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CHARACTERIZATION OF THE FOURIER SERIES OF A DISTRIBUTION HAVING A VALUE AT A POINT

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ABSTRACT. Let f be a periodic distribution of period 2π. Let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. We show that the distributional point value $f(\theta_0)$ exists and equals γ if and only if the partial sums $\sum_{-x\le n\le ax} a_n e^{in\theta_0}$ converge to γ in the Cesàro sense as $x\to\infty$ for each $a>0$.

1. INTRODUCTION

The study of the relationship between the local behavior of a periodic function at a point and the convergence and summability of the corresponding Fourier series has been one of the main concerns of classical harmonic analysis, a subject with many years of history.

In particular, the characterization of the Fourier series having a "value" at a point is a very interesting problem, whose solution depends on the notion of value used. Among the various useful notions of value we could mention the value of a continuous function at a point, the measure-theoretical approximate value of an integrable function, or the distributional point value of a generalized function.

In this article we shall be concerned with the characterization of the Fourier series of generalized functions having a distributional point value. The notion of distributional point value was introduced by Łojasiewicz [8] and corresponds, roughly, to the existence of value "on the average". If $f \in \mathcal{D}'(\mathbb{R})$ is a distribution and $x_0 \in \mathbb{R}$, we say that f has the distributional value γ at $x = x_0$, and write $f(x_0) = \gamma$ in $\mathcal{D}'$ if for each $\phi \in \mathcal{D}$ we have $\lim_{\varepsilon\to 0}\langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\, dx$ (if f is given by an integral and $\int_{-\infty}^{\infty} \phi(x)\, dx = 1$ this means that the "average" $\int_{-\infty}^{\infty} f(x_0+\varepsilon x)\phi(x)\, dx$ tends to γ). It can be shown that $f(x_0) = \gamma$ in $\mathcal{D}'$ if and only if there exists a primitive of order k of f, i.e., $F^{(k)} = f$, which is continuous in a neighborhood of $x = x_0$ and satisfies $\lim_{x\to x_0} k!\,(x-x_0)^{-k}F(x) = \gamma$. This corresponds to the notion of generalized derivatives (see [13] and the references therein) used in the theory of trigonometric series well before the definition of Łojasiewicz and even before the introduction of the notion of distributions by Schwartz [10].

It turns out that the existence of distributional point values is equivalent to the existence *in the Cesàro sense* of the limits of certain partial sums of the corresponding Fourier series. Convergence in the Cesàro sense is also a kind of "convergence

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on the average" and is defined as follows [6]: if f is an integrable function defined for $x \geq 0$ we say that $F(x)$ tends to L in the $(C,1)$ sense as $x \to +\infty$, and write $\lim_{x\to+\infty} F(x) = L\ (C,1)$ if $\lim_{x\to+\infty} \frac{1}{x}\int_0^x F(t)\,dt = L$. Convergence in the (C,k) sense, $k = 2, 3, \ldots$, is defined recursively by $\lim_{x\to\infty} F(x) = L\ (C,k)$ if $\lim_{x\to+\infty} \frac{1}{x}\int_0^x F(t)\,dt = L\ (C,k-1)$. We say that $F(x)$ converges to L in the Cesàro sense and write $\lim_{x\to+\infty} F(x) = L\ (C)$ if $\lim_{x\to+\infty} F(x) = L\ (C,k)$ for some k.

The Cesàro convergence of a sequence $\{x_n\}$ can be defined similarly and it is equivalent to the Cesàro convergence of the function F defined by $F(t) = x_{[\![t]\!]}$. In particular a series $\sum_{n=1}^{\infty} a_n$ is Cesàro summable to the sum S, written as $\sum_{n=1}^{\infty} a_n = S\ (C)$, if $\lim_{n\to\infty} \sum_{j=1}^{n} a_j = S\ (C)$. A series $\sum_{n=-\infty}^{\infty} a_n$ is (C) summable to S if both series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are (C) summable to S_1 and S_2, respectively, with $S = S_1 + S_2$. A series $\sum_{n=-\infty}^{\infty} a_n$ is principal value Cesàro summable to S, written as $\text{p.v.}\sum_{n=-\infty}^{\infty} a_n = S\ (C)$, if $\lim_{n\to\infty} \sum_{j=-n}^{n} a_j = S\ (C)$.

Now let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be the associated Fourier series. Let $\theta_0 \in \mathbb{R}$. Then the following two results are known [3, 12]:

(1) If $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma\ (C)$, i.e., if the partial sums $\sum_{n=-N}^{M} a_n e^{in\theta_0}$ tend to γ in the Cesàro sense as N and M tend to infinity independently, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$. The converse does not hold ($f(\theta) = \sum_{n=2}^{\infty} \frac{e^{in\theta}}{n\log n}$, at $\theta = 0$, is an example).

(2) If $f(\theta_0) = \gamma$ in $\mathcal{D}'$, then $\text{p.v.}\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma\ (C)$, i.e.,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n e^{in\theta_0} = \gamma\ (C).$$

The converse does not hold ($f(\theta) = \sum_{n=1}^{\infty} n \sin n\theta$, at $\theta = 0$, is an example).

Here we prove the following

Theorem. *Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then*

$$f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}', \tag{1.1}$$

if and only if there exists k such that

$$\lim_{x\to+\infty} \sum_{-x\leq n\leq ax} a_n e^{in\theta_0} = \gamma, \quad (C,k) \tag{1.2}$$

for each $a > 0$.

Our approach is based on the theory of distributional asymptotic expansions [4, 5, 11] and is inspired by the work of Ramanujan [9] who was one of the first to study a sequence $\{a_n\}$ by studying the asymptotic behavior of series of the type $\sum_{n=1}^{\infty} a_n \phi(n\varepsilon)$, as $\varepsilon \to 0^+$, for smooth ϕ. See also [1, 2]. Here we study the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0}$ by analysing the behavior as $\varepsilon \to 0$ of the series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \phi(n\varepsilon)$ for $\phi \in \mathcal{S}$.

Notice that we use the standard notation concerning spaces of distributions and test functions [7, 10].

2. Asymptotically homogeneous functions

In this section we define and give the basic properties of the asymptotically homogeneous functions of degree 0. These functions play a central role in our analysis.

Lemma 1. *Let τ be a real-valued continuous function defined in an interval of the form $[A,\infty)$ for some $A\in\mathbb{R}$. Suppose*

$$\tau(ax)=a^{\mu}\tau(x)+o(1),\quad \text{as } x\to+\infty, \tag{2.1}$$

for each $a>0$. If $\mu<0$, then

$$\tau(x)=o(1),\quad \text{as } x\to+\infty. \tag{2.2}$$

Proof. Let $\varepsilon>0$. Let $x_0>0$ be such that $|\tau(2x)-2^{\mu}\tau(x)|\le\varepsilon$ for $x>x_0$. Let $M=\max\{|\tau(x)| : x_0\le x\le 2x_0\}$. An inductive argument shows that if $x\in[2^n x_0, 2^{n+1}x_0]$, $n=0,1,2,\dots$, then $|\tau(x)|\le 2^{n\mu}M+\sum_{j=1}^{n-1}2^{j\mu}\varepsilon$. Therefore, $\lim_{x\to+\infty}|\tau(x)|\le\frac{2^{\mu}}{1-2^{\mu}}\varepsilon$ and (2.2) follows. □

The lemma does not hold if $\mu=0$. Indeed, there are functions like $\ln(\ln x)$, $|\ln x|^{\alpha}$, $\alpha<1$, or $\cos\sqrt{|\ln x|}$ that satisfy $\tau(ax)=\tau(x)+o(1)$, as $x\to+\infty$, for each $a>0$, but which do not tend to zero at infinity.

Definition. Let τ be a continuous function defined in an interval of the form $[A,\infty)$ for some $A\in\mathbb{R}$. We say that τ is asymptotically homogeneous of degree 0 if for each $a>0$ we have

$$\tau(ax)=\tau(x)+o(1),\quad \text{as } x\to+\infty. \tag{2.3}$$

The asymptotically homogeneous functions of degree 0 are related to the slowly oscillating functions [5], also known as regularly varying functions of order 0. These are positive functions that satisfy $\rho(ax)=\rho(x)+o(\rho(x))$, as $x\to\infty$, for each $a>0$. However, the two concepts are different: $\ln x$ is slowly oscillating but not asymptotically homogeneous of degree 0 while $\cos\sqrt{|\ln x|}$ is asymptotically homogeneous of degree 0 but not slowly oscillating.

Observe that the argument of the proof of Lemma 1 shows that if τ is an asymptotically homogeneous function of order 0, then $\tau(x)=o(\ln x)$, as $x\to+\infty$.

Notice also that we did not ask for any uniform behavior with respect to a in (2.3). The fact that (2.3) holds uniformly on $a\in[A,B]$ if $[A,B]\subseteq(0,\infty)$ follows from the definition, as we shall see.

Lemma 2. *Let τ be an asymptotically homogeneous function of degree 0, continuous in $[0,\infty)$. Let H be the Heaviside function. Then*

$$\tau(\lambda x)H(x)=\tau(\lambda)H(x)+o(1),\quad \text{as } \lambda\to\infty,\ \text{in } \mathcal{S}'. \tag{2.4}$$

If $[A,B]\subseteq(0,\infty)$, then (2.3) holds uniformly for $a\in[A,B]$.

Proof. Suppose first that τ is bounded in $[0,\infty)$. Then if $\phi\in\mathcal{S}'$ we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\lambda\to\infty}\int_0^{\infty}(\tau(\lambda x)-\tau(\lambda))\phi(x)\,dx=0.$$

This is (2.4).

The uniform convergence follows from the distributional formula (2.4). Indeed, weak convergence in $\mathcal{S}'$ implies strong convergence [10]. Thus if K is a compact subset of $\mathcal{S}$, then $\langle \tau(\lambda x)H(x) - \tau(\lambda)H(x), \phi(x)\rangle = o(1)$ uniformly for $\phi \in K$. Let $\phi_0 \in \mathcal{S}$ be a function that satisfies $\int_0^\infty \phi_0(x)\,dx = 1$. Then if $[A, B] \subseteq (0, \infty)$ the set $K = \{a^{-1}\phi_0(a^{-1}x) : a \in [A, B]\}$ is a compact set of $\mathcal{S}$. Then $\langle \tau(\lambda x)H(x), \phi_0(x)\rangle = \tau(\lambda) + o(1)$ as $\lambda \to \infty$, and so $\tau(\lambda a) = \langle \tau(\lambda a x), \phi_0(x)\rangle + o(1) = \langle \tau(\lambda x), a^{-1}\phi_0(a^{-1}x)\rangle + o(1) = \tau(\lambda) + o(1)$ uniformly for $a \in [A, B]$.

Let us now return to the general case when τ is not necessarily bounded. We shall first show that if $[A, B] \subseteq (0, \infty)$, then $\tau(ax) = \tau(x) + o(1)$, as $x \to +\infty$, uniformly on $a \in [A, B]$. Observe first that the functions $\cos\tau(x)$ and $\sin\tau(x)$ are bounded asymptotically homogeneous functions. By what we have already proven it follows that if $[A, B] \subseteq (0, \infty)$, then $e^{i\tau(ax)} = e^{i\tau(x)} + o(1)$, as $x \to \infty$, uniformly on $a \in [A, B]$. Let $\varepsilon > 0$. Suppose $\varepsilon < \pi$ and $A < 1 < B$. There exists $x_0 > 0$ such that $|e^{i\tau(ax)} - e^{i\tau(x)}| \le |1 - e^{i\varepsilon}|$ for $x \ge x_0$ and $a \in [A, B]$. For each $x \ge x_0$ the set $\{\tau(ax) : A \le a \le B\}$ is a connected set contained in $\bigcup_{n=-\infty}^{\infty}[\tau(x) - \varepsilon + 2n\pi, \tau(x) + \varepsilon + 2n\pi]$ and it follows that it is contained in the component that contains $\tau(x)$; that is, $|\tau(ax) - \tau(x)| \le \varepsilon$, for $a \in [A, B]$.

To prove (2.4) it would be enough to prove that there are constants A_0, A_1 such that $|\tau(\lambda x) - \tau(\lambda)| \le A_0|\ln x| + A_1$ for $x > 0$ and $\lambda > \lambda_0$ for then the dominated convergence theorem could be invoked again.

Let $\lambda_0 > 0$ be such that $|\tau(\lambda x) - \tau(\lambda)| \le 1$ if $\lambda \ge \lambda_0$ and $x \in [1/2, 2]$. Then it follows by induction that $|\tau(\lambda x) - \tau(\lambda)| \le n + 1$ if $x \in [2^n, 2^{n+1}]$, $\lambda \ge \lambda_0$, $n = 0, 1, 2 \ldots$. Therefore $|\tau(\lambda x) - \tau(\lambda)| \le \frac{|\ln x|}{\ln 2} + 1$ if $x \ge 1$, $\lambda \ge \lambda_0$. Proceeding similarly, if $x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, $\lambda \ge 2^n\lambda_0$, $n = 0, 1, 2, \ldots$, then $|\tau(\lambda x) - \tau(\lambda)| \le n + 1 \le \frac{|\ln x|}{\ln 2} + 1$. Recall now that $\tau(x) = o(\ln x)$, as $x \to \infty$. Then we can find constants M_0, M_1 such that $|\tau(x)| \le M_0|\ln x| + M_1$, $x > 0$. Therefore, if $x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ but $\lambda < 2^n\lambda_0$, then $\lambda < \lambda_0/x$ and so $|\tau(\lambda x) - \tau(\lambda)| \le 2(M_0|\ln\lambda| + M_1) \le 2(M_0|\ln x| + M_0|\ln\lambda_0| + M_1)$. Summarizing, if $A_0 = \max\{2M_0, 1/\ln 2\}$, $A_1 = \max\{2(M_0|\ln\lambda_0| + M_1), 1\}$, then $|\tau(\lambda x) - \tau(\lambda)| \le A_0|\ln x| + A_1$ for $x > 0$ and $\lambda \ge \lambda_0$. □

The asymptotically homogeneous functions of degree 0 do not have to be smooth, i.e., C^∞. However, we have

Lemma 3. *Let τ be an asymptotically homogeneous function of degree 0, continuous in $[0, \infty)$. Then there exists a function σ, asymptotically homogeneous of degree 0, smooth in $[0, \infty)$, such that*

$$\tau(x) = \sigma(x) + o(1), \quad \text{as } x \to \infty. \tag{2.5}$$

Proof. Define $\sigma(\lambda) = \int_0^\infty \tau(\lambda x)\phi_0(x)\,dx$, $\lambda \ge 1$, where $\phi_0 \in \mathcal{S}$ satisfies $\int_0^\infty \phi_0(x)\,dx = 1$, and extend to $[0, +\infty)$ in any smooth way. Then (2.4) gives (2.5). □

3. Primitives of a distributionally null sequence

If $\{f_n\}$ is a sequence of continuous functions that converge uniformly to zero on an interval of the form $[-A, A]$, then the sequence of primitives $\{\int_0^x f_n(t)\,dt\}$ also converges uniformly to zero while, in general, the sequence of derivatives $\{f_n'\}$, even if defined, does not. On the other hand, the situation with the distributional convergence is the opposite: if $f_n \to 0$ in $\mathcal{S}'$, then also $f_n' \to 0$ in $\mathcal{S}'$ but the sequence of primitives $\{\int_0^x f_n(t)\,dt\}$, even if defined, might be divergent. We now turn our attention to the study of this last problem.

Let $F \in \mathcal{S}'$. If F is integrable near $x = 0$, then we can define the primitive $\int_0^x F(t)\,dt$. If F is more singular near $x = 0$, then there is no canonical way to define $\int_0^x F(t)\,dt$, that is, there is no canonical way to single out a primitive of F that vanishes at $x = 0$. However, if F is locally even near $x = 0$, it has a unique locally odd primitive, which we denote as $\int_0^x F(t)\,dt$. Observe that with this notation $\int_0^x \delta(t)\,dt = \frac{1}{2}\operatorname{sgn} x$, where $\operatorname{sgn} x = x/|x|, x \neq 0$, is the sign function. We shall use the notation $\int_0^x F(t)\,dt$ so long as $F = F_0 + F_1$, where F_0 is locally integrable near $x = 0$ and where F_1 is locally even.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distributions of $\mathcal{S}'$. Suppose that $\lim_{n\to\infty} F_n = 0$ in $\mathcal{S}'$. Then [10] there are primitives of the F_n that also tend to zero. In particular, if $\int_0^x F_n(t)\,dt$ is defined for each n, there are constants c_n such that $\int_0^x F_n(t)\,dt = c_n + o(1)$, as $n \to \infty$ in $\mathcal{S}'$. The example $F_n(x) = -2nxe^{-x^2}(1-e^{-x^2})^{n-1}$, $\int_0^x F_n(t)\,dt = 1-(1-e^{-x^2})^n = 1 + o(1)$ shows that the c_n's cannot be replaced by 0, in general. Observe also that if the F_λ depend smoothly on the parameter λ and $F_\lambda = o(1)$, as $\lambda \to \infty$, then $\int_0^x F_\lambda(t)\,dt = \sigma(\lambda) + o(1)$, where σ is smooth.

Lemma 4. *Let $F_0 \in \mathcal{S}'$ be a Radon measure such that $\int_0^x F_0(t)\,dt$ is defined. Define the distributions F_n, $n \geq 1$, recursively by $F_n(x) = \int_0^x F_{n-1}(t)\,dt$, $n \geq 1$. Suppose*

$$F_0(\lambda x) = o(1/\lambda), \quad \text{as } \lambda \to +\infty, \ \text{in } \mathcal{S}'. \tag{3.1}$$

Then there exists an asymptotically homogeneous function of degree $0, \sigma(\lambda)$, such that

$$F_n(\lambda x) = \frac{\lambda^{n-1}x^{n-1}\sigma(\lambda)}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \to +\infty, \ \text{in } \mathcal{S}', \tag{3.2}$$

for $n \geq 1$. There exists n_0 such that the convergence in (3.2) is uniform on compacts for $n \geq n_0$. Conversely, if (3.2) holds for some $n \geq 1$, then $F_0(\lambda x) = o(1/\lambda)$ in $\mathcal{S}'$.

Proof. Suppose $F_0(\lambda x) = o(1/\lambda)$ in $\mathcal{S}'$. Then there exists a smooth function $\sigma(\lambda)$ such that $F_1(\lambda x) = \sigma(\lambda) + o(1)$, as $\lambda \to \infty$, in $\mathcal{S}'$. Replacing λx by $\lambda x a$ and grouping in two different ways, we obtain $\sigma(a\lambda) = \sigma(\lambda) + o(1)$, as $\lambda \to \infty$ for each $a > 0$. Thus σ is asymptotically homogeneous of degree 0. Hence (3.2) holds for $n = 1$. Suppose now it holds for some $n \geq 1$. Then integrating again we obtain $F_{n+1}(\lambda x) = \lambda^n x^n \sigma(\lambda)/n! + \rho(\lambda) + o(\lambda^n)$, as $\lambda \to \infty$, for some function ρ. Evaluating at $\lambda a x$ thus yields $\rho(\lambda a) = \rho(\lambda) + o(\lambda^n)$ and thus by Lemma 1, applied to $\lambda^{-n}\rho(\lambda)$, it follows that $\rho(\lambda) = o(\lambda^n)$ and (3.2) is obtained for $n + 1$.

That the convergence in (3.2) is uniform for x in compacts if n is large follows by the definition of the convergence of distributions.

The converse is obtained by differentiating (3.2) n times with respect to x. □

Observe that if F_0 is even, then F_1 is odd and thus $\sigma(\lambda) = o(1)$, as $\lambda \to \infty$, and (3.2) becomes $F_n(\lambda x) = o(\lambda^{n-1})$, as $\lambda \to \infty$. The same conclusion is obtained if $\operatorname{supp} F_0 \subseteq [0, \infty)$.

The sequence of functions $\{F_n\}_{n=0}^{\infty}$ is also related to Cesàro summability. The following lemma is an immediate corollary of the results proved in [6, p. 110].

Lemma 5. *Let F_0 be a function defined for $x > 0$ that, suitably extended to $\mathbb{R}$, defines an element of $\mathcal{S}'$ for which $\int_0^x F_0(t)\,dt$ is defined. Then $\lim_{x\to+\infty} F_0(x) = \gamma$ (C, n) if and only if $\lim_{x\to\infty} n!\,x^{-n}F_n(x) = \gamma$.* □

4. The main result

In this section we apply the results of the previous sections to characterize the Fourier series of the periodic distributions having a distributional point value.

Lemma 6. *Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then*

$$f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}', \tag{4.1}$$

if and only if

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(\lambda x - n) = \frac{\gamma\delta(x)}{\lambda} + o(1/\lambda), \quad \text{as } \lambda \to \infty, \text{ in } \mathcal{S}'. \tag{4.2}$$

Proof.

$$\begin{aligned}
&f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}' \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \langle f(\theta_0 + \varepsilon\theta), \phi(\theta)\rangle = \langle \gamma, \phi(\theta)\rangle \quad \forall \phi \in \mathcal{D} \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \langle f(\theta_0 + \varepsilon\theta), \phi(\theta)\rangle = \langle \gamma, \phi(\theta)\rangle \quad \forall \phi \in \mathcal{S} \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \langle e^{in\varepsilon\theta}, \phi(\theta)\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\, dx \quad \forall \phi \in \mathcal{S} \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \hat{\phi}(n\varepsilon) = \gamma\hat{\phi}(0) \quad \forall \phi \in \mathcal{S} \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \psi(n\varepsilon) = \gamma\psi(0) \quad \forall \psi \in \mathcal{S} \\
&\quad \Leftrightarrow \lim_{\varepsilon\to 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(x - n\varepsilon) = \gamma\delta(x) \quad \text{in } \mathcal{S}' \\
&\quad \Leftrightarrow \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(\lambda x - n) = \frac{\gamma\delta(x)}{\lambda} + o(1/\lambda), \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'. \quad \square
\end{aligned}$$

We are now ready to give our main result.

Theorem. *Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then*

$$f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}', \tag{4.3}$$

if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x \le n \le ax} a_n e^{in\theta_0} = \gamma \quad (C, k) \tag{4.4}$$

for each $a > 0$.

Proof. Let $F_0(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(x - n)$ and define $F_1, F_2, \dots$ recursively by $F_{n+1}(x) = \int_0^x F_n(t)\, dt$. Let $a > 0$, put $G_0(x) = G_0(a, x) = \sum_{-x \le n \le ax} a_n e^{in\theta_0}$ and define $G_1, G_2, \dots$ recursively by $G_{n+1}(x) = \int_0^x G_n(t)\, dt$. Observe that (4.4) means that $\lim_{x\to\infty} k!\, x^{-k} G_k(x) = \gamma$.

Suppose first that $f(\theta_0) = \gamma$. Then $F_0(\lambda x) = \gamma\delta(\lambda x) + o(1/\lambda)$, as $\lambda \to \infty$, in $\mathcal{S}'$. Using Lemma 4, there exists an asymptotically homogeneous function of degree 0 such that if $n \ge 1$, then

$$F_n(\lambda x) = \frac{\gamma \operatorname{sgn}(\lambda x)(\lambda x)^{n-1}}{2(n-1)!} + \frac{\sigma(\lambda)(\lambda x)^{n-1}}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \to \infty, \text{ in } \mathcal{S}'$$

and there exists n_0 such that if $n \geq n_0$ this holds uniformly on $x \in [-A, A]$ for $A > 0$. Thus, if $n \geq n_0$,

$$\begin{aligned} G_{n-1}(x) &= a^{1-n} F_n(ax) - (-1)^{1-n} F_n(-x) \\ &= \frac{\gamma x^{n-1}}{2(n-1)!} + \frac{\sigma(x)}{(n-1)!} + \frac{\gamma}{2(n-1)!} - \frac{\sigma(x)}{(n-1)!} + o(x^{n-1}) \\ &= \frac{\gamma x^{n-1}}{(n-1)!} + o(x^{n-1}) \end{aligned}$$

and (4.4) follows with $k = n - 1$.

Conversely, suppose (4.4) holds. Let $n = k + 1$, so that

$$\lim_{x\to\infty} (n-1)!\, x^{1-n} G_{n-1}(x) = \gamma.$$

Define $\tau(x) = (n-1)!\, x^{1-n} F_n(x) - \gamma/2$. Then τ is asymptotically homogeneous of degree 0. Thus there exists a smooth asymptotically homogeneous function of degree 0 such that $\tau(x) = \sigma(x) + o(1)$, as $x \to \infty$. It follows that

$$F_n(x) = \frac{\gamma \operatorname{sgn} x\, x^{n-1}}{2(n-1)!} + \frac{\sigma(x) x^{n-1}}{(n-1)!} + o(x^{n-1}), \quad \text{as } |x| \to \infty,$$

and therefore, by Lemma 2,

$$F_n(\lambda x) = \frac{\gamma \operatorname{sgn}(\lambda x)(\lambda x)^{n-1}}{2(n-1)!} + \frac{\sigma(\lambda)(\lambda x)^{n-1}}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \to \infty, \text{ in } \mathcal{S}'.$$

Differentiating n times we obtain $F_0(\lambda x) = \gamma\delta(\lambda x) + o(1/\lambda)$, as $\lambda \to \infty$, as required. □

It is interesting to observe that (4.4) holds uniformly on $a \in [A, B]$ if $[A, B] \subseteq (0, \infty)$ but that it is not necessary to assume such uniform behavior to obtain (4.3).

Notice also that the theorem remains valid if (4.4) holds uniformly for $a \in H$ where H is a set dense in some interval, $\overline{H} = [A, B]$, $0 < A < B < \infty$. In particular, $f(\theta_0) = \gamma$ if and only if

$$\lim_{N\to\infty} \sum_{n=-[\![N/p]\!]}^{[\![N/q]\!]} a_n e^{in\theta_0} = \gamma \quad (C, k) \tag{4.5}$$

for each $p, q \in \{1, 2, 3, \dots\}$, uniformly if $A < p/q < B$ for $[A, B] \subseteq (0, \infty)$. Observe that (4.5) considers the convergence, in the Cesàro sense, of sequences.

The following consequences are worth recording.

Corollary 1. *If f is symmetric about $\theta = \theta_0$, i.e., $f(\theta - \theta_0) = f(\theta_0 - \theta)$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma$ (C).* □

Corollary 2. *If f is antisymmetric about $\theta = \theta_0$, i.e., $f(\theta - \theta_0) = -f(\theta_0 - \theta)$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if there is a function σ, asymptotically homogeneous of degree 0, such that $\sum_{n=1}^{N} a_n e^{in\theta_0} = \sigma(N) + o(1)$ (C), as $N \to \infty$. In this case $\gamma = 0$.* □

Corollary 3. *If the Fourier series of f is of the power series type, i.e., $f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if $\sum_{n=0}^{\infty} a_n e^{in\theta_0} = \gamma$ (C).* □

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