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# QUASITRIANGULAR HOPF ALGEBRAS WHOSE GROUP-LIKE ELEMENTS FORM AN ABELIAN GROUP

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Abstract. In this paper we prove some properties of the set of group-like elements of $A$, $G(A)$, for a pointed minimal quasitriangular Hopf algebra $A$ over a field $k$ of characteristic 0, and for a pointed quasitriangular Hopf algebra which is indecomposable as a coalgebra. We first show that over a field of characteristic 0, for any pointed minimal quasitriangular Hopf algebra $A$, $G(A)$ is abelian. We show further that if $A$ is a quasitriangular Hopf algebra which is indecomposable as a coalgebra, then $G(A)$ is contained in $A_R$, the minimal quasitriangular Hopf algebra contained in $A$. As a result, one gets that over a field of characteristic 0, a pointed indecomposable quasitriangular Hopf algebra has a finite abelian group of group-like elements.

## Introduction

Cocommutative Hopf algebras have played a central role in the theory of Hopf algebras as they include the basic examples of group algebras and universal enveloping algebras. Basic to the theory of such algebras are the pointed irreducible ones, for which of course $G(A)$, the set of group-like elements of $A$, is the trivial group, in particular it is a finite abelian group. In recent years, extensive research has been carried out concerning a generalization of cocommutative Hopf algebras, that is, quasitriangular Hopf algebras. Here, as for cocommutative Hopf algebras, $G(A)$ is an important object, but the basic building blocks are the indecomposable subcoalgebras rather than the irreducible ones. Thus, a natural question to ask is:

If $(A, R)$ is a quasitriangular, pointed indecomposable Hopf algebra, is $G(A)$ finite abelian? In this paper we give a positive answer to the above question in characteristic 0 by showing that $G(A)$ is in fact contained in the so-called minimal quasitriangular Hopf algebra determined by $R$.

## 1. Preliminaries

In this section we discuss basic definitions and results used in the sequel. Basic references for the results on quasitriangular Hopf algebras is [R] and for indecomposable Hopf algebras are [K, M1, M2, ShM, XF].

A quasitriangular Hopf algebra over a field $k$ is a pair $(A, R)$ where $A$ is a Hopf algebra over $k$ and $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$ satisfies the following:

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1. $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$ $(r = R)$,
2. $\sum R^{(1)} \otimes \Delta(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
3. $\sum \varepsilon(R^{(1)}) R^{(2)} = \sum R^{(1)} \varepsilon(R^{(2)}) = 1$,
4. $R\Delta(a) = \Delta^{\text{cop}}(a)R$ for all $a \in A$.

($\Delta^{\text{cop}}(a) = \tau \circ \Delta(a)$ where $\tau$ is the twist map from $A \otimes A$ to $A \otimes A$). It follows that $R$ is invertible and $R^{-1} = \sum S(R^{(1)}) \otimes R^{(2)}$ where $S$ is the bijective antipode of $A$.

If $R = \sum_{i=1}^{m} x_i \otimes y_i$ where $m$ is as small as possible, then $H = \text{Sp}\{x_1, \dots, x_n\}$ and $B = \text{Sp}\{y_1, \dots, y_m\}$ are sub-Hopf algebras of $A$ that satisfy $HB = BH$ [R, Theorem 1]. The algebra $A_R = BH$ is a finite-dimensional sub-Hopf algebra of $A$ and is the smallest sub-Hopf algebra $C$ of $A$ such that $R \in C \otimes C$; furthermore, $B \cong H^{*\,\text{cop}}$ as Hopf algebra [R, Proposition 2]. A quasitriangular Hopf algebra $A$ is called minimal quasitriangular if $A = A_R$.

For a finite-dimensional Hopf algebra $H$, with antipode $S$, one can construct the Drinfeld-double $(D(H), R)$, which is the bicrossproduct $H^{*\,\text{cop}} \bowtie H$. As a coalgebra, $D(H) = H^{*\,\text{cop}} \otimes H$, and multiplication is given by:

$$(p \bowtie h)(p' \bowtie h') = \sum p(h_{(1)} \rightharpoonup p' \leftharpoonup S^{-1}(h_{(3)})) \bowtie h_{(2)} h'$$

for $p, p' \in H^*, h, h' \in H$ and actions defined by:

$$h \rightharpoonup p = \sum \langle h, p_{(2)} \rangle p_{(1)}, \qquad h \leftharpoonup p = \sum \langle p, h_{(1)} \rangle h_{(2)}.$$

If we consider $D(H)$, where $H$ is defined as above, then $A_R$ is a homomorphic image of $D(H)$.

A coalgebra $C$ is simple if $C$ has no proper subcoalgebras. It is cosemisimple if it is a direct sum of simple subcoalgebras. A coalgebra $C$ is indecomposable if $C$ cannot be written as a direct sum of proper subcoalgebras. For any coalgebra $C$, the set $G(C) = \{c \in C \mid c \neq 0 \text{ and } \Delta(c) = c \otimes c\}$ is called the set of group-like elements of $C$. A coalgebra is pointed if all its simple subcoalgebras are one-dimensional; they are of the form $k\sigma$ for some $\sigma \in G(C)$. If $A$ is a Hopf algebra, then $G(A)$ is a group, hence if $A$ is a cosemisimple pointed Hopf algebra, then it is a group algebra.

For $\sigma, \tau$ in $G(C)$ the set $P_{\sigma,\tau} = \{x \in C \mid \Delta(x) = x \otimes \sigma + \tau \otimes x\}$ is called the set of $\sigma - \tau$ primitive elements in $C$.

For any subspaces $S$ and $T$ of $C$ the wedge $S \wedge T$ is defined as $\Delta^{-1}(S \otimes T + T \otimes S)$. Let $S, T$ be simple subcoalgebras of $C$. We say that $S$ and $T$ are linked if $S \wedge T \neq T \wedge S$. This is equivalent to $S \wedge T$ or $T \wedge S$ properly contains $S \oplus T$ [M2, 1.8].

The coalgebra $C$ is called link-indecomposable if for any $S, T$ simple subcoalgebras of $C$, there exists a sequence of simple subcoalgebras $S = S_0, S_1, \dots, S_n = T$ such that $S_i$ and $S_{i+1}$ are linked, $0 \leq i \leq m - 1$. If $C$ is pointed, then each simple subcoalgebra has the form $k\sigma$, $\sigma \in G(C)$, and $k\sigma \wedge k\tau$ properly contains $k\sigma \oplus k\tau$ is equivalent to the existence of an $x$ in $P_{\sigma,\tau}$ so that $x \notin k\sigma \oplus k\tau$. This follows by the Taft-Wilson theorem [M1, 5.4.1].

**Theorem** ([M2, ShM, XF])**.** *$C$ is indecomposable iff $C$ is link-indecomposable.*

## 2. $G(A)$ IS ABELIAN

In this section we prove that under certain conditions on the Hopf algebra $A$, $G(A)$ is abelian. We first prove:

**Lemma.** *Let $H$ be a finite-dimensional Hopf algebra such that* $\mathrm{char}(k) \nmid |G(H)|$. *If $H^*$ is pointed, then $G(H)$ is abelian.*

*Proof.* Recall that every homomorphic image of a pointed coalgebra is pointed [M1, 5.3.5]. Since $kG(H) \subseteq H$, it follows that $(kG(H))^*$ is a homomorphic image of $H^*$, thus by the above $(kG(H))^*$ is pointed. If $\mathrm{char}(k) \nmid |G|$, then $kG(H)$ is a semisimple algebra, hence $(kG(H))^*$ being a pointed cosemisimple Hopf algebra is a group-algebra, hence cocommutative, and this implies that $kG(H)$ is commutative. Note that by [NZ] $\mathrm{char}(k) \nmid \dim(H)$ implies $\mathrm{char}(k) \nmid |G(H)|$.

We are now ready to prove our first theorem.

**Theorem 1.** *Let $(A, R)$ be a pointed minimal quasitriangular Hopf algebra, and assume* $\mathrm{char}(k) \nmid |G(A)|$. *Then $G(A)$ is abelian.*

*Proof.* Let $B$ and $H$ be as in the preliminaries. Then $B \cong H^{*\,\mathrm{cop}}$, $A = HB = BH$, and $A$ is a homomorphic image of $D(H)$. Now, $\mathrm{char}(k) \nmid |G(H)|$ and $\mathrm{char}(k) \nmid |G(H^*)|$ as $G(H)$ and $G(H^*)$ are subgroups of $G(A)$. Since any subcoalgebra of a pointed coalgebra is pointed, we have that $B = H^*$ and $H$ are pointed. Using the above lemma twice, for $H$ and for $H^*$, we get that $G(H^*)$ and $G(H)$ are abelian. By [R, Proposition 10] $G(D(H)) = G(H^*) \times G(H)$, hence $G(D(H))$ is abelian. Moreover, $D(H)$ is pointed by [M1, 5.1.10] and the fact that as a coalgebra $D(H) = H^{*\,\mathrm{cop}} \otimes H$. Since $A$ is a homomorphic image of the pointed $D(H)$, $G(A)$ is a homomorphic image of $G(D(H))$ ([A, 2.3.12]; [M1, 5.3.5]), hence $G(A)$ is abelian. Q.E.D.

*Remark.* The requirements appearing in the theorem are all necessary. (1) The Drinfeld-double of $k(G)$, where $G$ is a non-abelian $p$-group and $\mathrm{char}(k) = p$, is minimal quasitriangular pointed (since $k(G)^*$ is pointed by the fact that $k(G)$ has a unique maximal ideal of codimension 1). (2) The Drinfeld-double of any non-commutative $k(G)$ over a field $k$ of characteristic 0 is minimal quasitriangular but not pointed (since $k(G)^*$ is not pointed). (3) For any non-abelian group $G$, $k(G)$ is pointed quasitriangular but not minimal. In all cases $G(H)$ is not abelian.

Next we prove

**Theorem 2.** *Let $(A, R)$ be an indecomposable pointed quasitriangular Hopf algebra. Then $G(A) \subseteq G(A_R)$. In particular, $G(A)$ is finite.*

*Proof.* Assume $\sigma \in G(A) - G(A_R)$. Since $A$ is link indecomposable and pointed, there exists a sequence $1 = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma$, where $\sigma_i$ and $\sigma_{i+1}$ are linked.

But $1 \in A_R$ and $\sigma \notin A_R$, hence $\exists i, \sigma_i \in A_R$ while $\sigma_{i+1} \notin A_R$. Since $\sigma_i$ and $\sigma_{i+1}$ are linked, $\exists x, x \notin k\sigma_i \oplus k\sigma_{i+1}$ and $\Delta(x) = x \otimes \sigma_i + \sigma_{i+1} \otimes x$.

Thus $\Delta(\sigma_{i+1}^{-1}x) = \sigma_{i+1}^{-1}x \otimes 1 + \sigma_{i+1}^{-1}\sigma_i \otimes \sigma_{i+1}^{-1}x$.

Since $\sigma_i \in A_R$ and $\sigma_{i+1} \notin A_R$, $\sigma_{i+1}^{-1}\sigma_i \notin A_R$ and it is linked to 1. So without loss of generality we may assume that $\exists \sigma, \sigma \notin A_R$ and $\exists p, p \notin k \oplus k\sigma$ with $\Delta(p) = p \otimes 1 + \sigma \otimes p$. Take $R = \sum x_i \otimes y_i$ with $\{x_i\}, \{y_i\}$ linearly independent; then

$$R\Delta(p)R = \Delta^{\mathrm{cop}}(p),$$

thus

$$\sum (x_i \otimes y_i)(p \otimes 1 + \sigma \otimes p)(Sx_j \otimes y_j) = 1 \otimes p + p \otimes \sigma,$$

that is

$$(*)\qquad \sum x_i p S x_j \otimes y_i y_j + x_i \sigma S x_j \otimes y_i p y_j = 1 \otimes p + p \otimes \sigma.$$

First show that $\sigma$ and $p$ are linearly independent $\bmod A_R$. Assume $\alpha\sigma + \beta p \in A_R$ for some $\alpha, \beta \in k$; then $\Delta(\alpha\sigma + \beta p) \in A_R \otimes A_R$. But

$$(**)\qquad \Delta(\alpha\sigma + \beta p) = \alpha\sigma \otimes \sigma + \beta p \otimes 1 + \beta\sigma \otimes p = \sigma \otimes (\alpha\sigma + \beta p) + \beta p \otimes 1 \in A_R \otimes A_R.$$

By our assumption $p \notin k \oplus k\sigma$, hence $\{\alpha\sigma + \beta p, 1\}$ are linearly independent. Take $f \in A^*$ with $f(1) = 0, f(\alpha\sigma + \beta p) = 1$; then applying $\mathrm{id} \otimes f$ to $(**)$, one gets that $\sigma \in A_R$, a contradiction. So $\sigma$ and $p$ are linearly independent $\bmod A_R$.

Thus we can take $f \in A^*$ with $f(A_R) = 0, f(\sigma) = 0$ and $f(p) = 1$. Applying $\mathrm{id} \otimes f$ to $(*)$, one gets that $1 \in \mathrm{Sp}\{x_i \sigma S x_j\}$, $i, j = 1, \ldots, m$, and hence that $\sigma^{-1} \in \mathrm{Sp}\{\sigma^{-1} x_i \sigma S x_j\}$.

Our last step is to show that for any $i$, $\sigma^{-1} x_i \sigma \in A_R$.

Using $R\Delta(\sigma) = \Delta^{\mathrm{cop}}(\sigma) R$, we have

$$\sum \sigma^{-1} x_i \sigma \otimes \sigma^{-1} y_i \sigma = \sum x_i \otimes y_i.$$

Since the set of $\{y_i\}$ is linearly independent, it follows that the $\{\sigma^{-1} y_i \sigma\}$ are linearly independent, but this and the above equality imply that $\forall i, \sigma^{-1} x_i \sigma \in \mathrm{Sp}\{x_j\}$, $i, j = 1, \ldots, m$. Since $\sigma^{-1} \in \mathrm{Sp}\{\sigma^{-1} x_i \sigma S x_j\}$ contained in $A_R$, we get that $\sigma \in A_R$, a contradiction. Q.E.D.

As an immediate corollary of the above theorems we have

**Theorem 3.** *Let $(A, R)$ be a pointed quasitriangular indecomposable Hopf algebra over a field of characteristic $0$. Then $G(A)$ is a finite abelian group.*


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