PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

ON A GENERALISATION OF SELF-INJECTIVE VON NEUMANN REGULAR RINGS

GEORGE IVANOV

(Communicated by Ken Goodearl)

ABSTRACT. Apart from von Neumann regular rings, rings with infinite identities have not been studied in any detail. We take a first step in that direction by obtaining structure theorems for a class of self-injective rings with infinite identities. These extend the main structure theorems for self-injective von Neumann regular rings.

A ring's identity element may be infinitely decomposable in that it cannot be expressed as the sum of (a finite number of) primitive idempotents. For brevity we will call such identity elements *infinite* and the others *finite*. This article is the result of an attempt to understand (a little of) the structure of rings with infinite identity elements. Except for von Neumann regular rings such rings have received little attention from algebraists. Von Neumann regular rings with finite identity elements are the semisimple Artinian rings (and are injective). It appears that in the class of rings with infinite identities the self-injective von Neumann regular rings play the same role - as the simplest, most basic examples.

Therefore our point of departure is to study a class of rings which includes the injective von Neumann regular rings but is sufficiently larger to be interesting. We choose the rings whose finitely generated (left) ideals are quasi-injective and call these rings (*left*) *fQ-rings*. They are a generalisation of Q-rings which have been of some interest in their own right (see e.g [I1], [I2] or [J] (and their references) - the last paper has a survey of results in the area).

In §1 we show that fQ-rings with finite identities are Q-rings and therefore their structure is known completely (except for the rather uninteresting local case). The general case when the ring has infinite identity appears to be very difficult. To simplify the task we introduce two different concepts (*dense primitive idempotents* and *idempotent nonsingular*) both of which are intermediate between *finite identity* and arbitrary *infinite identity* but in different "directions". In §1 we show that the main structure theorem on injective von Neumann regular rings (the decomposition into Types) can be extended to fQ-rings which are idempotent nonsingular; and in §2 we determine the structure of indecomposable idempotent nonsingular fQ-rings with dense primitive idempotents and represent them as rings of matrices.

Throughout this paper all rings are unital, and all ideals, properties, etc., which are one-sided are left ideals, properties, etc. Right or two-sided ideals, properties,

Received by the editors June 7, 1994 and, in revised form, October 18, 1994.
1991 *Mathematics Subject Classification*. Primary 16D50, 16D70; Secondary 16E60.
Honorary Associate at Macquarie University.

etc., will specifically be so designated. The letters e, f with or without superscripts or subscripts will always stand for idempotents. *Idempotent* will mean nontrivial idempotent (i.e. $\neq 0$ or 1). The injective hull of a module M will be denoted by $E(M)$. In the interest of brevity we will from now on refer to *von Neumann regular rings* as simply *von Neumann rings*. For results on von Neumann rings the reader is referred to Goodearl's book [G] which also contains basic information about quasi-injective modules.

§1. Idempotent nonsingular fQ-rings

Definition. A ring's identity (element) is finite if it is a sum of primitive orthogonal idempotents, otherwise it is *infinite*. A set of idempotents is (*left*) *dense* in a ring if it generates an essential (left) ideal of the ring.

Clearly a ring has finite identity if and only if it is a (module) direct sum of indecomposable ideals. A ring with finite identity obviously has dense primitive idempotents. The structure of rings with identities is completely determined by their indecomposable ring summands. This is not true for rings with infinite identities and in fact such rings need not even have indecomposable summands.

In this section we introduce a concept of *Types* which is a natural extension of von Neumann's defintion but which is applicable to much larger classes of rings. We show that the main structure theory on injective von Neumann rings (the decomposition into Types) can be extended to a large class of fQ-rings under this new defintion. The example at the end of the paper shows that this class (the idempotent nonsingular rings) is the largest possible.

1.1 **Lemma.** *An indecomposable nonlocal fQ-ring with primitive idempotents has nonzero socle. The image of any homomorphism between a principal indecomposable ideal and a disjoint ideal is simple.*

Proof. Let R be such a ring and let $e \in R$ be a primitive idempotent. Then either $eR(1-e) \neq 0$ or $(1-e)Re \neq 0$. First assume the latter and let $a \in (1-e)Re$ be nonzero. Then Ra is essential in Re, since Re is an indecomposable injective, and $L = Ra \oplus R(1-e)$ is quasi-injective and therefore invariant under endomorphisms of $Re \oplus R(1-e)$. That is, $(1-e)Re \subseteq Ra$. As a is arbitrary, this implies that Ra is simple and so R has nonzero socle.

Now consider the case $eR(1-e) \neq 0$ and let $a \in eR(1-e)$ be any nonzero element. If Rf is the injective hull of Ra, then $Ra \oplus Re$ is quasi-injective and essential in $Rf \oplus Re$. It follows, as above, that $eRf \subseteq Ra$. Since eRe is local ([F1], Proposition 5.8) Ra is indecomposable and so Rf is indecomposable. Hence f is a primitive idempotent and so Ra is the unique subideal of Rf by the preceding paragraph. □

1.2 **Corollary.** *An indecomposable nonlocal fQ-ring with dense primitive idempotents has essential socle.*

Proof. Let e be a primitive idempotent in such a ring R. If Re is simple, then the injective hull of all its images in R is a ring summand of R and therefore must be R. So we can assume that Re is not simple. If $(1-e)Re \neq 0$, then, by Lemma 1.1, Re contains a simple ideal which must be essential in Re since Re is indecomposable and injective. So we need only consider the case $(1-e)Re = 0$. Then $eR(1-e) \neq 0$

since R is indecomposable and therefore the R-subideals of Re and the eRe-ideals of eRe coincide. If L is a cyclic ideal contained in the radical J of eRe, then since eRe is local, there is a nonzero homomorphism from L to $R(1-e)$. This homomorphism can be extended to a homomorphism $Re \to R(1-e)$ since $R(1-e)$ is injective. But by Lemma 1.1 every homomorphism $Re \to R(1-e)$ kills J - a contradiction. Therefore $(1-e)Re \neq 0$ and Re has nonzero socle. □

1.3 **Corollary.** *An indecomposable nonlocal fQ-ring with finite identity is a Q-ring.*

Proof. If e is a primitive idempotent in such a ring R, then Re has essential socle S (by Corollary 1.2) which must be simple and eRe is local with radical J consisting of zero divisors (Theorem 1.22 of [G]). As every endomorphic image of Re must contain S, this implies that $J = 0$. Hence by Lemma 1.1, Re has at most one submodule. Therefore R is a sum of Artinian ideals and so is itself Artinian. That is, R is a Q-ring. □

The structure of Q-rings with finite identities is determined in [I1] (except for the local case) so **from now** on we will only consider rings with infinite identities. Thus all our proofs will assume it and the interested reader is referred to [I1] for proofs of the finite identity case. But in the interest of completeness the statements of our results will be given in full generality.

1.4 **Lemma.** *A nonsingular fQ-ring is a von Neumann regular ring.*

Proof. If R is an fQ-ring, then $R \cong End(R)$ and so $R/J(R)$ is a von Neumann ring. But $J(R)$ is also the singular ideal of R so if R is nonsingular it is a von Neumann ring ([G], Theorem 1.22). □

1.5 **Lemma.** *Let e, f be orthogonal idempotents in an fQ-ring R and let T be a subideal in Re. If T has a nonzero image S in Rf, then T contains an idempotent. If S is cyclic, then it is the image of a subideal of T generated by an idempotent.*

Proof. Let $\phi : T \to S$ be an epimorphism. We may assume that S is cyclic. If S contains an idempotent f', then since Rf' is projective, the pre-image of Rf' under ϕ splits and therefore T contains a copy of Rf'. But Rf' is injective so this copy in T is generated by an idempotent. So we can assume that S contains no idempotents. Let $Re' = E(T)$. Since S is cyclic, $Re' \oplus S$ is quasi-injective and so ϕ extends to a homomorphism $\Phi : Re' \to S$. But then $T\Phi = S$ and $Re'\Phi = S$ and it follows that $T = Re'$. For if T is a proper subideal of Re', then, since $ker\Phi \subseteq T$, $T/ker\Phi$ is a proper subideal of $Re'/Ker\Phi = S$ - a contradiction. Hence $T = Re'$ and so T contains an idempotent. □

1.6 **Lemma.** *Let e, f be orthogonal idempotents in an fQ-ring R. If $Re \cong Rf$, then every cyclic ideal in Re is injective.*

Proof. Let $Ra \subseteq Rf$. Then $Ra \oplus Re$ is quasi-injective and so $eRf_1 \subseteq Ra$ where $Rf_1 = E(Ra)$. But Re contains an isomorphic copy of Rf_1, which must be a summand, and so there is an epimorphism $Re \to Rf_1$. Therefore eRf_1 generates Rf_1 and so $Ra = Rf_1$. Hence Ra is injective. □

1.7 **Lemma.** *Let e_1, e_2, e_3 be orthogonal idempotents in an fQ-ring R. If $\alpha : Re_1 \to Re_2$ and $\beta : Re_2 \to Re_3$ and the image of α does not contain injective ideals, then $\alpha\beta = 0$.*

Proof. $Re_1\alpha$ does not contain an idempotent as it contains no injective ideals. Therefore, by Lemma 1.5, $Re_1\alpha$ cannot have an image in Re_3. Hence $\alpha\beta = 0$. □

Definition. An idempotent e in an fQ-ring R is *abelian* if all idempotents in the ring eRe are central (in eRe). An fQ-ring is of *Type* I if it has dense abelian idempotents.

The standard definition of Type I for self-injective von Neumann rings (see [G]) is that such a ring is of Type I if it contains an abelian idempotent which is not orthogonal to any central idempotent (except zero).

1.8 **Lemma.** *An injective von Neumann regular ring is of Type I, in our sense, if and only if it is of Type I in von Neumann's sense.*

Proof. Let R be of Type I in our sense. Then the two sided ideal generated by the abelian idempotents is essential as a left ideal and so R is of Type I in von Neumann's sense by Theorem 10.4 of [G]. Now assume that R is of Type I in von Neumann's sense. Then every ideal contains an abelian idempotent (by Theorem 10.4 of [G]) and so R has an essential ideal generated by abelian idempotents. □

Definition. A ring is (left) *idempotent nonsingular* if none of its elements is annihilated (on the left) by an essential (left) ideal generated by idempotents.

It is clear that nonsingular rings and rings with finite identities are idempotent nonsingular.

1.9 **Theorem.** *An idempotent nonsingular fQ-ring is a direct sum of a ring with dense primitive idempotents and a ring with no primitive idempotents.*

Proof. Let Re be the injective hull of the ideal P generated by the primitive idempotents in an fQ-ring R. By Lemma 1.1 any image L in $R(1-e)$ of Rf, for a primitive idempotent f, is simple so its injective hull is generated by a primitive idempotent. But $R(1-e)$ has no primitive idempotents so any homomorphism from Re to $R(1-e)$ must kill P. This contradicts the assumption that R is idempotent nonsingular. Hence $eR(1-e) = 0$.

Conversely, if $f \in Re$ is a primitive idempotent, then any homomorphism $\alpha : R(1-e) \to Rf$ has a simple image (by Lemma 1.1). Then the kernel K of α must be essential since $R(1-e)$ has no simples. We now show that the ideal I generated by the idempotents in K is essential in K. Let $x \in K$ be any element and consider $Rf_1 = E(Rx)$. If Rf_1 is in I there is nothing to prove. If Rf_1 is not in K, then $Rf_1/(Rf_1 \cap K)$ is simple. But f_1 is not primitive so it is a sum of orthogonal idempotents g_1, g_2 and one of these is in K, say g_1. But $Rx \cap Rg_1 \neq 0$ so $I \cap Rx \neq 0$ as required. Thus we have shown that there can be no homomorphism $R(1-e) \to Rf$ for any primitive idempotent $f \in Re$ and so it follows that $(1-e)Re = 0$. Therefore R decomposes as claimed. □

1.10 **Theorem.** *An idempotent nonsingular fQ-ring is a direct sum of a ring with essential singular ideal and a von Neumann regular ring.*

Proof. Let $Z(R)$ be the singular ideal of an idempotent nonsingular fQ-ring R and let $Re = E(Z(R))$. As a homomorphic image of a singular module is singular, there are no images in $R(1-e)$ of Re. That is $eR(1-e) = 0$. As $R(1-e)$ is nonsingular, its endomorphism ring $(1-e)R(1-e)$ is a von Neumann ring. Assume that $(1-e)Re \neq 0$ and let $x \in (1-e)Re$ be nonzero. The annihilator $\alpha(x)$ of x in $R(1-e) = (1-e)R(1-e)$ is an essential subideal and must contain an essential subideal generated by idempotents since $(1-e)R(1-e)$ is a von Neumann ring. This contradicts the assumption that R is idempotent nonsingular. □

1.11 **Corollary.** *If an indecomposable idempotent nonsingular fQ-ring R has orthogonal idempotents e, f such that $Re \cong Rf$, then R is a von Neumann regular ring.*

Proof. By Lemma 1.6 every cyclic ideal in Re is injective and thus is generated by an idempotent. Hence Re is nonsingular. As R is indecomposable, Theorem 1.10 implies that all of R is nonsingular and thus a von Neumann ring. □

We now prove the first of the main decomposition theorems which show that the theory of types which applies to injective von Neumann rings can be extended to idempotent nonsingular fQ-rings.

1.12 **Theorem.** *An idempotent nonsingular fQ-ring is the direct sum of a ring of Type I and a ring with no abelian idempotents.*

Proof. Let A be the set of abelian idempotents in an idempotent nonsingular fQ-ring R and let $Re = E(RA)$. We want to show that $eR(1-e) = 0 = (1-e)Re$. First consider $eR(1-e)$ and assume that $eR(1-e) \neq 0$. Since R is idempotent nonsingular, there is an $f \in A$ such that $fR(1-e) \neq 0$. Pick any nonzero $a \in fR(1-e)$ and let $Re' = E(Ra)$. Then e' is infinite, for otherwise it would be a sum of abelian idempotents and would thus be in Re. So there are orthogonal idempotents $e_1, e_2 \in Re'$ such that $e_1Re_2 \neq 0$. Let $c \in e_1Re_2$ be nonzero. As $Rc \cap Ra \neq 0$, there is an $Rb \subseteq Rc$ which is the image of a $K \subseteq Rf$. By Lemma 1.5, Rb is an image of $Rf' \subseteq Rf$ and so $f'R(1-e) \neq 0$. But Rb is also an image of Re'_1 for an $e'_1 \in Re_1$ and therefore Rf' and Re'_1 have isomorphic summands. As f' is abelian, any summand of Rf' is generated by an abelian idempotent so Re'_1 contains an abelian idempotent - a contradiction to the fact that $e'_1 \in R(1-e)$. Therefore $eR(1-e) = 0$.

We now show that $(1-e)Re = 0$. Assume not and let $\alpha : R(1-e) \to Re$ be a nonzero homomorphism. We can regard α as an endomorphism of R. By Lemma 1.5, α kills every subideal of R which has no idempotents so α kills $Z(R)$, the singular ideal of R. Therefore α can be considered to be a homomorphism from $\overline{R} = R/Z(R)$. By Theorem 19.27 of [F2], $\overline{R}$ is a self-injective von Neumann ring. As R is idempotent nonsingular, $R\alpha = \overline{R}\alpha$ is a nonsingular $\overline{R}$-module. Therefore $\overline{\ker\alpha}$ is a summand of $\overline{R(1-e)}$ and so $R(1-e)\alpha = \overline{R(1-e)}\alpha$ is isomorphic to a summand of $\overline{R(1-e)}$. As idempotents lift from $\overline{R}$ to R, there is an idempotent $f \in R(1-e)$ such that $\overline{\ker\alpha} = \overline{R(1-e-f)}$. Then $\alpha|_{\overline{Rf}}$ is an isomorphism. Let $f_1 \in Rf$ be an idempotent and $f_2 = f - f_1$. Then $f_iRf_j = 0$, for $i \neq j$, as otherwise Rf_j contains a homomorphic image of $\overline{Rf_i} \cong Rf_i\alpha$, so there would be a homomorphism from a subideal of Re to Rf_j—a contradiction. Hence the idempotents f_1, f_2 are central in fRf which means that f is abelian—a contradiction. Therefore $(1-e)Re = 0$ and the required decomposition follows. □

Definition. A module is *finitary* or *von Neumann finite* if it cannot be expressed as a direct sum of two nonzero submodules, one of which is isomorphic to the module itself. An idempotent is *finitary* or *von Neumann finite* if the ideal generated by it is finitary. A module (idempotent) is *infinitary* or *von Neumann infinite* if it is not finitary. An fQ-ring is of *Type* II if it has dense finitary idempotents but has no abelian idempotents. An fQ-ring is of *Type* III if it contains no finitary idempotents.

1.13 **Lemma.** *An injective von Neumann regular ring is of Type II in our sense if and only if it is of Type II in von Neumann's sense.*

Proof. As for Lemma 1.8 using Theorem 10.5 of [G]. □

1.14 **Theorem.** *An fQ-ring of Type III is a von Neumann regular ring of Type III.*

Proof. Let R be an fQ-ring of Type III and let $R = L_1 \oplus L_2$ where $L_1 \cong R$. Then $L_1 = L_{11} \oplus L_{12}$ where $L_{11} \cong L_1$ and $L_{12} \cong L_2$. By Lemma 1.6 every cyclic subideal of L_2 is injective. Now let $a \in L_1$ be nonzero and assume that Ra contains no injective ideals. Let $Re = E(Ra)$. Then since R has no finitary idempotents, $Re = T_1 \oplus T_2$, where $T_1 \cong Re$. By the above argument every cyclic ideal in T_2 is injective. As Ra is essential in Re, $Ra \cap T_2 \neq 0$, which means that Ra contains injective ideals - a contradiction to our assumption. Therefore every cyclic ideal in L_1 is injective and so R is a von Neumann ring. □

We now prove the final theorem on decomposition into types.

1.15 **Theorem.** *An idempotent nonsingular fQ-ring with no abelian idempotents is a direct sum of a ring of Type II and a ring of Type III.*

Proof. Let R be an idempotent nonsingular fQ-ring without abelian idempotents and let X be the set of finitary idempotents in R. Let $Re = E(RX)$. We want to show that $eR(1-e) = (1-e)Re = 0$.

Let $f \in R(1-e)$. Then $Rf = Rf_1 \oplus Rf_2$ where $Rf \cong Rf_1$. By Lemma 1.6 every cyclic subideal of Rf is generated by an idempotent and is injective. Hence if $0 \neq a \in eR(1-e)$, then the pre-image P of Ra in Re under any homomorphism contains a copy of Ra. Now Ra is infinitary and all its cyclic subideals are injective and infinitary. Therefore $RX \cap P$ contains infinitary idempotents - a contradiction. Hence $eR(1-e) = 0$.

Since $R(1-e)$ has no finitary idempotents, the ring $(1-e)R(1-e)$ is of Type III and is a von Neumann ring by Theorem 1.14. Let $x \in (1-e)Re$ and assume x is nonzero. If the annihilator $\alpha(x)$ of x in $R(1-e)$ is not essential, then there is a $c \in R(1-e)$ such that $Rc \cap \alpha(x) = 0$. Then $Rc \cong Rcx$. As Rc is generated by an infinitary idempotent, it is injective and so Rcx is generated by an infinitary idempotent - a contradiction to the definition of Re. Therefore $\alpha(x)$ must be essential in $R(1-e)$. But then $\alpha(x)$ contains an essential subideal generated by idempotents, since $R(1-e) = (1-e)R(1-e)$ is a von Neumann ring, which contradicts the assumption that R is idempotent nonsingular. So $(1-e)Re = 0$ as required. □

§2. fQ-rings with dense primitive idempotents

Rings with dense primitive idempotents have indecomposable summands and in the case of fQ-rings these determine the structure of the rings. In this section we determine the structure of indecomposable idempotent nonsingular fQ-rings with dense primitive idempotents by representing them as rings of matrices. Example 2.6 shows that the idempotent nonsingular assumption is essential. Except for the statement of the main result (Theorem 2.3) we will only be concerned with rings which are not von Neumann rings and not local.

2.1 **Lemma.** *Every minimal ideal in an idempotent nonsingular fQ-ring is the image of an ideal generated by a primitive idempotent.*

Proof. Let R be an idempotent nonsingular fQ-ring and let M be a minimal ideal in R with injective hull Re. If $eM \neq 0$ there is nothing to prove since e must be a primitive idempotent. Therefore we need only consider the case when M is an image of $R(1-e)$. Since M is minimal, its annihilator $\alpha(M)$ in $R(1-e)$ is a maximal subideal. If $\alpha(M)$ is not essential in $R(1-e)$, then its complement is a minimal ideal (isomorphic to M) which is a summand of $R(1-e)$ and is therefore generated by a primitive idempotent as required. If $\alpha(M)$ is essential in $R(1-e)$, then it must contain an essential subideal generated by idempotents.

For let $x \in \alpha(M)$ be any element and let Rf be the injective hull of Rx. If $Rf \subseteq \alpha(M)$, then Rx intersects the subideal of $\alpha(M)$ generated by idempotents. So we can assume that $Rf \subsetneq \alpha(M)$. Then $Rf/(Rf \cap \alpha(M))$ is simple and isomorphic to M. If f is primitive, then there is nothing to prove so we assume that $f = f_1 + f_2$ for orthogonal idempotents f_1 and f_2. One of f_1, f_2 must be in $\alpha(M)$, say f_1. As Rx is essential in Rf, $Rx \cap Rf_1 \neq 0$ and so Rx intersects the subideal of $\alpha(M)$ generated by idempotents, as required. But this means that M is annihilated by an essential ideal generated by idempotents contrary to the assumption about R.□

Definition. An *idempotent chain* is a sequence of primitive idempotents $\{e_i \mid i \in I\}$ such that $Soc(Re_i)$ is an image of Re_{i+1} for all $i \in I$ where I is a segment in $\mathbb{Z}$ unbounded above.

2.2 **Lemma.** *Let R be an indecomposable idempotent nonsingular fQ-ring with dense primitive idempotents. If R is not local and not a von Neumann regular ring, then it has a unique idempotent chain.*

Proof. Let $e_0 \in R$ be a primitive idempotent and assume that R is not von Neumann and not local. Then one of $e_0R(1-e_0)$ and $(1-e_0)Re_0$ is nonzero. Assume the former. By Lemma 1.1 any image of Re_0 in $R(1-e_0)$ is simple. By Corollary 1.11 it is unique. Let Re_{-1} be its injective hull. If Re_{-1} has an image in $R(1-e_{-1})$, then that image is simple. Denote its injective hull by Re_{-2}. In this manner we can construct a sequence of disjoint ideals Re_n, $n = -1, -2, -3, \ldots$, such that Re_n has a nonzero image in Re_{n-1}. As we have assumed that R is not a von Neumann ring, Re_0 is not simple (Lemma 1.6) and so it has a proper subideal which must be its socle (Lemma 1.1). By Lemma 2.1, $Soc(Re_0)$ is an image of an ideal generated by a primitive idempotent. By Corollary 1.11 this ideal is unique. Denote it by Re_1. The socle of Re_1 is an image of an ideal Re_2 generated by a primitive idempotent, and so on. In this manner we obtain a sequence of primitive idempotents e_n for all integers n. This sequence $\mathcal{C}$ may be finite on the left, that is, $e_k = 0$ for some negative integer k in which case $e_i = 0$ for all $i < k$. As our standing assumption is that R does not have a finite identity, $\mathcal{C}$ will always be infinite on the right as we shall see below.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two idempotent sequences in R and assume that there is an idempotent e which belongs to both of them. It follows that the sequences $\mathcal{C}$ and $\mathcal{C}'$ coincide to the right of e since $Soc(Re)$ is the image of a unique $Rf \subseteq R(1-e)$. As Re has a unique image in $R(1-e)$, $\mathcal{C}$ and $\mathcal{C}'$ coincide to the left of e. That is, $\mathcal{C} = \mathcal{C}'$. Therefore the idempotent chains partition the set of primitive idempotents in R.

We now show that there can only be one idempotent chain in R. Let $Rf = E(R\mathcal{C})$ and consider $R(1-f)$. Since R is idempotent nonsingular and every Re, for $e \in \mathcal{C}$, has images only in $R\mathcal{C}$, there cannot be any homomorphisms from Rf to $R(1-f)$. That is, $fR(1-f) = 0$. Since every simple ideal in $R\mathcal{C}$ can be an image only of an Re for some $e \in \mathcal{C}$, there are no homomorphisms $R(1-f) \to Rf$. Therefore $(1-f)Rf = 0$ and so Rf is generated by a central idempotent. Since R is indecopmposable, this means that $Rf = R$.

Now assume that an idempotent e occurs twice in $\mathcal{C}$. Then $\mathcal{C}$ contains only a finite number of distinct idempotents and so R must have finite identity. Therefore all the idempotents in $\mathcal{C}$ are distinct and $\mathcal{C}$ is infinite - at least to the right. □

Definition. Let D be a sfield and V a null D-algebra one dimensional on both sides. If $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of positive, respectively all, integers, then the rings $H(\mathbb{N}, D, V)$ and $H(\mathbb{Z}, D, V)$ are defined to be the following matrix rings.

$$H(\mathbb{N}, D, V) = \begin{pmatrix} D & & & \\ V & D & & \\ & V & D & \\ & & \ddots & \ddots \end{pmatrix} \qquad \text{with } \mathbb{N} \text{ rows and columns.}$$

$$H(\mathbb{Z}, D, V) = \begin{pmatrix} \ddots & & & & \\ \ddots & D & & & \\ & V & D & & \\ & & V & D & \\ & & & \ddots & \ddots \end{pmatrix} \qquad \text{with } \mathbb{Z} \text{ rows and columns.}$$

2.3 **Theorem.** *Let R be an indecomposable idempotent nonsingular fQ-ring with infinite identity and dense primitive idempotents. If R is not a von Neumann regular ring, then it is isomorphic to one of the rings $H(\mathbb{N}, D, V)$, $H(\mathbb{Z}, D, V)$ for some sfield D.*

Proof. Assume that R is not a von Neumann ring. By Lemma 2.2, R has a unique idempotent chain $\mathcal{C}$ and R is the injective hull of $I = R\mathcal{C}$. For any $r \in R$ and $e \in \mathcal{C}$ the ideal Rer is an image of Re and is therefore in I. That is, I is a two sided ideal. Therefore any endomorphism of R induces an endomorphism of I. Conversely, since R is injective, every endomorphism of I extends to an endomorphism of R. This extension is unique. For if two endomorphisms of R, obtained by right multiplication by x and y, say, coincide on I, then $I(x-y) = 0$. Since R is idempotent nonsingular this is only possible if $x = y$. Therefore $R \cong End(R) \cong End(I)$ and we obtain the required representation of R from that of $End(I)$.

Let $e \in \mathcal{C}$ be any idempotent. We know Re has a unique subideal which is not an image of Re. Therefore $End(Re)$ is a sfield. The socle of Re is quasi-injective so is invariant under eRe and therefore endomorphims of $Soc(Re)$ lift uniquely to $End(Re)$. That is, $Soc(Re)$ is a one dimensional right vector space over eRe. Since $Soc(Re)$ is an image of Re', where e' is the successor of e in $\mathcal{C}$, it is a one dimensional vector space over $e'Re'$. Therefore $eRe \cong e'Re'$ as sfields. Hence we can denote all the $eRe's$ by D and all the bimodules $e'Re$ by V. Since $e''Re'.e'Re = 0$, we can regard the $V's$ as null algebras.

Now $I = \bigoplus_{e\in\mathcal{C}} Re$, so $End(I)$ can be represented as the ring of all $\mathbb{N}\times\mathbb{N}$ or $\mathbb{Z}\times\mathbb{Z}$ row finite matrices whose entries at the place (i,j) are arbitrary elements of $Hom(Re_i, Re_j)$. From Lemma 2.2 these are nonzero only if $i = j$ or $i = j+1$. Using the above identifications this gives that R is isomorphic to one of the rings $H(\mathbb{N}, D, V)$, $H(\mathbb{Z}, D, V)$ depending on the sequence type of $\mathcal{C}$. □

2.4 **Corollary.** *A nonlocal indecomposable idempotent nonsingular fQ-ring with infinite identity and dense primitive idempotents need not be right injective.*

Proof. The ring $H = H(\mathbb{N}, D, V)$ is not right injective since eH, where e is the matrix whose only nonzero entry is the identity of D at the place $(1,1)$, is not injective. But H is very close to being right injective since all the other principal indecomposable right ideals are injective. □

Definition. Let D be a sfield and V a null D-algebra one dimensional on both sides. The ring of all row finite $\omega\times\omega$-matrices over D is denoted by $M_\omega(D)$. For an integer $m \geq 2$, define $H(m, D, V)$ to be the following matrix ring.

$$H(m,D,V) = \begin{pmatrix} D & & & & V \\ V & D & & & \\ & V & \ddots & & \\ & & \ddots & D & \\ & & & V & D \end{pmatrix} \qquad \text{with } m \text{ rows and columns.}$$

It is well known that the only indecomposable self-injective von Neumann rings with dense primitive idempotents are the rings $M_\omega(D)$ ([G], Theorem 9.12). It is shown in [I1] that the rings $H(m, D, V)$ are the only indecomposable Q-rings with finite identities which are not von Neumann rings or local rings.

2.5 **Corollary.** *A ring is an idempotent nonsingular fQ-ring with dense primitive idempotents if, and only if, it is a product of rings of the following types: local fQ-rings, $M_\omega(D)$, $H(m, D, V)$, $H(\mathbb{N}, D, V)$ and $H(\mathbb{Z}, D, V)$.*

Proof. Every primitive idempotent is contained in an indecomposable (ring) summand. Each of these is isomorphic to one of the rings listed above. The product of these rings is their injective hull. Therefore the ring is isomorphic to the product of the subrings. A standard argument shows that a product of such rings is an fQ-ring. □

2.6 **Example.** Let A be a von Neumann ring, let V be a singular simple A-module and let D be the endomorphism sfield of V. Let R be the matrix ring $\begin{pmatrix} D & 0 \\ V & A \end{pmatrix}$. If A is of Type III, then R is not idempotent nonsingular, it cannot be decomposed into Types and it is not a direct sum of a nonsingular ring and a ring with essential singular ideal. It does not decompose into a ring with dense primitive idempotents and a ring with no primitive idempotents. This shows that the idempotent nonsingular assumptions cannot be dropped from Theorems 1.9, 1.10, 1.12 and 1.15. If $A = M_\omega(D)$, for infinite ω, then R has dense primitive idempotents but it does not satisfy Theorem 2.3.

Acknowledgement

The author is indebted to the referee for his many helpful suggestions about improving the presentation of this paper; and to Safdar Raza Syed for pointing out an error in a proof.

References

[F1] C. Faith, *Lectures on Injective Modules and Quotient Rings*, Lecture Notes in Mathematics **49** (1967), Springer-Verlag. MR **37:**2791

[F2] ——, *Algebra* II. *Ring Theory*, Springer-Verlag, New York, 1976. MR **55:**383

[G] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979. MR **80e:**16011

[I1] G. Ivanov, *Non-local rings whose ideals are quasi-injective*, Bull. Aust. Math. Soc. **6** (1972), 45–52. MR **45:**311

[I2] G. Ivanov, *Rings with quasi-injective ideals*, Bull. Aust. Math. Soc. **50** (1994), 197–204. CMP 95:02

[J] S. K. Jain, *Rings whose cyclic modules have certain properties and the duals*, Lecture Notes Series **25** (1976), Marcel Dekker, New York, 143–160.

Department of Mathematics, Macquarie University, Sydney, Australia 2109

E-mail address: `ivanov@mpce.mq.edu.au`