \{x\}$ for all $x \in \mathbb{S}^n$) such that $\varphi(x)$ is UV^n for every $x \in \mathbb{B}^{n+1}\backslash\{0\}$ but $\varphi(0)$ is only UV^{n-1}. This is, in fact, a good example, showing that both Theorems 1.1 and 1.2 become false if "$\varphi(x)$ is UV^n" fails for at least one point $x \in X$.

A compact metric space A is UV^ω provided it is UV^n for all $n \geq -1$. Another consequence of Theorem 1.1, which seems also especially interesting, is the following generalization of Kakutani's theorem [3] (when there is no dimensional requirement).

Theorem 1.3. *Let X be a compact metric AR and let $\varphi : X \to \mathcal{F}(X)$ be u.s.c. such that $\varphi(x)$ is UV^ω for all $x \in X$. Then there is a point $x_0 \in X$ such that $x_0 \in \varphi(x_0)$.*

The proof of Theorem 1.3 is contained in Section 6. The proof of Theorem 1.1 takes up most of this paper—Sections 2-5. In Section 7, we apply Theorems 1.1, 1.2 and 1.3 to prove a list of fixed-point theorems for UV^n and UV^ω decompositions of compact metric spaces.

2. UV^n sets in compact metric ARs

Let $n \geq -1$. For subsets U and V of a space Y we shall write that $V \overset{n}{\hookrightarrow} U$ if every continuous image of a k-sphere ($k \leq n$) in V is contractible in U.

Proposition 2.1. *Let $U, G, W, V \subset Y$ be such that $V \subset W \overset{n}{\hookrightarrow} G \subset U$. Then $V \overset{n}{\hookrightarrow} U$.*

Proof. Routine verification. □

Let (Y, d) be a compact metric AR. For $A \in \mathcal{F}(Y)$ and $\varepsilon > 0$, we use $B_\varepsilon(A)$ to denote the ε-neighbourhood of A in (Y, d), i.e. $B_\varepsilon(A) = \{y \in Y : d(y, A) < \varepsilon\}$. Denote $UV^n(Y) = \{A \in \mathcal{F}(Y) : A \text{ is } UV^n\}$. Note that $A \in UV^n(Y)$ if and only if every neighbourhood U of A in Y contains this one V such that $V \overset{n}{\hookrightarrow} U$.

Proposition 2.2. *Let (Y, d) be a compact metric AR and let $A \in \mathcal{F}(Y)$. Then the following two conditions are equivalent*:

(a) *$A \in UV^n(Y)$.*

(b) *To every $\varepsilon > 0$ there corresponds a $\delta(\varepsilon) \in (0, \varepsilon)$ such that $B_{\delta(\varepsilon)}(A) \overset{n}{\hookrightarrow} B_\varepsilon(A)$.*

Proof. That (b) → (a) is obvious.

(a) → (b). Let $\varepsilon > 0$. By definition, there is a neighbourhood V_ε of A such that $V_\varepsilon \overset{n}{\hookrightarrow} B_\varepsilon(A)$. Next, for every $a \in A$, fix a $\delta(\varepsilon, a) \in (0, \varepsilon)$ with $B_{2\delta(\varepsilon,a)}(a) \subset V_\varepsilon$. Since A is compact, there is a finite $A_0 \subset A$ such that $A \subset \bigcup\{B_{\delta(\varepsilon,a)}(a) : a \in A_0\}$. Then $\delta(\varepsilon) = \min\{\delta(\varepsilon, a) : a \in A_0\}$ works because

$$B_{\delta(\varepsilon)}(A) \subset \bigcup\{B_{\delta(\varepsilon,a)+\delta(\varepsilon)}(a) : a \in A_0\} \subset \bigcup\{B_{2\delta(\varepsilon,a)}(a) : a \in A_0\} \subset V_\varepsilon$$

and therefore, by Proposition 2.1, $B_{\delta(\varepsilon)}(A) \overset{n}{\hookrightarrow} B_\varepsilon(A)$. □

3. Some lemmas about u.s.c. mappings

For a space X, we denote:

$$\mathrm{Cov}(X) = \{\mathcal{W} : \mathcal{W} \text{ is an open cover of } X\}$$

and

$$\text{f-Cov}(X) = \{\mathcal{W} \in \mathrm{Cov}(X) : \mathcal{W} \text{ is finite}\}.$$

Let $\mathcal{W} \in \mathrm{Cov}(X)$. We shall say that a map $c : \mathcal{W} \to X$ is $\mathcal{W}$-*cross* provided $c(W) \in W$ for every $W \in \mathcal{W}$.

Lemma 3.1. *Let X be a compact space, (Y, d) a metric space, $\mathcal{V} \in \mathrm{Cov}(X)$, and $\varphi : X \to \mathcal{F}(Y)$ be u.s.c. Then for every map $\mu : X \to (0, +\infty)$ there exists a star-refinement $\mathcal{W} \in \text{f-Cov}(X)$ of $\mathcal{V}$ and a $\mathcal{W}$-cross map $c : \mathcal{W} \to X$ such that*

$$\varphi(x) \subset B_{\mu(c(W))}(\varphi(c(W))) \quad \textit{for every } x \in W \in \mathcal{W}.$$

Proof. Let $\mathcal{U} \in \mathrm{Cov}(X)$ be a star-refinement of $\mathcal{V}$ which exists because of the compactness of X. Next, for every $x \in X$, pick a fixed $U_x \in \mathcal{U}$ with $x \in U_x$, and then set

$$W_x = \{z \in U_x : \varphi(z) \subset B_{\mu(x)}(\varphi(x))\} = U_x \cap \varphi^{\#}(B_{\mu(x)}(\varphi(x))).$$

Since φ is u.s.c., W_x is a neighbourhood of x. Therefore, there is a finite subset $A \subset X$ such that $X = \bigcup\{W_a : a \in A\}$. Then set $\mathcal{W} = \{W_a : a \in A\}$. As for the map $c : \mathcal{W} \to X$, for every $W \in \mathcal{W}$, take $c(W) \in X$ to be such that $W = W_{c(W)}$. That this works follows immediately from the definition of the sets $W \in \mathcal{W}$. □

Lemma 3.2. *Let X be a compact space, (Y, d) a compact metric AR, and $\varphi : X \to UV^n(Y)$ be u.s.c. Then for every $\mathcal{V} \in \mathrm{Cov}(X)$ and every $\varepsilon > 0$ there is a star-refinement $\mathcal{W} \in \text{f-Cov}(X)$ of $\mathcal{V}$, a $\mathcal{W}$-cross map $c : \mathcal{W} \to X$, and a map $\gamma : \mathcal{W} \to (0, \varepsilon)$ such that, for every $W \in \mathcal{W}$,*

$$B_{\gamma(W)}(\varphi(c(W))) \overset{n}{\hookrightarrow} B_{\varepsilon}(\varphi(c(W)))$$

and

$$B_{\gamma(W)/2}(\varphi(x)) \subset B_{\gamma(W)}(\varphi(c(W))) \quad \textit{whenever } x \in W.$$

Proof. Since $\varphi(x) \in UV^n(Y)$, by Proposition 2.2, there is a $\delta(\varepsilon, x) \in (0, \varepsilon)$ for which $B_{\delta(\varepsilon,x)}(\varphi(x)) \overset{n}{\hookrightarrow} B_{\varepsilon}(\varphi(x))$. Next, by Lemma 3.1 with $\mu(x) = \delta(\varepsilon, x)/2$, we get a star-refinement $\mathcal{W} \in \text{f-Cov}(X)$ of $\mathcal{V}$ and a $\mathcal{W}$-cross map $c : \mathcal{W} \to X$ such that, for every $x \in W \in \mathcal{W}$,

$$B_{\delta(\varepsilon,c(W))/2}(\varphi(x)) \subset B_{\delta(\varepsilon,c(W))}(\varphi(c(W))).$$

Then setting $\gamma(W) = \delta(\varepsilon, c(W))$, we finish the proof. □

Let $\mathcal{W}, \mathcal{V} \in \mathrm{Cov}(X)$. For a subset $A \subset X$, we use $\mathrm{St}_{\mathcal{W}}(A)$ to denote the *star* of A with respect to $\mathcal{W}$, i.e. $\mathrm{St}_{\mathcal{W}}(A) = \bigcup\{W \in \mathcal{W} : W \cap A \neq \varnothing\}$. We shall say that a map $t : \mathcal{W} \to \mathcal{V}$ is *star-refining*, or *s.r.*, if $\mathrm{St}_{\mathcal{W}}(W) \subset t(W)$ for every $W \in \mathcal{W}$.

Lemma 3.3. *Let X be a compact space, (Y, d) a compact metric AR, and $\varphi : X \to UV^n(Y)$ be u.s.c. Suppose $\mathcal{W}_{n+2} \in \text{f-Cov}(X)$ and $\gamma_{n+2} : \mathcal{W}_{n+2} \to (0, +\infty)$. Then, for every $k = 0, 1, \dots, n+1$, there exist*

(i) *a $\mathcal{W}_k \in \text{f-Cov}(X)$,*
(ii) *an s.r. map $t_k : \mathcal{W}_k \to \mathcal{W}_{k+1}$,*

(iii) *a* $\mathcal{W}_k$*-cross map* $c_k : \mathcal{W}_k \to X$*, and*
(iv) *a map* $\gamma_k : \mathcal{W}_k \to (0, \min\{\gamma_{k+1}(W)/2 : W \in \mathcal{W}_{k+1}\})$

such that, for every $W \in \mathcal{W}_k$*,*

(a) $B_{\gamma_k(W)}(\varphi(c_k(W))) \overset{n}{\hookrightarrow} B_{\gamma_{k+1}(t_k(W))/2}(\varphi(c_k(W)))$*, and*
(b) $B_{\gamma_k(W)/2}(\varphi(x)) \subset B_{\gamma_k(W)}(\varphi(c_k(W)))$ *whenever* $x \in W$*.*

Proof. By finite induction. Using Lemma 3.2, with $\mathcal{V} = \mathcal{W}_{n+2}$ and with $\varepsilon = \min\{\gamma_{n+2}(W)/2 : W \in \mathcal{W}_{n+2}\}$, we find that a star-refinement $\mathcal{W}_{n+1} \in \text{f-Cov}(X)$ of $\mathcal{W}_{n+2}$, a $\mathcal{W}_{n+1}$-cross map $c_{n+1} : \mathcal{W}_{n+1} \to X$, and a map $\gamma_{n+1} : \mathcal{W}_{n+1} \to (0, \min\{\gamma_{n+2}(W)/2 : W \in \mathcal{W}_{n+2}\})$ such that, for every $W \in \mathcal{W}_{n+1}$,

$$B_{\gamma_{n+1}(W)}(\varphi(c_{n+1}(W))) \overset{n}{\hookrightarrow} B_\varepsilon(\varphi(c_{n+1}(W)))$$

and

$$B_{\gamma_{n+1}(W)/2}(\varphi(x)) \subset B_{\gamma_{n+1}(W)}(\varphi(c_{n+1}(W))) \quad \text{whenever } x \in W.$$

Next, for every $W \in \mathcal{W}_{n+1}$ pick a fixed $t_{n+1}(W) \in \mathcal{W}_{n+2}$ with $\text{St}_{\mathcal{W}_{n+1}}(W) \subset t_{n+1}(W)$. Thus, we get an s.r. map $t_{n+1} : \mathcal{W}_{n+1} \to \mathcal{W}_{n+2}$. Note that $\varepsilon \le \gamma_{n+2}(t_{n+1}(W))/2$ and therefore, by Proposition 2.1,

$$B_{\gamma_{n+1}(W)}(\varphi(c_{n+1}(W))) \overset{n}{\hookrightarrow} B_{\gamma_{n+2}(t_{n+1}(W))/2}(\varphi(c_{n+1}(W)))$$

which, in effect, completes the first step of our induction. Since the next steps are now obvious, the lemma is proved. □

4. A lemma about nerves of coverings

Whenever M is a finite simplicial complex, we use $|M|$ to denote the polytope on M and M^k to denote the k-skeleton of M. For a simplex $\sigma \in M$ we use $\partial|\sigma|$ to denote the boundary of σ. Note that $\partial|\sigma| = |\sigma \cap M^k|$ in case $\sigma \in M^{k+1} \setminus M^k$. Finally, for $\mathcal{W} \in \text{f-Cov}(X)$, by $\mathcal{N}(\mathcal{W})$ we denote the *nerve* of $\mathcal{W}$, i.e., the simplicial complex $\mathcal{N}(\mathcal{W}) = \{\sigma \subset \mathcal{W} : \bigcap \sigma \neq \varnothing\}$.

Lemma 4.1. *Let* X *be a compact space,* (Y, d) *a compact metric AR, and* $\varphi : X \to UV^n(Y)$ *be u.s.c. Then for every* $\mathcal{V} \in \text{f-Cov}(X)$ *and every* $\varepsilon > 0$ *there exists a* $\mathcal{W} \in \text{f-Cov}(X)$*, an s.r. map* $p : \mathcal{W} \to \mathcal{V}$*, a continuous* $\mathrm{w} : |\mathcal{N}^{n+1}(\mathcal{W})| \to Y$*, and a map* $s : \mathcal{N}(\mathcal{W}) \to X$ *such that, for every simplex* $\sigma \in \mathcal{N}^{n+1}(\mathcal{W})$*,*

$$s(\sigma) \in \bigcap p(\sigma) \text{ and } \mathrm{w}(|\sigma|) \subset B_\varepsilon(\varphi(s(\sigma))).$$

Proof. Let $\mathcal{W}_k$, t_k, c_k, and γ_k ($k = 0, 1, \dots, n+1$) be as in Lemma 3.3 applied with $\mathcal{W}_{n+2} = \mathcal{V}$ and with $\gamma_{n+2}(W) = \varepsilon$, $W \in \mathcal{W}_{n+2}$. Set $\mathcal{W} = \mathcal{W}_0$, and let $p_0 : \mathcal{W} \to \mathcal{W}_0$ be the identity. Also, let $q_0 : \mathcal{N}(\mathcal{W}) \to \mathcal{W}_0$ be such that $q_0(\sigma) \in p_0(\sigma)$, $\sigma \in \mathcal{N}(\mathcal{W})$. Next, for every $k = 0, 1, \dots, n+1$, we define the following:

(p) s.r. maps $p_{k+1} : \mathcal{W} \to \mathcal{W}_{k+1}$ by $p_{k+1} = t_k \circ p_k$;
(q) maps $q_{k+1} : \mathcal{N}(\mathcal{W}) \to \mathcal{W}_{k+1}$ such that, for every $\sigma \in \mathcal{N}(\mathcal{W})$,

$$q_{k+1}(\sigma) \in p_{k+1}(\sigma) \quad \text{and} \quad \gamma_{k+1}(q_{k+1}(\sigma)) = \max\{\gamma_{k+1}(W) : W \in p_{k+1}(\sigma)\};$$

(s) maps $s_k : \mathcal{N}(\mathcal{W}) \to X$ by $s_k = c_k \circ q_k$; and
(r) maps $r_k : \mathcal{N}(\mathcal{W}) \to (0, \min\{r_{k+1}(\sigma)/2 : \sigma \in \mathcal{N}(\mathcal{W})\})$ by

$$r_k = \gamma_k \circ q_k \quad \text{and} \quad r_{n+2} = \gamma_{n+2} \circ q_{n+2}.$$

Note, first of all, that the definition of (r) is correct. Indeed, by 3.3(iv), $\kappa \in \mathcal{N}(\mathcal{W})$ implies

$$\begin{aligned} r_k(\kappa) = \gamma_k(q_k(\kappa)) &\le \max\{\gamma_k(W) : W \in \mathcal{W}_k\} \\ &< \min\{\gamma_{k+1}(W)/2 : W \in \mathcal{W}_{k+1}\} \\ &\le \min\{\gamma_{k+1}(q_{k+1}(\sigma))/2 : \sigma \in \mathcal{N}(\mathcal{W})\} \\ &= \min\{r_{k+1}(\sigma)/2 : \sigma \in \mathcal{N}(\mathcal{W})\}. \end{aligned}$$

Now let $0 \le k \le n$, and let $\kappa, \sigma \in \mathcal{N}(\mathcal{W})$ with $\kappa \subset \sigma$. The following holds:

(1) $$r_{k+1}(\kappa) \le r_{k+1}(\sigma).$$

Indeed, $p_{k+1}(\kappa) \subset p_{k+1}(\sigma)$ implies

$$\begin{aligned} r_{k+1}(\kappa) &= \gamma_{k+1}(q_{k+1}(\kappa)) = \max\{\gamma_{k+1}(W) : W \in p_{k+1}(\kappa)\} \\ &\le \max\{\gamma_{k+1}(W) : W \in p_{k+1}(\sigma)\} = \gamma_{k+1}(q_{k+1}(\sigma)) = r_{k+1}(\sigma). \end{aligned}$$

(2) $$B_{r_{k+1}(\kappa)/2}(\varphi(s_k(\kappa))) \subset B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))).$$

Note, first of all, that $\bigcup p_k(\sigma) \subset \bigcap p_{k+1}(\sigma)$. Indeed, whenever $W \in p_k(\sigma)$, 3.3(ii) implies $\bigcup p_k(\sigma) \subset \mathrm{St}_{\mathcal{W}_k}(W) \subset t_k(W)$ and therefore (see (p))

$$\bigcup p_k(\sigma) \subset \bigcap \{t_k(W) : W \in p_k(\sigma)\} = \bigcap t_k(p_k(\sigma)) = \bigcap p_{k+1}(\sigma).$$

According then to 3.3(iii), this implies that

$$s_k(\kappa) = c_k(q_k(\kappa)) \in q_k(\kappa) \subset \bigcup p_k(\kappa) \subset \bigcup p_k(\sigma) \subset \bigcap p_{k+1}(\sigma) \subset q_{k+1}(\sigma).$$

Finally, by (1) and 3.3(b), we get

$$\begin{aligned} B_{r_{k+1}(\kappa)/2}(\varphi(s_k(\kappa))) &\subset B_{r_{k+1}(\sigma)/2}(\varphi(s_k(\kappa))) = B_{\gamma_{k+1}(q_{k+1}(\sigma))/2}(\varphi(s_k(\kappa))) \\ &\subset B_{\gamma_{k+1}(q_{k+1}(\sigma))}(\varphi(c_{k+1}(q_{k+1}(\sigma)))) = B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))). \end{aligned}$$

(3) $$B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))) \overset{n}{\hookrightarrow} B_{r_{k+2}(\sigma)/2}(\varphi(s_{k+1}(\sigma))).$$

To show this we first note that $t_{k+1}(q_{k+1}(\sigma)) \in p_{k+2}(\sigma)$ (see (p) and (q)), and therefore $\gamma_{k+2}(t_{k+1}(q_{k+1}(\sigma))) \le \gamma_{k+2}(q_{k+2}(\sigma)) = r_{k+2}(\sigma)$. Then, by 3.3(a),

$$\begin{aligned} B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))) &= B_{\gamma_{k+1}(q_{k+1}(\sigma))}(\varphi(c_{k+1}(q_{k+1}(\sigma)))) \\ &\overset{n}{\hookrightarrow} B_{\gamma_{k+2}(t_{k+1}(q_{k+1}(\sigma)))/2}(\varphi(c_{k+1}(q_{k+1}(\sigma)))) \\ &\subset B_{r_{k+2}(\sigma)/2}(\varphi(s_{k+1}(\sigma))). \end{aligned}$$

So, Proposition 2.1 completes the verification of (3).

Now take $p = p_{n+2} : \mathcal{W} \to \mathcal{W}_{n+2} = \mathcal{V}$ and $s = s_{n+1} : \mathcal{N}(\mathcal{W}) \to X$. Suppose $\sigma \in \mathcal{N}(\mathcal{W})$. Since $q_{n+1}(\sigma) \in p_{n+1}(\sigma)$, we get that $p_{n+1}(W) \cap q_{n+1}(\sigma) \ne \varnothing$ for every $W \in \sigma$, and therefore

$$\begin{aligned} s(\sigma) = c_{n+1}(q_{n+1}(\sigma)) \in q_{n+1}(\sigma) &\subset \mathrm{St}_{\mathcal{W}_{n+1}}(p_{n+1}(W))) \\ &\subset t_{n+1}(p_{n+1}(W)) = p_{n+2}(W) = p(W). \end{aligned}$$

That is, $s(\sigma) \in \bigcap\{p(W) : W \in \sigma\}$, which completes the first part of the proof.

As for now the map $w : |\mathcal{N}^{n+1}(\mathcal{W})| \to Y$, we shall construct this map by induction. First, we define $w^0 : |\mathcal{N}^0(\mathcal{W})| \to Y$ by $w^0(W) \in \varphi(c_0(W))$, $W \in \mathcal{W} = \mathcal{N}^0(\mathcal{W}) = |\mathcal{N}^0(\mathcal{W})|$. Since $q_0(W) = W$ for every $W \in \mathcal{W}$, it follows that

$$w^0(W) \in \varphi(c_0(W)) = \varphi(c_0(q_0(W))) = \varphi(s_0(W)) \subset B_{r_1(W)/2}(\varphi(s_0(W))).$$

Let us now suppose that, for some $0 \le k \le n$, $w^k : |\mathcal{N}^k(\mathcal{W})| \to Y$ is continuous such that

$$w^k(|\sigma|) \subset B_{r_{k+1}(\sigma)/2}(\varphi(s_k(\sigma))), \qquad \sigma \in \mathcal{N}^k(\mathcal{W}),$$

and let us extend w^k to a continuous $w^{k+1} : |\mathcal{N}^{k+1}(\mathcal{W})| \to Y$ such that

$$w^{k+1}(|\sigma|) \subset B_{r_{k+2}(\sigma)/2}(\varphi(s_{k+1}(\sigma))), \qquad \sigma \in \mathcal{N}^{k+1}(\mathcal{W}).$$

To this end, take a $\sigma \in \mathcal{N}^{k+1}(\mathcal{W})$. If $\sigma \in \mathcal{N}^k(\mathcal{W})$, let $w^{k+1}_\sigma = w^k||\sigma|$. Then, by (2),

$$\begin{aligned} w^{k+1}_\sigma(|\sigma|) = w^k(|\sigma|) \subset B_{r_{k+1}(\sigma)/2}(\varphi(s_k(\sigma))) &\subset B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))) \\ &\subset B_{r_{k+2}(\sigma)/2}(\varphi(s_{k+1}(\sigma))) \end{aligned}$$

because, by definition (see (r)), $r_{k+1}(\sigma) \le r_{k+2}(\sigma)/2$. In case $\sigma \in \mathcal{N}^{k+1}(\mathcal{W}) \setminus \mathcal{N}^k(\mathcal{W})$ note that, by (2), $\kappa \in \mathcal{N}^k(\mathcal{W})$ and $\kappa \subset \sigma$ implies

$$w^k(|\kappa|) \subset B_{r_{k+1}(\kappa)/2}(\varphi(s_k(\kappa))) \subset B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma))),$$

and therefore $w^k(\partial|\sigma|) \subset B_{r_{k+1}(\sigma)}(\varphi(s_{k+1}(\sigma)))$. Then, by virtue of (3), there is a continuous extension $w^{k+1}_\sigma : |\sigma| \to B_{r_{k+2}(\sigma)/2}(\varphi(s_{k+1}(\sigma)))$ of $\omega^k|\partial|\sigma|$. Finally, we define $w^{k+1} : |\mathcal{N}^{k+1}(\mathcal{W})| \to Y$ by letting $w^{k+1}||\sigma| = w^{k+1}_\sigma$ for every $\sigma \in \mathcal{N}^{k+1}(\mathcal{W})$. Thus, in effect, we have already defined a continuous $w^{n+1} : |\mathcal{N}^{n+1}(\mathcal{W})| \to Y$ such that, for every $\sigma \in \mathcal{N}^{n+1}(\mathcal{W})$, $w^{n+1}(|\sigma|) \subset B_{r_{n+2}(\sigma)/2}(\varphi(s_{n+1}(\sigma))) \subset B_\varepsilon(\varphi(s(\sigma)))$. Then, setting $w = w^{n+1}$, we finish the proof. □

Remark. The idea for Lemma 4.1 is taken from [4, Lemma 6.1]. Concerning the proof of this lemma, the author would like to express his sincere appreciation to the referee for the helpful suggestions that fixed up some of its elements.

5. Proof of Theorem 1.1

Lemma 5.1. *Let (Z, ρ) be a metric space with $\dim(Z) \le n+1$, (Y, d) be a compact metric AR, $g : Y \to Z$ be continuous, and $\varphi : Z \to UV^n(Y)$ be u.s.c. Then to every $\varepsilon > 0$ there corresponds a continuous map $f_{1/\varepsilon} : g(Y) \to Y$ and a map $h_{1/\varepsilon} : g(Y) \to g(Y)$ such that $\rho(x, h_{1/\varepsilon}(x)) < \varepsilon$ and $f_{1/\varepsilon}(x) \in B_\varepsilon(\varphi(h_{1/\varepsilon}(x)))$ for every $x \in g(Y)$.*

Proof. Let $\varepsilon > 0$. Set $X = g(Y)$ and let $\mathcal{V} \in$ f-Cov(X) be such that, with respect to ρ, $\operatorname{diam}(V) < \varepsilon$ for every $V \in \mathcal{V}$. Such $\mathcal{V}$ certainly exists because X is compact. Now using Lemma 4.1 with these particular X and $\mathcal{V}$, we get a $\mathcal{W} \in$ f-Cov(X), an s.r. map $p : \mathcal{W} \to \mathcal{V}$, a continuous $w : |\mathcal{N}^{n+1}(\mathcal{W})| \to X$, and a map $s : \mathcal{N}(\mathcal{W}) \to X$ such that

$$s(\sigma) \in \bigcap p(\sigma) \quad \text{and} \quad w(|\sigma|) \subset B_\varepsilon(\varphi(s(\sigma))) \quad \text{for every } \sigma \in \mathcal{N}^{n+1}(\mathcal{W}).$$

Since $\dim(X) \le n+1$, $\mathcal{W}$ has an open index-refinement $\{U_W : W \in \mathcal{W}\}$ of (indexed) order $\le n+2$. That is, $\sigma_x = \{W \in \mathcal{W} : x \in U_W\} \in \mathcal{N}^{n+1}(\mathcal{W})$ for every point $x \in X$. Then define $h_{1/\varepsilon} : X \to X$ by $h_{1/\varepsilon}(x) = s(\sigma_x)$, $x \in X$. Next, take a point $x \in X$ and let $W_x \in \mathcal{W}$ be such that $x \in U_{W_x}$. Then,

$$h_{1/\varepsilon}(x) = s(\sigma_x) \in \bigcap p(\sigma_x) = \bigcap \{p(W) : x \in U_W\} \subset p(W_x)$$

and therefore $\rho(x, h_{1/\varepsilon}(x)) < \varepsilon$ because $\operatorname{diam}(p(W_x)) < \varepsilon$.

As for the map $f_{1/\varepsilon} : X \to Y$, define first a canonical map $\xi : X \to |\mathcal{N}(\mathcal{W})|$ obtained by using a partition of unity $\{\xi_W : W \in \mathcal{W}\}$ on X subordinated to $\{U_W :$

$W \in \mathcal{W}\}$. That is, $\xi(x) = \sum\{\xi_W(x) \cdot W : W \in \mathcal{W}\}$. Then $\xi\colon X \to |\mathcal{N}^{n+1}(\mathcal{W})|$ because $\xi(x) \in |\sigma_x|$ for every $x \in X$. Thus, we can define $f_{1/\varepsilon} = \mathcal{w} \circ \xi$. Hence,

$$f_{1/\varepsilon}(x) = \mathcal{w}(\xi(x)) \in \mathcal{w}(|\sigma_x|) \subset B_\varepsilon(\varphi(s(\sigma_x))) = B_\varepsilon(\varphi(h_{1/\varepsilon}(x))),$$

which completes the proof. □

Having established Lemma 5.1, we finish the proof of Theorem 1.1 following the proof of Kakutani's theorem [3]. Namely, suppose X, Y, and φ are as in Theorem 1.1 and let $g : Y \to X$ be continuous. Let, in addition, ρ be a metric on X agreeing with its topology and, respectively, d be a metric on Y agreeing with the topology of Y. Let $k > 0$. First, by Lemma 5.1 with $Z = X$ and with $\varepsilon = 1/k$, we find a continuous map $f_k : g(Y) \to Y$ and a map $h_k : g(Y) \to g(Y)$ such that

$$\rho(x, h_k(x)) < 1/k \quad \text{and} \quad f_k(x) \in B_{1/k}(\varphi(h_k(x))) \quad \text{for every } x \in g(Y).$$

Next, let $y_k \in Y$ be such that $f_k(g(y_k)) = y_k$, which exists because Y is a compact metric AR. We may now assume that, without loss of generality, the so obtained sequence $\{y_k\}$ converges to some point $y_0 \in Y$. Claim that $y_0 \in \varphi(g(y_0))$. Indeed, let $\varepsilon > 0$. Since $\rho(g(y_k), h_k(g(y_k))) < 1/k$, $\{h_k(g(y_k))\}$ converges to $g(y_0)$. Therefore, there is an $m > 0$ such that $1/m < \varepsilon/4$, $d(y_0, y_m) < \varepsilon/4$ and $h_m(g(y_m)) \in \varphi^{\#}(B_{\varepsilon/2}(\varphi(g(y_0))))$. For this particular m we have:

$$\begin{aligned} d(y_0, \varphi(h_m(g(y_m)))) &\le d(y_0, y_m) + d(y_m, \varphi(h_m(g(y_m)))) \\ &= d(y_0, y_m) + d(f_m(g(y_m)), \varphi(h_m(g(y_m)))) \\ &< \varepsilon/4 + 1/m \\ &< \varepsilon/4 + \varepsilon/4 \qquad = \varepsilon/2 \end{aligned}$$

Hence $y_0 \in B_{\varepsilon/2}(\varphi(h_m(g(y_m)))) \subset B_\varepsilon(\varphi(g(y_0)))$, and therefore $y_0 \in \varphi(g(y_0))$ because $\varphi(g(y_0))$ is closed. This completes the proof of Theorem 1.1. □

6. Proof of Theorem 1.3

Since every compact metric space X is, in effect, a closed subset of the Hilbert cube Q, Theorem 1.3 is a simple consequence of the following fixed-point theorem.

Theorem 6.1. *Let $\varphi : Q \to \mathcal{F}(Q)$ be u.s.c. such that $\varphi(x)$ is UV^ω for all $x \in Q$. Then there is a point $x_0 \in Q$ such that $x_0 \in \varphi(x_0)$.*

In preparation for the proof of Theorem 6.1, we need some notation. As usual, $\mathbb{N}$ denotes the set of all natural numbers and J denotes the interval $[-1, 1]$. The Hilbert cube Q is the countable infinite product $\prod\{J_m : m \in \mathbb{N}\}$, where each J_m is a copy of J. For each $n \in \mathbb{N}$, denote π_n to be the projection from Q onto its nth factor. Set, moreover, $g_n : Q \to \prod\{J_m : m \le n\}$ to be the projection, and let $J^n = g_n(Q)$. Finally, let $h_n : J^n \to Q$ be the standard inclusion map (that is, $g_n(h_n(x)) = x$ and $\pi_m(h_n(x)) = 0$ for $m > n$).

Proof of Theorem 6.1. Suppose $\varphi : Q \to \mathcal{F}(Q)$ is as in the theorem. Let $n \in \mathbb{N}$. Define a set-valued mapping $\varphi_n : J^n \to \mathcal{F}(Q)$ by letting $\varphi_n(x) = \varphi(h_n(x))$ for every $x \in J^n$. First, note that φ_n is u.s.c. because h_n is continuous. Next, note that, in particular, $\varphi_n : J^n \to UV^{n-1}(Q)$. Then, by Theorem 1.1 with $X = J^n$, $Y = Q$, $g = g_n$, and $\varphi = \varphi_n$, there is a point $x_n \in Q$ such that $x_n \in \varphi_n(g_n(x_n))$. Because of the compactness of Q, there now is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ which converges to some point $x_0 \in Q$. Claim that $x_0 \in \varphi(x_0)$. Indeed, since

$\pi_m(x_{k_n}) = \pi_m(h_{k_n}(g_{k_n}(x_{k_n})))$ for every $n \geq m$, it follows that $\{h_{k_n}(g_{k_n}(x_{k_n}))\}$ converges to x_0 too. Therefore $x_0 \in \varphi(x_0)$ because φ is u.s.c. closed-valued and because $x_{k_n} \in \varphi_{k_n}(g_{k_n}(x_{k_n})) = \varphi(h_{k_n}(g_{k_n}(x_{k_n})))$. This completes the proof of Theorem 6.1. ☐

7. UV^n and UV^ω decompositions

For a space X, the statement that G is a UV^n (resp., UV^ω) decomposition of X means that G is an upper semi-continuous decomposition of X into compact sets, each with property UV^n (resp., UV^ω). If G is a decomposition of a space X, then X/G will denote the associated decomposition space, and P the natural projection from X onto X/G.

The theorems to be proved in this section all sharpen (in some aspects) results of [1].

Theorem 7.1. *Let X be a compact metric AR with $\dim(X) \leq n+1$ and let G be a UV^n decomposition of X. Then X/G has the fixed-point property.*

Proof. Suppose $f : X/G \to X/G$ is continuous. Then $\varphi = P^{-1} \circ f \circ P : X \to UV^n(X)$ is u.s.c. and therefore, by Theorem 1.2, there is a point $x_0 \in X$ such that $x_0 \in \varphi(x_0) = P^{-1}(f(P(x_0)))$. Hence $P(x_0) = f(P(x_0))$, which completes the proof. ☐

Theorem 7.2. *Let X be a compact metric AR and let G be a UV^ω decomposition of X. Then X/G has the fixed-point property.*

Proof. Repeat precisely the proof of Theorem 7.1 but now using Theorem 1.3 instead of Theorem 1.2. ☐

Theorem 7.3. *Let Y be a compact metric AR and let G be a UV^n decomposition of Y such that $\dim(Y/G) \leq n+1$. Then Y/G has the fixed-point property.*

Proof. Suppose $f : Y/G \to Y/G$ is continuous. Then $\varphi = P^{-1} \circ f : Y/G \to UV^n(Y)$ is u.s.c. and therefore, by Theorem 1.1 with $X = Y/G$ and $g = P$, there is a point $y_0 \in Y$ such that $y_0 \in \varphi(P(y_0)) = P^{-1}(f(P(y_0)))$. Hence $P(y_0) = f(P(y_0))$, which completes the proof. ☐

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