PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

# A CHARACTERIZATION OF REFLEXIVE BANACH SPACES

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(Communicated by Dale Alspach)

ABSTRACT. A Banach space $Z$ is not reflexive if and only if there exist a closed separable subspace $X$ of $Z$ and a convex closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$. If, moreover, $Z$ is separable, then it is possible to put $X = Z$.

We consider the following problem: When does a Banach space contain a closed convex set $Q$ with empty interior which contains a translate of any compact set in $X$? The basic example of such a Banach space is the space $C(\mathcal{K})$ of continuous functions on a compact infinite space $\mathcal{K}$. Indeed, it is enough to choose a point $p \in \mathcal{K}$ which is not isolated and define $Q$ as the set of all functions in $C(\mathcal{K})$ which attain their minima at $p$. Since $p$ is not isolated, $Q$ has empty interior. If $K$ is a compact subset of $C(\mathcal{K})$, then by the Banach-Dieudonné theorem [3] there exists a sequence $\{f_n\}$ of functions in $C(\mathcal{K})$ converging to zero such that $K$ is contained in its closed convex hull. If we define the function $g$ by

$$g(t) := \sup\{|f_n(t) - f_n(p)| : \ n \in N\} \quad \text{for} \quad t \in \mathcal{K},$$

then it is easy to check that $g$ is continuous and each function $g + f_n$ is in $Q$. Consequently, since $Q$ is convex, the translate $g + K$ is contained in $Q$.

If a Banach space $Z$ can be mapped linearly onto a Banach space $X$ containing the required set $Q$, then $Z$ also contains such a set. Namely, by the open mapping theorem, it is enough to take the preimage of $Q$. Therefore, for example, $\ell_1$ contains the required set because it can be mapped onto any separable Banach space, in particular, $C[0, 1]$.

In this note we show that, in fact, any separable nonreflexive Banach space $X$ contains a closed convex set with empty interior which contains a translate of any compact set in $X$.

Borwein and Noll observed in [1] that there exist a convex continuous function on the space $c_0$ of null sequences and a closed subset $Q$ of $c_0$ which is not a Haar null set so that $f$ fails to be Fréchet differentiable on $Q$. They define $f$ as the distance from the positive cone $Q := \{\{x_n\} \in c_0;\ x_n \geq 0,\ n = 1, 2, \dots\}$. As $Q$ has no interior points, $f$ fails to be Fréchet differentiable at all points of $Q$. The set $Q$ contains a translate of any compact set in $c_0$, and, therefore, for any

Received by the editors May 24, 1994 and, in revised form, August 18, 1994.
1991 *Mathematics Subject Classification.* Primary 46B10; Secondary 46B20.
*Key words and phrases.* Banach spaces, reflexivity, convexity.
The first author was partially supported by a grant of the Ősterreichische Akademische Austauschdienst.

probability Borel measure $\mu$ on $c_0$ there exists some $x \in c_0$ such that $\mu(Q+x) > 0$. Consequently, $Q$ is not Haar null (for the definition see [2]). They conjecture in [1] that also in $\ell_2$ there exists a closed convex set $C$ with empty interior which contains a translate of any compact set. We show that this is not the case in any reflexive Banach space, but on the other hand every nonreflexive Banach space has a closed subspace containing such a set.

By $B_X$ we denote the open unit ball of a Banach space $X$, and $B_X(x,r)$ is the usual notation for the open ball with center $x$ and diameter $r$; the subscript will be often omitted. We denote the closure of a set $A$ by $\bar{A}$ or cl$A$.

We will make use of the following variation of the Banach-Dieudonné theorem: Let $X$ be a Banach space, $K$ a compact subset of $B_X(0,c)$ and $E$ a dense subset of $B_X(0,2c)$. Then there exists a sequence $\{F_n\}$ of finite sets in $E$ so that

$$K \subset \operatorname{cl}\left(\sum_{n=1}^{\infty} 2^{-n} F_n\right). \tag{1}$$

This follows from the fact that there exist a sequence $\{F_n\}$ of finite sets in $E$ and a sequence of compact sets $\{K_n\}$ in $\bar{B}_X(0,c)$ so that

$$K \subset \sum_{i=1}^{n} 2^{-i} F_i + 2^{-n} K_n \quad \text{ for } \; n \in N. \tag{2}$$

Indeed, if $n=1$, choose $F_1 \subset E$ so that $2^{-1}F_1$ is a $\frac{c}{2}$-net for $K$. Then the set

$$K_1 := 2\left((K - 2^{-1}F_1) \cap \bar{B}(0, \frac{c}{2})\right)$$

is a compact subset of $\bar{B}(0,c)$ and $K \subset 2^{-1}F_1 + 2^{-1}K_1$. Now we can continue by induction. Suppose that $F_i$ and $K_i$ for $i = 1,\ldots,n$ so that (2) holds have been already constructed. Choosing $F_{n+1} \subset E$ so that $2^{-1}F_{n+1}$ is a $\frac{c}{2}$-net for the set $K_n$ and defining

$$K_{n+1} := 2\left((K_n - 2^{-1}F_{n+1}) \cap \bar{B}(0, \frac{c}{2})\right)$$

completes the proof. The following lemmata are possibly not the most efficient way to our main result, but we think that they may be of independent interest.

**Lemma 1.** *Let $Z$ be a Banach space, $U$ an open convex subset of $Z$ and $f$ a continuous real valued function defined on $U$. Then, either $f$ is affine or the convex hull $G$ of the graph of $f$ has nonempty interior.*

*Proof.* Suppose that $f$ is not affine. Then there exist $x$ and $y$ in $U$ such that $1/2(f(x)+f(y)) \neq f((x+y)/2)$. Define $z_0 := (x+y)/2$ and $c := (f(x)+f(y))/2$. We can suppose by replacing $f$ by $-f$ and adding a constant, if necessary, that

$$(f(x)+f(y))/2 - f(z_0) = \alpha > 0 \;\text{ and }\; f(z_0) > 0.$$

Choose some $\varepsilon > 0$ so that $0 < f(v) < f(z_0) + \alpha/2$ for every $v \in Z$ for which $\|v - z_0\| < \varepsilon$. Clearly the interior of the cone cap

$$\begin{aligned} M := \{x_{z,t} = t(z,0) + (1-t)(z_0,c) :\; z \in Z,\; \|z - z_0\| < \varepsilon,\; 0 \le t \le \alpha/(2c)\} \\ \subset Z \times R \end{aligned}$$

is nonempty. Let some $x_{z,t} \in M$ be given, we will show that $x_{z,t} \in G$. Consider the function

$$g(s) := (1-s)c - f((1-s)z_0 + sz), \quad t \le s \le 1.$$

The function $g$ is continuous, $g(t) > 0$, and $g(1) < 0$. Therefore, there exists some $r \in (t, 1)$ for which $g(r) = 0$. Hence, $x_{z,r}$ is contained in the graph of $f$ and since

$$x_{z,t} = \frac{t}{r} x_{z,r} + (1 - \frac{t}{r})(z_0, c)$$

we have $x_{z,t} \in G$. □

We say that a convex subset $Q$ of a Banach space $X$ is spanning if it contains a line segment in every direction, that is $X = \bigcup_{t>0} t(Q - Q)$. Observe that if a convex set $Q$ contains a translate of every finite subset of the unit ball, then $Q$ is spanning. If $Q$ contains translates of all compact sets in $X$ (or, for that matter, of all line segments), then $X = Q - Q$. Indeed, if $x \in X$ is given, then there exists $z \in X$ so that $[z, z + x] \subset Q$, and $x = z + x - z \in Q - Q$.

**Lemma 2.** *Suppose that $X$ is a Banach space and $Q \subseteq X$ is a bounded, closed and convex set with empty interior that is also spanning. Then for any compact subset $K$ of $X$ it follows that $Q + K$ also has empty interior.*

*Proof.* First, we show that $Q \cap H$ is nowhere dense in $H$ if $H$ is any closed hyperplane. Suppose that $x^* \neq 0$, $w \in H = \{x^* = a\}$, $\delta > 0$ and

$$B(w, \delta) \cap H \subseteq Q \cap H.$$

Choose some $y \in X$ such that $\langle x^*, y\rangle > 0$. Since $Q$ is spanning there exist $t > 0$, $u$ and $v$, both in $Q$, so that $t(u-v) = y$. It follows that $\langle x^*, u-v\rangle > 0$ and one of $u$ or $v$, say $u$, is not in $H$. It is routine to check that the convex hull of $\{u\}\cup(B(w, \delta)\cap H)$ has an interior point relative to $X$ (try $\frac{1}{2}(u + w)$) which contradicts the fact that $Q$ has no interior. Suppose that $H \subseteq X$ is a closed hyperplane, $u \in X$, $x \notin H$ and suppose that $h^* \in H^*$. Then the set $\{y+\langle h^*, y\rangle x+u : y \in H\}$ is a hyperplane in $X$ and the transformation $y \mapsto y+\langle h^*, y\rangle x+u$ is an affine homeomorphism. If $x \in X$, then the set $Q' = Q + [-x, x]$ is also bounded, closed, convex and spanning. We will show that it has empty interior. Suppose that the interior of $Q'$ is nonempty. Then $x \neq 0$; choose $x^* \in X^*$ so that $\langle x^*, x\rangle > 0$. Let $P$ be the projection on $X$ whose image is the kernel $H$ of $x^*$ and whose kernel is the span of $x$. The open mapping theorem says that $P(Q) = P(Q')$ has nonempty interior in $H$. Suppose that $w \in H$, $\delta > 0$ and $B(w, \delta) \cap H \subseteq P(Q)$. For $z \in B(w, \delta) \cap H$ define

$$f(z) := \inf\{t : z + tx \in Q\}.$$

It is easy to see that $f$ is bounded and convex, hence continuous. The mapping $(z, t) \mapsto z + tx$ is an isomorphism from $H \times R$ onto $X$ which maps the graph of $f$ onto the set $\{z + f(t)x : z \in B(w, \delta) \cap H\} \subset Q$. Because $Q$ has empty interior, Lemma 1 implies that $f$ must be affine, and we shall show that this leads to a contradiction. Since it is defined on an open convex subset of $H$, there exists an $h^* \in H^*$ and a real number $b$ such that $f(z) = \langle h^*, z\rangle + b$. Finally,

$$\{z + \langle h^*, z\rangle x + bx : z \in B(w, \delta) \cap H\} \subseteq Q$$

and this means that $Q$ contains a relatively open subset of a hyperplane, which is a contradiction. By induction, given $x_1, x_2, \ldots, x_n \in X$ we have that

$$Q + [-x_1, x_1] + \cdots + [-x_n, x_n]$$

has no interior point. The case of an arbitrary compact set $K$ can be verified by an application of (1). We give a few details. Suppose the interior of $Q + K$ is nonempty. By translating $Q+K$ if necessary we can suppose that $B(0, r) \subset Q+K$

for some $r > 0$. Choose a sequence $\{F_n\}$ of finite subsets of a ball in $X$ so that (1) holds. Choose $n_0 \in N$ so that

$$\sum_{i=n_0}^{\infty} 2^{-i} F_i \subset B(0, r/4). \tag{3}$$

Because the interior of the closed and convex set $Q_0 := Q + \sum_{i=1}^{n_0} 2^{-i} \mathrm{co} F_i$ is empty, there exists $v \in B_X(0, r)$ so that

$$\mathrm{dist}(v,\ Q_0) > r/2. \tag{4}$$

To see this choose a point $y \in B(0, r/4) \setminus Q_0$ and $x^*$ in the unit sphere of $X^*$ which separates $y$ from $Q_0$, namely $r/4 \geq \langle x^*, y\rangle \geq \langle x^*, u\rangle$ for any $u \in Q_0$. Choose $x \in B(0, r)$ so that $\langle x^*, x\rangle > 3r/4$. Then $v = x$ satisfies the required inequality. From (3) and (4) follows that

$$\mathrm{dist}(v, Q + \sum_{i=1}^{\infty} 2^{-i} F_i) \geq r/4,$$

which is a contradiction. □

With the hypothesis above, observe that if $T : X \to Y$ is a surjective linear operator with finite-dimensional kernel $F$, then $T(Q)$ is a bounded, closed and convex set with empty interior that is also spanning; this is because $T^{-1}(T(X)) = Q + F$ is a first category set.

In connection with the next theorem observe that the positive cone of $\ell_2$ is a closed convex set with empty interior which contains a translate of any finite subset $F$ of $\ell_2$. (Indeed, if for $x = \{x_n\} \in \ell_2$ we define $x^- = \{x_n^-\}$ so that $x_n^- = -x_n$ if $x_n < 0$ and $x_n^- = 0$ otherwise, then the set $F + \sum_{x \in F} x^-$ is contained in the positive cone.) However, as we will see later, because $\ell_2$ is reflexive it does not contain a closed convex set with empty interior containing a translate of every compact set. Hence the boundedness hypothesis in (iv) of the next theorem is needed.

**Theorem 3.** *Let $X$ be a Banach space. Then the following are equivalent*:

- (i) *there exists a convex and closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$; i.e. whenever $K$ is a compact subset of $X$ there exists $x_K \in X$ so that $K + x_K \subset Q$;*
- (ii) *there exists a convex and closed subset $P$ of $X$ with empty interior such that if $K$ is a compact subset of the unit ball of $X$, then there exists $x_K \in X$ so that $K + x_K \subset P$;*
- (iii) *there exists a convex, closed and bounded subset $C$ of $X$ with empty interior such that if $K$ is a compact subset of the unit ball of $X$, then there exists $x_K \in X$ so that $K + x_K \subset C$; and*
- (iv) *there exist a dense subset $E$ of the unit ball of $X$ and a convex, closed and bounded subset $D$ of $X$ with empty interior so that whenever $F$ is a finite set contained in $E$, there exists $x_F \in X$ so that $F + x_F \subset D$.*

*Proof.* Clearly (i) implies (ii). To prove that (ii) implies (iii), it is enough to show that there exists $1 \geq r > 0$ and $c > 0$ so that for any compact set $K \subset \bar{B}(0, r)$ there exist $z_K \in B(0, c)$ so that $K + z_K \subset P$, for then we may define

$$C := \frac{1}{r} \left(P \cap \bar{B}(0, r + c)\right).$$

For a contradiction, suppose that for every $n \in N$ there exists a compact set $K_n \subset \bar{B}(0, 1/n)$ so that

$$\text{if } K_n + x \subset P, \text{ then } \|x\| \geq n. \tag{5}$$

Define

$$K := \bigcup_{n=1}^{\infty} K_n \cup \{0\}.$$

The set $K$ is a compact subset of the unit ball, hence there exists $z \in X$ such that $K + z \subset P$. Because $K_n \subset K$ for $n \in N$, we have $\|z\| \geq n$ for all $n$, which is nonsense.

Let us show now that (iii) implies (i). We can suppose that zero is not contained in $C$ and define

$$Q := \bigcup_{\lambda \geq 0} \lambda C.$$

The set $Q$ is convex and contains translates of all compact sets in $X$. To show that $Q$ is closed, let $z \in X$, $x_n \in C$ and $\lambda_n \geq 0$ such that $\lim_{n\to\infty} \lambda_n x_n = z$ be given. Because the sequence $\{x_n\}$ is bounded away from zero, the sequence $\{\lambda_n\}$ is bounded, and consequently it has a converging subsequence $\lambda_{n_k} \to \lambda \geq 0$. If $\lambda = 0$, then from the boundedness of $C$ it follows that $z = 0 \in Q$. Otherwise the sequence $\{x_{n_k}\}$ converges to $z/\lambda$. Because $C$ is closed we get that $z = \lim_{k\to\infty} \lambda_{n_k} x_{n_k} = z \in \lambda C$. Finally, let us show that the set $Q$ has empty interior. Choose some $z \in C$. The set $\tilde{C} := C + [-z, 0]$ is closed and convex, and because $C$ is spanning $\tilde{C}$ has empty interior by Lemma 2. Since

$$Q = \bigcup_{\lambda \geq 0} \lambda C \subset \bigcup_{n \in N} n\tilde{C},$$

it follows from the Baire theorem that the interior of $Q$ is empty.

Clearly (iii) implies (iv), so let us show that the opposite implication also holds. Let $K$ be a compact subset of $B_X(0, 2^{-1})$. We will show that $K$ can be translated into $D$. Then $C := 2D$ will satisfy (iii). Let $\{F_n\}$ be a sequence of finite sets in $E$ so that (1) holds. Choose $z_n \in X$ so that $z_n + F_n \subset D$. Because $D$ is bounded, the sequence $\{z_n\}$ is bounded. If we define $z := \sum_{n=1}^{\infty} (1/2^n) z_n$, we get

$$z + K \subset z + \operatorname{cl} \sum_{n=1}^{\infty} 2^{-n} F_n \subset \operatorname{cl} \left( \sum_{n=1}^{\infty} 2^{-n} z_n + 2^{-n} F_n \right) \subset D,$$

where the last inclusion follows from the fact that $D$ is convex and closed. □

It should be remarked here that from the proof of equivalence of (i) and (iii) of the previous theorem it follows that if a Banach space $X$ contains a closed and convex set with empty interior containing the translates of all compacts, then $X$ contains a closed and convex cone $Q$ with empty interior which contains the translates of all compacts.

**Corollary 4.** *Let $Z$ be a Banach space and $Y$ be a separable subspace of $Z$. Let $Z$ contain a convex closed set $Q$ with empty interior which contains translates of all compact sets in $Z$. Then there exist a closed, separable and linear subspace $X$ of $Z$ containing $Y$ and a convex closed subset $P$ of $X$ with empty interior which contains translates of all compact sets in $X$.*

*Proof.* By Theorem 3 there exists a convex closed bounded subset $C$ of $Z$ with empty interior which contains translates of all compact subsets of $B_Z$. Using induction we construct an increasing sequence $\{X_n\}$ of closed separable subspaces of $Z$. Define $X_1 := Y$ and choose a dense countable subset $S_1$ of the unit ball of $X_1$. Choose a countable set $T_1 \subset Z$ such that whenever $F$ is a finite subset of $S_1$ there exists $v \in T_1$ for which $v + F \subset C$. Choose a countable set $C_1 \subset Z \setminus C$ such that $\overline{C_1} \supset C \cap X_1$. Suppose $X_n$, $S_n$, $T_n$ and $C_n$ for some $n \in N$ have been already constructed. Define

$$X_{n+1} := \overline{\operatorname{span}}(X_n \cup T_n \cup C_n),$$

and choose a countable dense subset $S_{n+1} \supset S_n$ of the unit ball of $X_{n+1}$. Choose a countable set $T_{n+1} \subset Z$ such that whenever $F$ is a finite subset of $S_{n+1}$ there exists $v \in T_{n+1}$ for which $v + F \subset C$. Choose a countable set $C_{n+1} \subset Z \setminus C$ such that $\overline{C_{n+1}} \supset C \cap X_{n+1}$. Define

$$X := \overline{\bigcup_{n=1}^{\infty} X_n} \quad \text{and} \quad D := \overline{\bigcup_{n=1}^{\infty} (X_n \cap C)}.$$

The set $E := \bigcup_{n=1}^{\infty} S_n$ is dense in $\bar{B}_X$ and from the construction it follows that any finite set contained in $E$ can be translated into $D$. The set $D$ is closed and convex, and it has empty interior because $\bigcup_{n=1}^{\infty} C_n \subset X \setminus D$ and $\overline{\bigcup_{n=1}^{\infty} C_n} \supset D$. An application of Theorem 3 completes the proof. □

The following lemma is essentially due to James [4].

**Lemma 5.** *Let $X$ be a nonreflexive Banach space. Then there exists a sequence $\{x_n\}$ in the unit ball of $X$ and $\varepsilon > 0$ so that for any finite-dimensional subspace $Y$ of $X$ there exists $n \in N$ so that*

$$\operatorname{dist}(Y,\ \operatorname{co}\{x_i\}_{i=n}^{\infty}) > \varepsilon.$$

*Proof.* The unit ball $\bar{B}_X$ of $X$ is not weakly compact, therefore by the Gantmacher-Smulyan theorem [3] there exists a decreasing sequence $\{C_n\}$ of nonempty, closed and convex subsets of $\bar{B}_X$ such that $\bigcap_{n=1}^{\infty} C_n = \emptyset$. We will show that there exist $\varepsilon > 0$ and a decreasing sequence of convex nonempty sets $\{D_n\}$ so that $D_n \subset C_n$ for $n \in N$ and for any compact set $K \subset X$ there exists $m \in N$ such that

$$(K + B(0, \varepsilon)) \cap D_m = \emptyset.$$

Suppose for a contradiction that the required sequence $\{D_n\}$ does not exist. Let $C_{1,n} := C_n$ for $n \in N$. There exists a compact convex set $K_1$ so that

$$C_{1,n} \cap (K_1 + B(0, 2^{-1})) \neq \emptyset \quad \text{for} \quad n \in N.$$

Let $C_{2,n} := C_{1,n} \cap (K_1 + B(0, 2^{-1}))$ for $n \in N$. In general, if the sequence $\{C_{k,n}\}$ and the compact convex set $K_k$ have been already constructed, define

$$C_{k+1,n} := C_{k,n} \cap (K_k + B(0, 2^{-k})) \quad \text{for} \quad n \in N,$$

and choose a compact convex set $K_{k+1}$ so that

$$C_{k+1,n} \cap (K_{k+1} + B(0, 2^{-(k+1)})) \neq \emptyset \quad \text{for} \quad n \in N.$$

Then $C_{k+1,n} \subset C_{k,n}$, and by induction $C_{k,n+1} \subset C_{k,n}$. In particular if we define $G_n := C_{n,n}$, then the sequence $\{G_n\}$ is decreasing, $G_n \subset C_n$ and

$$G_{n+1} \subset K_n + B(0, 2^{-n}).$$

Choose some $y_n \in G_n$. The sequence $\{y_n\}$ has a finite $\delta$-net for any $\delta > 0$. Therefore it has a converging subsequence. The limit point of this subsequence is contained in $\bigcap_{n=1}^{\infty} C_n$, which is a contradiction. Now that we have shown the existence of the sequence $\{D_n\}$, to finish the proof simply choose any $x_n \in D_n$. □

**Theorem 6.** *Let $Z$ be a Banach space. The following are equivalent*:

(i) *$Z$ is not reflexive*;

(ii) *there exist a nontrivial closed subspace $X$ of $Z$ and a convex closed subset $Q$ of $X$ with empty interior which contains translates of all compact sets in $X$, i.e. whenever $K$ is a compact subset of $X$ there exists $x_K \in X$ so that $K + x_K \subset Q$.*

*Moreover, if $Z$ is separable, then* (ii) *holds with $X = Z$.*

*Proof.* To show that (i) implies (ii) choose any separable nonreflexive subspace $X$ of $Z$; such a space exists by the Eberlein-Smulyan theorem. If $Z$ is separable let $X := Z$. Choose an increasing sequence $\{X_n\}$ of finite-dimensional subspaces of $X$ so that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$. Choose a sequence $\{x_n\}$ in the unit ball of $X$ and $\varepsilon > 0$ as in Lemma 5. By passing to a subsequence of $\{x_n\}$ if necessary we may suppose that

$$\operatorname{dist}\left(\operatorname{span}(X_n \cup \{x_i\}_{i=1}^{n}),\ \operatorname{co}\{x_i\}_{i=n+1}^{\infty}\right) > \varepsilon \quad \text{for} \quad n \in N. \tag{6}$$

Put $K_n := X_n \cap B_Z$, and define

$$D := \overline{\operatorname{co}} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i).$$

The convex, closed and bounded set $\tilde{D} := (4/\varepsilon)D$ contains a translate of any finite subset of $\bar{B}_X \cap \bigcup_{n=1}^{\infty} X_n$. By Theorem 3, it only remains to show that the interior of $\tilde{D}$ is empty. For a contradiction, suppose that the interior of $D$ is nonempty. Because $\operatorname{co} \bigcup_{i=1}^{\infty}(x_i + (\varepsilon/4)K_i)$ is dense in $D$ there exist $n \in N$, $\alpha_i \geq 0$ and $u_i \in (\varepsilon/4)K_i$, $i = 1, \dots n$, so that $\sum_{i=1}^{n} \alpha_i = 1$ and the point $z := \sum_{i=1}^{n} \alpha_i(x_i + u_i)$ is contained in the interior of $D$. From (6) it follows that there exists a point $x^*$ in the unit sphere of $X^*$ so that

$$\begin{array}{rcll} \langle x^*, x\rangle & = & 0 & \text{for} \quad x \in X_n, \\ \langle x^*, x\rangle & \leq & -\varepsilon/2 & \text{for} \quad x \in \operatorname{co}\{x_i\}_{i=n+1}^{\infty}. \end{array}$$

Choose a point $w$ in the unit sphere of $X$ for which

$$\langle x^*, w\rangle \geq 1/2.$$

Since $z$ is an interior point of $D$, there exists an $r > 0$ so that $z + rw \in D$. Consequently, there exist $m \in N$, $m > n$, $\beta_i \geq 0$ and $v_i \in (\varepsilon/4)K_i$, $i = 1, \dots m$, so that $\sum_{i=1}^{m} \beta_i = 1$ and if we define $y := \sum_{i=1}^{m} \beta_i(x_i + v_i)$, then

$$\|z + rw - y\| < r/2.$$

From the definition of $x^*$ it follows that

$$\begin{aligned} &\langle rw + z - y, x^*\rangle \\ &= r\langle w, x^*\rangle + \langle \textstyle\sum_{i=1}^{n} \alpha_i(x_i + u_i) - \beta_i(x_i + v_i), x^*\rangle - \langle \textstyle\sum_{i=n+1}^{m} \beta_i(x_i + v_i), x^*\rangle \\ &\geq r/2 + 0 - \textstyle\sum_{i=n+1}^{m} \beta_i(\langle x_i, x^*\rangle + \langle v_i, x^*\rangle) \\ &\geq r/2 - \textstyle\sum_{i=n+1}^{m} \beta_i(-\varepsilon/2 + \varepsilon/4) \\ &\geq r/2, \end{aligned}$$

which is a contradiction.

Now, let us prove that (ii) implies (i). By Corollary 4, we may suppose that $X$ is separable. We will show that $X$ is nonreflexive and therefore $Z$ is also nonreflexive. For a contradiction suppose that $X$ is reflexive. Choose a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ that is dense in the unit sphere of $X$. Denote

$$K_n := \operatorname{span}\{x_i\}_{i=1}^{n} \cap \bar{B}_X.$$

Clearly $\{K_n\}$ is an increasing sequence of compact subsets of the unit ball of $X$ for which

$$\overline{\bigcup_{n=1}^{\infty} K_n} = \bar{B}_X. \tag{7}$$

By Theorem 3 there exists a closed, convex and bounded subset $C$ of $X$ with empty interior which contains translates of all compact subsets of the unit ball of $X$. For $n \in N$ choose $z_n \in X$ so that $z_n + K_n \subset C$. The sequence $\{z_n\}$ is bounded, therefore it has a weakly converging subsequence $\{z_{n_k}\}$. Denote $z := w\text{-}\lim_{k\to\infty} z_{n_k}$. Because the set $C$ is convex and closed, it is also weakly closed. Consequently, because the sets $K_n$ are increasing, if there exists a $k \in N$ so that $y \in K_{n_k}$, then $y + z \in C$. Hence,

$$z + \bar{B}_X = z + \overline{\bigcup_{k=1}^{\infty} K_{n_k}} \subset C, \tag{8}$$

which, of course, means that the interior of $C$ is nonempty, which is a contradiction. □

## Acknowledgment

The authors thank the participants of the semiannual k&k analysis meeting and namely L. Zajíček for helpful discussions. We would also like to thank the referee for his close reading of the first submitted version of this paper.

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