

## TRACE CLASS BACKWARD WEIGHTED SHIFTS ARE QUASISUBSCALAR

EUNGIL KO

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ABSTRACT. Recently, the author generalized Putinar techniques. In this paper we use those recent techniques and results to show (Theorem 3.1) that every trace class backward weighted shift with a monotone decreasing weight sequence is quasisubscalar.

### 1. INTRODUCTION

Let  $H$  and  $K$  be separable, complex Hilbert spaces and let  $\mathcal{L}(H, K)$  denote the space of all linear bounded operators from  $H$  to  $K$ . If  $H = K$ , we write  $\mathcal{L}(H)$  in place of  $\mathcal{L}(H, K)$ . An operator  $T$  is trace class if there is a basis  $\{e_n\}$  such that  $\sum \langle |T|e_n, e_n \rangle < \infty$  where  $|T|$  is the unique positive square root of  $T^*T$ .

In [Ko2] we made a different technique and proof to show that every operator on a finite dimensional complex Hilbert space is subscalar. So it is natural to ask the following question. Can we apply this technique to some operators on an infinite dimensional Hilbert space? That is, can we apply this technique to show that any operator on an infinite dimensional Hilbert space is represented using a scalar operator? As an example, this technique applies a certain trace class. Thus every trace class backward weighted shift with a monotone decreasing weight sequence is quasisubscalar. For purposes of completeness, we include some preliminary material of a general nature as well as some material on the trace class.

### 2. GENERAL PRELIMINARIES

Let  $z$  be the coordinate in the complex plane  $\mathbf{C}$  and let  $d\mu(z)$  denote the planar Lebesgue measure. Fix a bounded, connected open subset  $D$  of  $\mathbf{C}$ .

We shall denote by  $L^2(D, H)$  the Hilbert space of measurable functions  $f: D \rightarrow H$ , such that

$$\|f\|_{2,D} = \left( \int_D \|f(z)\|^2 d\mu(z) \right)^{1/2} < \infty.$$

The space of functions  $f \in L^2(D, H)$  which are analytic in  $D$  (i.e.,  $\bar{\partial}f = 0$ ) is denoted by  $A^2(D, H)$ .  $A^2(D, H)$  is called the Bergman space for  $D$ .

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Let us define now a Sobolev type space.  $D$  will be a bounded open subset of complex plane and  $m$  will be a fixed nonnegative integer. Let  $W^m(D, H)$  consist of all functions  $f = (f_1, f_2, \dots) \in L^2(D, H)$  such that

$$\|f\|_{W^m}^2 = \sum_{i=0}^m \|\bar{\partial}^i f\|_{2,D}^2$$

is finite. Then  $W^m(D, H)$  is a Hilbert space. Note that  $W^\infty(D, H)$  can be defined using  $m = \infty$  and is a Hilbert space.

An operator  $S$  in  $\mathcal{L}(H)$  is called  $C^m$ -scalar if there is a continuous unital morphism,

$$\Phi: C_0^m(\mathbf{C}) \rightarrow \mathcal{L}(H),$$

such that  $\Phi(z) = S$  where as usual  $z$  stands for the identity function on  $\mathbf{C}$ , and  $C_0^m(\mathbf{C})$  stands for the space of compactly supported functions on  $\mathbf{C}$ , continuously differentiable of order  $m$ ,  $0 \leq m \leq \infty$ .

We next discuss some facts concerning the multiplication operator by  $z$  on  $W^\infty(D, H)$ . The linear operator  $M$  of multiplication by  $z$  on  $W^\infty(D, H)$  is continuous and it has the relation defined by

$$\Phi: C_0^m(\mathbf{C}) \rightarrow \mathcal{L}(W^\infty(D, H)), \quad \Phi(f) = M_f.$$

Therefore,  $M$  is a scalar operator.

An operator  $T$  in  $\mathcal{L}(K)$  is said to be  $B$ -quasisubscalar if there exists a nonzero  $V$  in  $\mathcal{L}(K, H)$  such that  $VT = SV$  where  $S (= \Phi(z))$  is a scalar operator. If, in addition,  $V$  is one-to-one, then  $T$  is called a quasisubscalar operator.

We next recall some facts concerning the trace class. A compact operator  $A$  on  $H$  is said to be in the  $\mathcal{C}^p$  class ( $0 < p < \infty$ ) if the eigenvalues of  $(A^*A)^{1/2}$  are in  $l^p$ . In particular, if  $p = 1$ , it is called the trace class. Let  $T \in \mathcal{L}(H)$  be a backward weighted shift defined by  $Te_n = \alpha_n e_{n-1}$  if  $n \geq 1$  and  $Te_n = 0$  if  $n = 0$ . Then we know that  $T$  is in the  $\mathcal{C}^p$  class if and only if  $\{\alpha_n\} \in l^p$ . Let  $P_n: H \rightarrow \bigvee_{i=0}^{n-1} \{e_i\}$  be an orthogonal projection. Throughout, the symbol  $T_n$  will be denoted by  $P_n T P_n \oplus 0$ .

### 3. TRACE CLASS BACKWARD WEIGHTED SHIFTS

In this section we finally prove the main theorem of the paper, which is the following.

**Theorem 3.1.** *If  $T \in \mathcal{L}(H)$  is a trace class backward weighted shift such that  $\{\alpha_n\}_{n=1}^\infty$  is a monotone decreasing sequence of nonzero complex numbers, then  $T$  is a quasisubscalar operator.*

We now begin the proof of this theorem. Throughout, unless stated otherwise,  $T$  will denote a trace class backward weighted shift.

**Lemma 3.2** ([Ko2], Theorem 4.5). *There exists  $\varepsilon_0$  sufficiently small such that*

$$\|(T_2 - z)f - e_0\|_{W^4} > \varepsilon_0$$

for any  $f \in W^4(D, H)$ .

**Lemma 3.3.** *If  $g \in W^{2m}(D, H)$  with  $\| -zg \|_{W^{2m}} < \delta$ , then for  $i = 0, 1, \dots, 2m-2$  there exist  $h_i \in A^2(D, H)$  such that*

$$\|(g, \bar{\partial}g, \dots, \bar{\partial}^{2m-2}g) - (h_0, h_1, \dots, h_{2m-2})\|_{2,D} \leq 2C_D \delta$$

where  $C_D$  is a constant depending on  $D$ .

*Proof.* Let  $\bigoplus_m H$  denote the direct sum of  $m$  copies of  $H$ . Let  $P$  denote the orthogonal projection of  $\bigoplus_{2m-1} L^2(D, H)$  onto the Bergman space  $\bigoplus_{2m-1} A^2(D, H)$ . By Proposition 2.1 with  $\bigoplus_m T = (0)$  in [Pu],

$$\begin{aligned} \|(I - P)(g, \bar{\partial}g, \dots, \bar{\partial}^{2m-2}g)\|_{2,D} &\leq C_D(\| -z(\bar{\partial}g, \dots, \bar{\partial}^{2m-1}g)\|_{2,D} \\ &\quad + \| -z(\bar{\partial}^2, \dots, \bar{\partial}^{2m}g)\|_{2,D}) \\ &< 2C_D\delta. \end{aligned}$$

Set  $P(g, \bar{\partial}g, \dots, \bar{\partial}^{2m-2}g) = (h_0, h_1, \dots, h_{2m-2})$ . □

**Lemma 3.4.** For any  $f \in W^{2n}(D, H)$ ,

$$\|(T_n - z)f - e_0\|_{W^{2n}} > \varepsilon_0$$

where  $n = 2, 3, \dots$

*Proof.* We prove this lemma by induction. If  $n = 2$ , it is clear from Lemma 3.2. Assume that it is true for  $n = k$ . Then we can choose  $\beta > 0$  such that

$$(*) \quad | \|(T_k - z)f - e_0\|_{W^{2k}} - \varepsilon_0 | > \beta \sum_{i=1}^{\infty} |\alpha_i|$$

for any  $f \in W^{2k}(D, H)$ .

*Claim.*  $\|(T_{k+1} - z)f - e_0\|_{W^{2(k+1)}} > \varepsilon_0$  for any  $f \in W^{2(k+1)}(D, H)$ .

Now we verify the above claim. If not, there exists  $f \in W^{2(k+1)}(D, H)$  such that

$$\|(T_{k+1} - z)f - e_0\|_{W^{2(k+1)}} \leq \varepsilon_0.$$

Since  $W^{2n}(D, H) = \beta W^{2n}(D, H)$ ,  $f = \beta g$  for some  $g \in W^{2(k+1)}(D, H)$ .  $g(z) = g_1(z)e_0 + g_2(z)e_0 + \dots$  will be written by  $(g_1, g_2, \dots)$ . By Lemma 3.3 and equation (5) in [Ko2], we may assume that  $\|g_{k+1}\|_{W^{2k}} < 1$ . Then

$$\begin{aligned} \varepsilon_0 &\geq \|(T_{k+1} - z)\beta g - e_0\|_{W^{2(k+1)}} \\ &\geq \|(T_{k+1} - z)\beta g - e_0\|_{W^{2k}} \\ &= \|(T_k - z)\beta g - e_0 + \alpha_k(e_{k-1} \otimes e_k)\beta g\|_{W^{2k}} \\ &\geq \|(T_k - z)\beta g - e_0\|_{W^{2k}} - \beta|\alpha_k| \|g_{k+1}\|_{W^{2k}} \\ &> \varepsilon_0 + \beta \sum_{i=1}^{\infty} |\alpha_i| - \beta|\alpha_k| \\ &> \varepsilon_0. \end{aligned}$$

So we have a contradiction. □

**Lemma 3.5.**  $\|(T - z)f - e_0\|_{W^\infty} > \frac{\varepsilon_0}{2}$  for any  $f \in W^\infty(D, H)$ .

*Proof.* By Lemma 3.4, we know that for any  $f \in W^{2n}(D, H)$ ,

$$\|(T_n - z)f - e_0\|_{W^{2n}} > \varepsilon_0$$

where  $n = 2, 3, \dots$

Choose  $\hat{f} = (f_1, \dots, f_r, 0, \dots)^t$ , the projection of  $f$  such that

$$\|(T - z)f - (T - z)\hat{f}\|_{W^\infty} < \frac{\varepsilon_0}{4}.$$

Then

$$\begin{aligned}
 \|(T-z)f - e_0\|_{W^\infty} &\geq \|(T-z)\hat{f} - e_0\|_{W^\infty} - \|(T-z)\hat{f} - (T-z)f\|_{W^\infty} \\
 &> \|(T-z)\hat{f} - e_0\|_{W^\infty} - \frac{\varepsilon_0}{4} \\
 &= \|(T_r - z)\hat{f} - e_0\|_{W^\infty} - \frac{\varepsilon_0}{4} \\
 &> \frac{3}{4}\varepsilon_0. \quad \square
 \end{aligned}$$

**Lemma 3.6** ([RR], Theorem 4.12). *If  $T \in \mathcal{L}(H)$  is a trace class backward weighted shift such that  $\{|\alpha_n|\}_{n=1}^\infty$  is a monotone decreasing sequence of nonzero complex numbers and  $\mathcal{M}$  is a nontrivial invariant subspace of  $T$ , then  $\mathcal{M} = \mathcal{M}_n$  for some  $n$ , where  $\mathcal{M}_n = \bigvee_{k=0}^n \{e_k\}$ .*

*Proof of Theorem 3.1.* Let  $D$  be a bounded open disk in  $\mathbf{C}$  which contains  $\sigma(T)$ . Consider the quotient space

$$H(D) = \frac{W^\infty(D, H)}{\text{cl}(T-z)W^\infty(D, H)}$$

where cl denotes norm closure.

Let  $M = M_z$  be the multiplication operator on  $W^\infty(D, H)$ . Then  $M$  is a scalar operator with the relation, defined by

$$\Phi_M: C_0^\infty(\mathbf{C}) \rightarrow \mathcal{L}(W^\infty(D, H)), \quad \Phi_M(f) = M_f.$$

Since  $\overline{\text{ran}(T-z)}$  is invariant under  $M_z$ ,  $\widetilde{M}_z$  is still a scalar operator with  $\widetilde{\Phi}_M$  as a spectral distribution.

Define the map  $V: H \rightarrow H(D)$  by  $V(h) = 1 \otimes h + \overline{(T-z)W^\infty(D, H)}$ . Then

$$VT = \widetilde{M}_z V.$$

The proof is complete once the following lemma has been established.

**Lemma 3.7.** *Let  $D$  be a bounded disk which contains  $\sigma(T)$ . Then the operator  $V: H \rightarrow H(D)$  is one-to-one.*

*Proof.* If  $Vh = \tilde{0}$  and  $h \neq 0$ ,  $\ker V$  is an invariant subspace for  $T$ . Since  $e_0$  does not belong to  $\ker V$  by Lemma 3.5,  $\ker V \neq H$ . Hence  $\ker V$  is nontrivial. By Lemma 3.6,  $\ker V = \bigvee_{k=0}^n \{e_k\}$  for some  $n$ . Thus  $e_0 \in \ker V$ . So we have a contradiction.  $\square$

*Remarks.* (1) Every  $C^p$  class backward weighted shift is  $B$ -quasisubscalar.

(2) It would be interesting to know whether or not every compact, weighted shift is  $B$ -quasisubscalar.

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DEPARTMENT OF MATHEMATICS, EWHA WOMEN'S UNIVERSITY, SEOUL 120-750, KOREA