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BEST BOUNDS FOR THE APPROXIMATE UNITS FOR CERTAIN IDEALS OF $L^1(G)$ AND OF $A_p(G)$

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Abstract. We compute the best bound for the approximate units of the augmentation ideal of the group algebra $L^1(G)$ of a locally compact amenable group G. More generally such a calculation is performed for the kernel of the canonical map from $L^1(G)$ onto $L^1(G/H)$, H being a closed amenable subgroup of G. Analogous results involving certain ideals of the Fourier algebra of an amenable group are also discussed.

1. Introduction

Let $T_{H,q}$ be the canonical map from $L^1(G)$ onto $L^1(G/H)$ where H is a closed subgroup of a locally compact group G. In 1968, Reiter [15] proved that if H is amenable the kernel of $T_{H,q}$ admits bounded approximate right units. He showed moreover that for $H = G$ this property characterizes the amenability of H. In 1978 the second author [3] obtained that this is also the case for a large class of subgroups of G (including all lattices in G). But at the present time a full converse is still in doubt.

In *loco citato* Reiter more precisely proved that the amenability of H implies the existence of approximate right units for $\ker T_{H,q}$ bounded by 2. One of the main results of this work is that 2 is the **best bound** if H is infinite. For H finite the best bound is $\dfrac{2(|H|-1)}{|H|}$. We also investigate the corresponding results for the Fourier algebra $A(G)$ of a locally compact **amenable** group G. The best bound for approximate units of the ideal $I(H)$ of all $u \in A(G)$ vanishing on a closed normal subgroup H of G is 2 if G/H is infinite. It is $\dfrac{2(|G/H|-1)}{|G/H|}$ otherwise (for H open in G it is not necessary to assume the normality of H in G!).

In section 2 we essentially develop the tools which permit estimates from above and from below for the bounds of approximate units. In section 3 we obtain new bounds for ideals in $L^1(G)$ of the form $T_H^{-1}(I)$, and section 4 is devoted to the corresponding results in the Figà-Talamanca Herz algebra $A_p(G)$ (recall that $A_2(G) = A(G)$). Our main results (Theorems 5, 10 and 11) are contained in sections 5 and 6.

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2. Bounded approximate units and projections

We collect in this rather technical section some results concerning an arbitrary normed algebra A. We will apply them later to $L^1(G)$ and $A_p(G)$.

The dual A^* of A carries a right A-module structure given by $(fa)(b) = f(ab)$ for $f \in A^*$ and $a, b \in A$. We denote by $\mathrm{Hom}_A A^*$ the Banach algebra of all bounded linear operators T of A^* with $T(fa) = (Tf)a$ for $f \in A^*$ and $a \in A$. For $A = L^1(G)$, $\mathrm{Hom}_A A^*$ is the algebra $\mathrm{Hom}_{L^1(G)} L^\infty(G)$ of all bounded operators T of $L^\infty(G)$ with $T(f * \varphi) = f * T\varphi$ for $f \in L^1(G)$, $\varphi \in L^\infty(G)$ (to $\varphi \in L^\infty(G)$, we associate F_φ the linear functional on $L^1(G)$ defined by $F_\varphi(f) = \int_G f(x)\overline{\varphi(x)}dx$ for $f \in L^1(G)$; the above right $L^1(G)$-module structure on $L^1(G)^*$ is given by $F_\varphi g = F_{g^* * \varphi}$ where $g^*(x) = \overline{g(x^{-1})}\Delta_G(x^{-1})$).

If $A = A_p(G)$, A^* is the space $PM_p(G)$ of all p-pseudomeasures on G and $\mathrm{Hom}_A A^*$ is the Banach algebra $\mathrm{Hom}_{A_p(G)} PM_p(G)$ of all bounded linear operators Φ of $PM_p(G)$ with $\Phi(uS) = u\Phi(S)$ for all $u \in A_p(G)$ and $S \in PM_p(G)$.

Proposition 1. *Let I be a closed left ideal of A having approximate right units bounded by $C \geq 0$. Then there exists a projection $P \in \mathrm{Hom}_A A^*$ from A^* onto $I^\perp$ with $||\,\mathrm{Id} - P|| \leq C$.*

Proposition 2. *Let $C \geq 0$. Assume that A admits two-sided approximate units bounded by C. Let I be a closed left ideal of A. Assume the existence of $P \in \mathrm{Hom}_A A^*$ which is a projection from A^* onto $I^\perp$. Then for $u \in A$, $v \in I$, $f \in A^*$ and $\varepsilon > 0$ there is $w \in I$ with $||w|| \leq C||\,\mathrm{Id} - P||$, $||v - vw|| < \varepsilon$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.*

Both propositions are essentially known. The commutative case is due to Lust-Piquard [13, pp. 7 and 15] and the general case to Forrest [7, Proposition 6.4, p. 17]. However the estimates of $||\,\mathrm{Id} - P||$ (Prop. 1) and $||w||$ (Prop. 2) being probably new and *a fortiori* the condition involving f, u and P, we present a complete proof of Proposition 2.

If $C = 0$, then $A = \{0\}$. Suppose $C > 0$ and $||\,\mathrm{Id} - P|| = 0$. We have $I = \{0\}$; it suffices to choose $w = 0$. We therefore suppose $C > 0$ and $||\,\mathrm{Id} - P|| > 0$. Let F be a finite nonempty subset of A^* and $\eta > 0$. We denote by $E(F, \eta)$ the set of all $w \in I$ with $||w|| \leq C||\,\mathrm{Id} - P||$, $|g(v) - g(vw)| < \eta$ for every $g \in F$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.

We first show that $E(F, \eta) \neq \emptyset$. There exists $u_1 \in A$ with $||u_1|| \leq C$ and

$$||u - uu_1|| < \frac{\varepsilon}{2(1 + ||(\mathrm{Id} - P)f||)},$$
$$||v - vu_1|| < \frac{\eta}{4(1 + \max_{g \in F} ||Pg||)(1 + \max_{g \in F} ||g||)},$$
$$||v - u_1 v|| < \frac{\eta}{4(1 + \max_{g \in F} ||Pg||)}.$$

For $g \in I^*$, we set $\gamma(g) = g_1(u_1) - (Pg_1)(u_1)$ where $g_1 \in A^*$ is such that the restriction of g_1 to I is g. We have $\gamma \in I^{**}$ and $||\gamma|| \leq C||\,\mathrm{Id} - P||$. By the theorem of Goldstine, there is $w \in I$ with $||w|| \leq ||\gamma||$, $|(fu)(w) - \gamma(\mathrm{Res}_I(fu))| < \frac{\varepsilon}{2}$ and $|(gv)(w) - \gamma(\mathrm{Res}_I(gv))| < \frac{\eta}{4}$ for every $g \in F$. We obtain

$$|f(uw) - f(uu_1) + (Pf)(uu_1)| < \frac{\varepsilon}{2}$$

and therefore $|f(uw) - f(u) + (Pf)(u)| < \frac{\varepsilon}{2} + |(f - Pf)(uu_1 - u)| < \varepsilon$.

For $g \in F$, we have $|g(vw) - g(vu_1) + (Pg)(vu_1)| < \frac{\eta}{4}$. Taking into account that $u_1 v \in I$ and $(Pg)(u_1 v) = 0$, we have

$$\begin{aligned} &|g(vw) - g(v)| \\ &\leq |g(vw) - g(vu_1) + (Pg)(vu_1)| + |g(vu_1) - g(v)| + |Pg(u_1 v) - (Pg)(vu_1)| \\ &< \frac{\eta}{4} + ||g||\,||vu_1 - v|| + ||Pg||\,||u_1 v - vu_1|| < \eta. \end{aligned}$$

This proves that $w \in E(F, \eta)$.

Let B be the set $\bigcup\{vE(F,\eta) \mid F$ nonempty finite subset of A^* and $\eta > 0\}$. It is clear that v lies in the $\sigma(A, A^*)$-closure of B in A. Therefore v lies in the norm closure in A of the convex hull of B. Consequently we can find $m \in \mathbb{N}$, $F_1, \ldots, F_m$ finite nonempty subsets of A^*, $\eta_1, \ldots, \eta_m, c_1, \ldots, c_m > 0$, $w_1, \ldots, w_m \in I$ such that $c_1 + \ldots + c_m = 1$, $w_j \in E(F_j, \eta_j)$ $(1 \leq j \leq m)$ and $||v - \sum_{j=1}^{m} c_j v w_j|| < \varepsilon$. Consider $w = \sum_{j=1}^{m} c_j w_j$, indeed we obtain $w \in I$, $||w|| \leq C||\operatorname{Id} - P||$, $||v - vw|| < \varepsilon$ and $|f(uw) - f(u) + (Pf)(u)| < \varepsilon$.

We say that $C \geq 0$ is a bound of approximate right units of A if for every $\varepsilon > 0$ and every $a \in A$ there is a $b \in A$ with $||a - ab|| < \varepsilon$ and $||b|| \leq C$. Let $\mathcal{C}$ be the set of all bounds of approximate right units. Then the infimum D of $\mathcal{C}$ also is a bound of approximate right units. Let $\varepsilon > 0$ and $a \in A$; there is $C \in \mathcal{C}$ with $C < D + \eta$ where $0 < \eta < \frac{\varepsilon}{1 + ||a||}$; there is also $b \in A$ with $||b|| \leq C$ and $||a - ab|| < \eta$, we have $||a - ab_1|| < \varepsilon$ and $||b_1|| \leq D$ for $b_1 = \frac{Db}{D + \eta}$. We call D the *best bound for the approximate right units of* A.

Assume that A admits two-sided approximate units bounded by one. Let I be a closed left ideal of A having bounded approximate right units. Then the best bound for the approximate right units of I is $\min\{||\operatorname{Id} - P|| \mid P \in \operatorname{Hom}_A A^*$, P is a projection from A^* onto $I^{\perp}\}$.

3. Bounds for approximate units of $T_H^{-1}(I)$

Let q be a continuous strictly positive function on G with

$$q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1}) \quad \text{for all } x \in G \text{ and } h \in H.$$

We choose measures $dx, dh, d_q\dot{x}$ on $G, H, G/H$ as in [14, p. 158]. For every $f \in L^1(G)$, we define $T_{H,q}f(\dot{x}) = \int_H \frac{f(xh)}{q(xh)} dh$ where $\dot{x} = xH = \omega(x)$. When we can choose $q = 1$ (this is the case if H is normal in G) we will write T_H instead of $T_{H,1}$.

Theorem 3. *Let H be a closed normal amenable subgroup of G and I a closed left ideal of $L^1(G/H)$ having approximate right units bounded by $C \geq 0$. Then $T_H^{-1}(I)$ has approximate right units bounded by $C + 2$.*

The existence of bounded right units in $T_H^{-1}(I)$ is due to Reiter [16, p. 70]. He found for $T_H^{-1}(I)$ the bound $3C + 5$ (see pp. 31-33). Later, Doran and Wichman [6, pp. 43–44] obtained the bound $3C + 2$ with the same method.

For $F \in C^b_{\ell u}(G)^*$ ($C^b_{\ell u}(G)$ is the Banach space of all bounded left uniformly continuous functions on G) the relation

$$\langle f, \tau_G(F)t\rangle_{L^1(G),L^\infty(G)} = \int_G f(x)\overline{(\tau_G(F)t)(x)}dx = \overline{F(f^* * t)}$$

for $f \in L^1(G)$ and $t \in L^\infty(G)$ defines an element $\tau_G(F)$ of $\operatorname{Hom}_{L^1(G)} L^\infty(G)$. We recall that τ_G is a Banach algebra isomorphism from $C^b_{\ell u}(G)^*$ (with the Arens product) onto $\operatorname{Hom}_{L^1(G)} L^\infty(G)$; for $\Phi \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$ and $t \in C^b_{\ell u}(G)$, we also have $\tau_G^{-1}(\Phi)(t) = \Phi(t)(e)$.

Let M be a left-invariant mean on $C^b_{\ell u}(H)$. For $\varphi \in C^b_{\ell u}(G)$ and $x \in G$, we put $\gamma(\varphi)(\dot{x}) = M(\varphi_{x,H})$, where $\varphi_{x,H}(h) = \varphi(xh)$ for $h \in H$. It is straightforward to verify that $\alpha = \tau_G \circ \gamma^* \circ \tau_{G/H}^{-1}$ is a Banach algebra isometry from $\operatorname{Hom}_{L^1(G/H)} L^\infty(G/H)$ into $\operatorname{Hom}_{L^1(G)} L^\infty(G)$ (note the analogy with [1, Theorem 8, p. 501]). But in general $\alpha(\operatorname{Id}_{L^\infty(G/H)}) \neq \operatorname{Id}_{L^\infty(G)}$! According to Proposition 1 there is $P \in \operatorname{Hom}_{L^1(G/H)} L^\infty(G/H)$, a projection from $L^\infty(G/H)$ onto $I^\perp$, with $\|\operatorname{Id}_{L^\infty(G/H)} - P\| \le C$. Let $t \in C^b_{\ell u}(G)$. For $f \in T_H^{-1}(I)$ we have

$$\langle f, \alpha(P)(t)\rangle_{L^1(G),L^\infty(G)} = \langle T_H f, P(\gamma(t))\rangle_{L^1(G/H),L^\infty(G/H)} = 0$$

and therefore $\alpha(P)(t) \in T_H^{-1}(I)^\perp$. From this we deduce $\alpha(P)(t) \in T_H^{-1}(I)^\perp$ for every $t \in L^\infty(G)$.

Let $t \in T_H^{-1}(I)^\perp$; for $f, g \in L^1(G)$ we have

$$\langle g * f, \alpha(P)t\rangle_{L^1(G),L^\infty(G)} = \langle T_H f, P(\gamma(g^* * t))\rangle_{L^1(G/H),L^\infty(G/H)} .$$

There is $u \in I^\perp \cap C^b_{\ell u}(G/H)$ with $u \circ \omega = g^* * t$, and therefore

$$\begin{aligned}\langle g * f, \alpha(P)t\rangle_{L^1(G),L^\infty(G)} &= \langle T_H f, Pu\rangle_{L^1(G/H),L^\infty(G/H)} \\ &= \langle T_H f, u\rangle_{L^1(G/H),L^\infty(G/H)} = \langle g * f, t\rangle_{L^1(G),L^\infty(G)} ,\end{aligned}$$

so we obtain $\alpha(P)t = t$. We have proved that $\alpha(P)$ is a projection from $L^\infty(G)$ onto $T_H^{-1}(I)^\perp$. The inequality

$$\|\operatorname{Id}_{L^\infty(G)} - \alpha(P)\| \le \|\operatorname{Id}_{L^\infty(G)} - \alpha(\operatorname{Id}_{L^\infty(G/H)})\| + \|\operatorname{Id}_{L^\infty(G/H)} - P\|$$

permits us to conclude.

It is possible to avoid Propositions 1, 2 and the use of $\operatorname{Hom}_{L^1(G)} L^\infty(G)$. Nevertheless the following more direct proof gives perhaps less insight into the question.

Let $f \in T_H^{-1}(I)$ and $\varepsilon > 0$. There is $u \in L^1(G)$ with $\|f - f * u\|_1 < \dfrac{\varepsilon}{C+5}$ and $\|u\|_1 = 1$. By assumption there is $r \in I$ with $\|T_H f - T_H f * r\|_1 < \dfrac{\varepsilon}{C+5}$ and $\|r\|_1 \le C$. Choose $s \in T_H^{-1}(I)$ with $T_H s = r$. Denote by $\mathcal{A}_H$ the convex hull of $\{A_h \mid h \in H\}$ where $(A_h\varphi)(x) = \varphi(xh)\Delta_G(h)$ for $\varphi \in \mathbb{C}^G$, $x \in G$ and $h \in H$. Using the amenability of H we can find $A \in \mathcal{A}_H$ with $\|As\|_1 < \|r\|_1 + \eta$ where $\eta = \dfrac{\varepsilon}{(C+5)(1+\|f\|_1)}$ (see [14, p. 174]). There is also $B \in \mathcal{A}_H$ with

$$\|B(f * u - f * u * As)\|_1 < \frac{\varepsilon}{C+5} + \|T_H(f * u - f * u * As)\|_1 .$$

We have $u - Bu + u * BAs \in T_H^{-1}(I)$, $||f - f * (u - Bu + u * BAs)||_1 < \dfrac{\varepsilon(C+4)}{C+5}$ and $||u - Bu + u * BAs||_1 < 2 + C + \eta$. It suffices now to put

$$k = \frac{C+2}{C+2+\eta}(u - Bu + u * BAs)$$

to conclude $||k||_1 \leq C + 2$ and $||f - f * k||_1 < \varepsilon$.

4. Analogous result for the Figà-Talamanca Herz algebra $A_p(G)$

Theorem 4. *Let G be an amenable locally compact group, H a closed normal subgroup of G and I a closed ideal of $A_p(H)$. We assume that I has approximate units bounded by C ($C \geq 0$). Then the closed ideal $\{u \in A_p(G) \mid \mathrm{Res}_H u \in I\}$ has approximate units bounded by $C + 2$.*

For $p = 2$, the special case $I = \{0\}$, with the bound 3, was already obtained by Forrest [7, p. 6, Proposition 3.7]. More recently, Forrest [8, Proposition 3.4] treated also the corresponding result in $A_p(G)$ with a less explicit and certainly less precise constant.

Instead of using $C^b_{\ell u}(G)$, we consider $cv_p(G)$, the norm closure in the space $\mathcal{L}(L^p(G))$ of all bounded operators of all p-convolution operators with compact support. Let $\mathcal{P}$ be the map from $\mathcal{L}(L^p(G))$ into $\mathcal{L}(L^p(H))$ constructed in Theorem 3 of [2]. Let also σ_G be the canonical Banach algebra isometry from $cv_p(G)^*$ onto $\mathrm{Hom}_{A_p(G)} PM_p(G)$ (see for example [1, p. 501]). The map $\lambda = \sigma_G \circ \mathcal{P}^* \circ \sigma_H^{-1}$ is a Banach algebra isometry from $\mathrm{Hom}_{A_p(H)} PM_p(H)$ into $\mathrm{Hom}_{A_p(G)} PM_p(G)$. In analogy with the L^1-case, we have in general $\lambda(\mathrm{Id}_{PM_p(H)}) \neq \mathrm{Id}_{PM_p(G)}$!

Let i be the canonical map from $PM_p(H)$ into $PM_p(G)$ defined in [4, p. 76]. Then $i(I^\perp)$ coincides with $J^\perp$ where $J = \{u \in A_p(G) \mid \mathrm{Res}_H u \in I\}$. To verify this, consider $T \in J^\perp$, the support of T lies in H. According to [12, p. 190, Théorème 5], there is $S \in PM_p(H)$ such that $i(S) = T$. We obtain $S \in I^\perp$. Conversely assume that $T = i(S)$ with $S \in I^\perp$. For $v \in J$ we have $\langle v, T\rangle_{A_p(G),PM_p(G)} = \langle \mathrm{Res}_H v, S\rangle_{A_p(H),PM_p(H)} = 0$ and thus $T \in J^\perp$.

For $\Phi \in \mathrm{Hom}_{A_p(H)} PM_p(H)$ we have $\lambda(\Phi) = i \circ \Phi \circ \mathcal{P}$. Take indeed $u \in A_p(G)$ and $T \in PM_p(G)$; then

$$\begin{aligned}\langle u, \lambda(\Phi)T\rangle_{A_p(G),PM_p(G)} &= \overline{\mathcal{P}^*(\sigma_H^{-1}(\Phi))(uT)} = \overline{\sigma_H^{-1}(\Phi)(\mathrm{Res}_H u\, \mathcal{P}(T))} \\ &= \langle \mathrm{Res}_H u, \Phi(\mathcal{P}(T))\rangle_{A_p(H),PM_p(H)} \\ &= \langle u, i(\Phi(\mathcal{P}(T)))\rangle_{A_p(G),PM_p(G)}\,.\end{aligned}$$

There exists $P \in \mathrm{Hom}_{A_p(H)} PM_p(H)$, a projection from $PM_p(H)$ onto $I^\perp$, with $||\mathrm{Id}_{PM_p(H)} - P|| \leq C$. The map $\lambda(P)$ is a projection from $PM_p(G)$ onto $J^\perp$. Let $T \in PM_p(G)$. For $w \in J$,

$$\langle w, \lambda(P)(T)\rangle_{A_p(G),PM_p(G)} = \langle \mathrm{Res}_H w, P(\mathcal{P}(T))\rangle_{A_p(H),PM_p(H)} = 0.$$

Therefore $\lambda(P)(T) \in J^\perp$. Let $T \in J^\perp$. There is $S \in I^\perp$ with $T = i(S)$. For

$w \in A_p(G)$ we have

$$\begin{aligned}\langle w, \lambda(P)(T)\rangle_{A_p(G),PM_p(G)} &= \langle \operatorname{Res}_H w, P(\mathcal{P}(i(S)))\rangle_{A_p(H),PM_p(H)}\\ &= \langle \operatorname{Res}_H w, P(S)\rangle_{A_p(H),PM_p(H)}\\ &= \langle \operatorname{Res}_H w, S\rangle_{A_p(H),PM_p(H)}\\ &= \langle w, i(S)\rangle_{A_p(G),PM_p(G)}\ .\end{aligned}$$

Finally, the inequality $|| \operatorname{Id}_{PM_p(G)} - \lambda(P)|| \leq 2 + C$ permits us as above to conclude.

It is also possible to write another more direct proof:

Let $u \in J$ and $\varepsilon > 0$. There is $v \in A_p(G)$ with $||v||_{A_p(G)} = 1$ and $||u - uv||_{A_p(G)} < \frac{\varepsilon}{4}$. By assumption, there exists $w \in I$ with $|| \operatorname{Res}_H(uv) - w \operatorname{Res}_H(uv)||_{A_p(H)} < \frac{\varepsilon}{4}$ and $||w||_{A_p(H)} \leq C$. Using [9, p. 92, Theorem 1b] there is $a \in A_p(G)$ with $\operatorname{Res}_H a = w$ and $||a||_{A_p(G)} < ||w||_{A_p(H)} + \eta$ where $0 < \eta < \min\{1, \frac{\varepsilon}{4(C+3)(1+||u||_{A_p(G)})}\}$. By [5, p. 102, Proposition 10] there is $b \in A_p(G/H)$ with $b(\dot{e}) = 1$, $||b||_{A_p(G/H)} < 1+\eta$ and $||b\circ\omega(uv - uva)||_{A_p(G)} < \frac{\varepsilon}{4} + || \operatorname{Res}_H(uv - uva)||_{A_p(H)}$. Let $d = v - b\circ\omega\, v + b \circ \omega\, va$. We have $d \in J$; from

$$||u - ud||_{A_p(G)} \leq ||u - uv||_{A_p(G)} + ||b \circ \omega(uv - uva)||_{A_p(G)},$$

we deduce $||u - ud||_{A_p(G)} < \frac{3\varepsilon}{4}$. Moreover we have $||d||_{A_p(G)} < C + 2 + 2\eta + \eta C + \eta^2$. Consider $f = \frac{(C+2)d}{C+2+2\eta+\eta C+\eta^2}$. We have $f \in J$ and $||f||_{A_p(G)} \leq C + 2$. From $||u - uf||_{A_p(G)} \leq ||u - ud||_{A_p(G)} + ||ud - uf||_{A_p(G)}$ and

$$||ud - uf||_{A_p(G)} \leq (2\eta + \eta C + \eta^2)||u||_{A_p(G)}$$

we obtain $||ud - uf||_{A_p(G)} < \frac{\varepsilon}{4}$ and finally $||u - uf||_{A_p(G)} < \varepsilon$.

5. Best bound for the approximate units of the ideal $\ker T_{H,q}$

Theorem 5. *Let H be a closed amenable subgroup of G. The best bound for the right approximate units of $\ker T_{H,q}$ is 2 if H is infinite. It is $\frac{2(|H|-1)}{|H|}$ otherwise.*

This theorem is a consequence of the following two propositions.

Proposition 6. *Let $P \in \operatorname{Hom}_{L^1(G)} L^\infty(G)$ which is a projection from $L^\infty(G)$ onto $\ker {T_{H,q}}^\perp$, H being a closed noncompact subgroup of G. Then we have*

1) *$P(f) = 0$ for all $f \in C_0(G)$ (the set of all continuous functions on G vanishing at infinity).*
2) *$|| \operatorname{Id} - P|| \geq 2$.*

The existence of P is, for H amenable, a consequence of Proposition 1.

To prove 1), observe first that $P(t) \in C^b_{\ell u}(G)$ for $t \in C^b_{\ell u}(G)$. For every $r \in C_{00}(G)$ (i.e. r is a continuous function with compact support on G), $t \in C^b_{\ell u}(G)$, $h \in H$, we have $\langle r - r_{h^{-1}}\Delta_G(h^{-1}), P(t)\rangle_{L^1(G),L^\infty(G)} = 0$ (for $a, x \in G$, $\varphi \in \mathbb{C}^G$,

${}_a\varphi(x) = \varphi(ax)$, $\varphi_a(x) = \varphi(xa)$). This implies $P(t)_h = P(t)$ and consequently $P({}_h t)(e) = P(t)(e)$. Now let $f \in C_{00}(G)$. There is a sequence $(h_n)_{n=1}^{\infty} \subset H$ such that $\mathrm{supp}_{h_n} f \cap \mathrm{supp}_{h_m} f = \emptyset$ for $n \neq m$. For $N \in \mathbb{N}$ we have

$$\left| P\left(\sum_{k=1}^{N} {}_{h_k} f \right)(e) \right| = |N(Pf)(e)| \leq ||P||\, ||f||_\infty \ ,$$

which implies $P(f)(e) = 0$. Consequently, for every $x \in G$, $P({}_x f)(e) = 0$ and therefore $P(f) = 0$.

To prove 2), it suffices to choose $f \in C_{00}(G)$ with $0 \leq f \leq 1_G$ and $f(e) = 1$. The function $g = 2f - 1_G$ satisfies the following properties: $g \in C^b_{\ell u}(G)$, $||g||_\infty = 1$ and $(\mathrm{Id} - P)g = 2f$. We finally obtain $||\,\mathrm{Id} - P|| \geq 2$.

Proposition 7. *Let H be a compact subgroup of G and $P \in \mathrm{Hom}_{L^1(G)} L^\infty(G)$ a projection from $L^\infty(G)$ onto $\ker T_H{}^\perp$. If H is infinite, then $||\,\mathrm{Id} - P|| \geq 2$ and $||\,\mathrm{Id} - P|| \geq \dfrac{2(|H|-1)}{|H|}$ otherwise.*

Preliminary remark. If H is a finite, $P(t) = \dfrac{1}{|H|} \sum_{h \in H} t_h$ (for $t \in L^\infty(G)$) defines $P \in \mathrm{Hom}_{L^1(G)} L^\infty(G)$, which is a projection from $L^\infty(G)$ onto $\ker T_H{}^\perp$. We have $||\,\mathrm{Id} - P|| \leq \dfrac{2(|H|-1)}{|H|}$.

Proof. 1) We have $(Pf)(e) = \displaystyle\int_H f(h)dh$ for all $f \in C_0(G)$ such that ${}_h f = f_h$ for every $h \in H$.

Using the continuity of the map $h \mapsto f_h$ from H into $C_0(G)$, we obtain the existence and the unicity of $g \in C_0(G)$ such that $L(g) = \displaystyle\int_H L(f_h)dh$ for every $L \in C_0(G)^*$. It follows that $g(x) = \displaystyle\int_H f(xh)dh$ for every $x \in G$ and consequently $P(g) = g$. We also have

$$(Pg)(e) = \int_H P(f_h)(e)dh = \int_H P({}_h f)(e)dh = \int_H (Pf)(h)dh = \int_H P(f)(e)dh.$$

We therefore conclude that $g(e) = (Pf)(e)$.

2) Assume that H is infinite. Let $\varepsilon > 0$. There is an open neighbourhood U of e in G such that $m_H(H \cap U) < \dfrac{\varepsilon}{2}$ where m_H is the normalized Haar measure of H. There is also a compact neighbourhood V of e in G with $V = V^{-1}$, $V^2 \subset U$ and $hV = Vh$ for every $h \in H$. Consider then $\varphi = \dfrac{1_V * 1_V}{m_G(V)}$. We have ${}_h\varphi = \varphi_h$ for every $h \in H$, $2\varphi - 1_G \in C^b_{\ell u}(G)$, $(2\varphi - 1_G)(e) = 1$, $||2\varphi - 1_G||_\infty = 1$ and $(\mathrm{Id} - P)(2\varphi - 1_G)(e) = 2 - 2P(\varphi)(e)$. From

$$P(\varphi)(e) = \int_H \varphi(h)dh \leq m_H(U \cap H) < \frac{\varepsilon}{2},$$

we deduce $||\,\mathrm{Id} - P|| > 2 - \varepsilon$.

3) In the finite case it suffices to consider an open neighbourhood U of e in G with $U \cap H = \{e\}$. We choose then V and φ as in 2). We obtain

$$(\mathrm{Id} - P)(2\varphi - 1_G)(e) = 2 - 2P(\varphi)(e) = 2 - \frac{2}{|H|}\sum_{h \in H} \varphi(h) = 2 - \frac{2}{|H|}$$

and therefore $||\,\mathrm{Id} - P|| \geq \dfrac{2(|H|-1)}{|H|}$.

It would be interesting (for H amenable) to obtain a description of the set of all projections P from $L^\infty(G)$ onto $\ker T_H{}^\perp$ with $P \in \mathrm{Hom}_{L^1(G)} L^\infty(G)$. For H compact and *normal* in G we can show that this set consists of a unique element given by $P(t)(x) = \int_H t(xh)dh$ for $t \in C_{lu}^b(G)$.

Let $f \in L^1(G)$, $t \in L^\infty(G)$ and $\varepsilon > 0$. By Proposition 2 there is $g \in \ker T_H$ with $||g||_1 \leq ||\,\mathrm{Id} - P||$, $||f - T_H f \circ \omega - (f - T_H f \circ \omega) * g|| < \dfrac{\varepsilon}{2(1 + ||t||_\infty)}$ and $|\langle f, P(t)\rangle_{L^1(G), L^\infty(G)} - \langle f, t\rangle_{L^1(G), L^\infty(G)} + \langle f * g, t\rangle_{L^1(G), L^\infty(G)}| < \dfrac{\varepsilon}{2}$. The subgroup H being normal in G, for every $x \in G$ we have

$$(T_H f \circ \omega) * g(x) = \int_{G/H} (T_H f)(\omega(x)\dot{y}^{-1})\Delta_G(y^{-1})\bigg(\int_H g(yh)dh\bigg)d\dot{y}$$

and therefore $(T_H f \circ \omega) * g = 0$. Taking into account that

$$\begin{aligned} &|\langle f, P(t)\rangle_{L^1(G), L^\infty(G)} - \langle T_H f \circ \omega, t\rangle_{L^1(G), L^\infty(G)}| \\ &\quad \leq |\langle f, P(t)\rangle_{L^1(G), L^\infty(G)} - \langle f, t\rangle_{L^1(G), L^\infty(G)} + \langle f * g, t\rangle_{L^1(G), L^\infty(G)}| \\ &\qquad + |\langle f - T_H f \circ \omega - (f - T_H f \circ \omega) * g, t\rangle_{L^1(G), L^\infty(G)}| \end{aligned}$$

we obtain that $|\langle f, P(t)\rangle_{L^1(G), L^\infty(G)} - \langle T_H f \circ \omega, t\rangle_{L^1(G), L^\infty(G)}| < \varepsilon$. In other words $\langle f, P(t)\rangle_{L^1(G), L^\infty(G)} = \langle T_H f \circ \omega, t\rangle_{L^1(G), L^\infty(G)}$. This implies for $t \in C_{lu}^b(G)$ and $x \in G$ that $P(t)(x) = \int_H t(xh)dh$.

6. Best bound for the approximate units of certain ideals of the Fourier algebra

For an arbitrary subset F of G we denote by $I(F)$ the closed ideal of $A_p(G)$ consisting of those functions vanishing on F. Motivated by the assertion 1) of Proposition 6 we first prove the following result.

Proposition 8. *Let G be amenable and let H be a closed normal nonopen subgroup of G. Let $P \in \mathrm{Hom}_{A_p(G)} PM_p(G)$ be a projection from $PM_p(G)$ onto $I(H)^\perp$. Then $P(T) = 0$ for every $T \in PF_p(G)$.*

$PF_p(G)$ is the norm closure of $L^1(G)$ in $PM_p(G)$. For G abelian $PF_2(G)$ is, via the Fourier transform, isomorphic to $C_0(\hat{G})$. To P there corresponds $\hat{P} \in \mathrm{Hom}_{L^1(\hat{G})} L^\infty(\hat{G})$, a projection from $L^\infty(\hat{G})$ onto $(\ker T_{H^\perp})^\perp$ where $H^\perp$ is the set of all continous characters of G equal to 1 on H. Moreover H is nonopen if and only if $H^\perp$ is noncompact.

The existence of P (in Proposition 8) is a consequence of Proposition 1 and Theorem 4.

Let $f \in C_{00}(G)$, $K = \operatorname{supp} f^*$, $u \in A_p(G)$, $\varepsilon > 0$ and U be an open neighbourhood of $H \cap K$ in G such that

$$m_G(U) < \frac{\varepsilon}{4(||f^*||_\infty + 1)(||u||_{A_p(G)} + 1)(||\operatorname{Id} - P|| + 1)}\,.$$

It is possible to choose $v \in I(H) \cap C_{00}(G)$ with $v = 1$ on $K_1 = K - U$. By Proposition 2 there is $w \in I(H)$ with $||w||_{A_p(G)} \leq ||\operatorname{Id} - P||$,

$$||vw - v||_{A_p(G)} < \frac{\varepsilon}{4(||f||_1 + 1)(||u||_{A_p(G)} + 1)}$$

and

$$\begin{aligned} &|\langle uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)} \\ &\quad - \langle u, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)} + \langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2} \end{aligned}$$

(for a bounded measure μ, $\lambda_G^p(\mu)$ is the convolution operator defined by $\lambda_G^p(\mu)(\varphi)(x) = \int_G \varphi(xy)\Delta_G(y)^{1/p} d\mu(y)$). We obtain the estimate

$$|\langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2} + |\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}|\,.$$

Taking into account that

$$\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)} = \int_G (u(x) - u(x)w(x))f^*(x)dx\,,$$

we can write

$$\begin{aligned} &|\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}| \\ &\quad \leq \int_{K_1} |u(x)v(x) - v(x)u(x)w(x)|\,|f^*(x)|\,dx \\ &\qquad + \int_{K\cap U} |u(x) - u(x)w(x)|\,|f^*(x)|\,dx\,. \end{aligned}$$

We estimate $\int_{K_1} |u(x)v(x) - v(x)u(x)w(x)|\,|f^*(x)|\,dx$ by $||u||_{A_p(G)}||v - vw||_{A_p(G)}||f||_1$ and $\int_{K\cap U} |u(x) - u(x)w(x)|\,|f^*(x)|\,dx$ by $m_G(K\cap U)(||u||_\infty + ||u||_\infty||w||_\infty)||f^*||_\infty$. We obtain therefore that $|\langle u - uw, \lambda_G^p(f)\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2}$ and finally

$$|\langle u, P(\lambda_G^p(f))\rangle_{A_p(G),PM_p(G)}| < \varepsilon.$$

We also need an analog of the assertion 2) of Proposition 6:

Proposition 9. *There is* $T \in PF_2(G)$ *with* $|||T|||_2 = 1$ *and* $||| \operatorname{Id}_{L^2(G)} - 2T|||_2 = 1$.

Let $f \in C_{00}(G)$ such that $f \neq 0$ and $f = f^*$. Consider the C^*-algebra $\overline{\lambda_G^2(M^1(G))}$ and denote by A and B the C^*-subalgebras generated by $\lambda_G^2(f)$, respectively $\lambda_G^2(f)$ and $\operatorname{Id}_{L^2(G)}$. Of course A and B are abelian and B is unital. Let $\Omega(A) \subset \Omega(B)$ be the spectra of A and B. The space $\Omega(A)$ is locally compact (and nonempty), $\Omega(B)$ is compact (it is actually the one-point compactification of $\Omega(A)$). It follows that the Gelfand transformation $\mathcal{F} : B \longrightarrow C_0(\Omega(B)) = C(\Omega(B))$ is an isometric isomorphism. Moreover the restriction of $\mathcal{F}$ to A is an isomorphism onto $C_0(\Omega(A))$. Choose now $\varphi \in C(\Omega(B))$ such that $0 \leq \varphi \leq 1$, $||\varphi||_\infty = 1$ and $\operatorname{supp} \varphi \subset \Omega(A)$. Let $T = \mathcal{F}^{-1}(\varphi) \in A \subset PF_2(G)$ and $|||T|||_2 = ||\varphi||_\infty = 1 = ||| \operatorname{Id} - 2T|||_2$.

Theorem 10. *Suppose that* G *is amenable, and let* H *be a closed normal* **nonopen** *subgroup of* G. *The best bound for approximate units of* $I(H)$ *(in* $A_2(G)$*) is* 2.

Let $P \in \operatorname{Hom}_{A(G)} PM(G)$ be a projection from $PM(G)$ onto $I(H)^\perp$. There is $T \in PF(G)$ with $|||T|||_2 = 1$ and $||| \operatorname{Id}_{L^2(G)} - T|||_2 = 1$. We have

$$(\operatorname{Id} - P)(\operatorname{Id}_{L^2(G)} - T) = -2T.$$

This implies $|| \operatorname{Id} - P|| \geq 2$.

Remark. We are unable to prove the corresponding result in $A_p(G)$ for $p \neq 2$!

Theorem 11. *Let* H *be an open (not necessarily normal) subgroup of an amenable group* G. *The best bound for approximate units of* $I(H)$ *(in* $A_2(G)$*) is* 2 *if* G/H *is infinite, and is* $\dfrac{2(|G/H| - 1)}{|G/H|}$ *otherwise.*

Let $MA(G)$ be the Banach algebra of all pointwise multipliers of $A(G)$ with the multiplier norm. We have $1_H \in MA(G)$. Clearly $P_0(T) = 1_H T$ defines a map which belongs to $\operatorname{Hom}_{A(G)} PM(G)$ and projects $PM(G)$ onto $I(H)^\perp$. From the decomposition $T = 1_H T + 1_{G-H} T$ it follows that any $P \in \operatorname{Hom}_{A(G)} PM(G)$ which projects $PM(G)$ onto $I(H)^\perp$ coincides with P_0. Therefore the best bound for the approximate units of $I(H)$ is precisely $|| \operatorname{Id} - P_0||$, i.e. $||1_{G-H}||_{MA(G)}$.

Now G being amenable $MA(G)$ coincides (isometrically) with the intricate Banach algebra $B_2(G)$ introduced by C. Herz [10]. We recall the necessary notions. Let X be a nonempty set (with the discrete topology). Every $k \in C_{00}(X \times X)$ is the kernel of a bounded operator of $\ell^2(X)$. The corresponding norm is denoted $|||k|||_2$. $V_2(X)$ is the space of all $\varphi \in \mathbb{C}^{X \times X}$ for which there is $C > 0$ with $|||\varphi k|||_2 \leq C|||k|||_2$ for every $k \in C_{00}(X \times X)$. The smallest possible C is $||\varphi||_{V_2(X)}$. By definition $B_2(G)$ is the set of all $\varphi \in C(G)$ for which $M_G \varphi \in V_2(G_d)$ where $M_G \varphi(x, y) = \varphi(y^{-1}x)$ and $||\varphi||_{B_2(G)} = ||M_G \varphi||_{V_2(G_d)}$.

Moreover, by an important result of C. Herz [11, Theorem 5],

$$||1_{G-H}||_{B_2(G)} = ||1_{G/H \times G/H - \Delta(G/H)}||_{V_2((G/H)_d)}$$

where $\Delta(G/H)$ is the diagonal in $G/H \times G/H$. Now G/H carries a structure of abelian group. Let L_d be this group with the discrete topology. We have $||1_{G-H}||_{B_2(G)} = ||1_{L_d - \{e\}}||_{B_2(L_d)}$. From above $||1_{L_d - \{e\}}||_{B_2(L_d)}$ is the best bound for approximate units of $\ker T_{\widehat{L_d}}$. We conclude then with the help of Theorem 5.

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