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SINGULAR INTEGRALS WITH EXPONENTIAL WEIGHTS

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ABSTRACT. We study the operators

$$\overline{V}f(t) = \frac{1}{w(t)}V(f(r)w(r))(t)$$

where V is the Hardy-Littlewood maximal function, the Hilbert transform or Carleson operator.

Under suitable conditions on the weight $w(t)$ of exponential type, we prove boundedness of $\overline{V}$ from L^p spaces, defined on $[1,+\infty)$ with respect to the measure $w^2(t)dt$, to $L^p + L^2$, $1 < p \leq 2$, with the same density measure. These operators, that arise in questions of harmonic analysis on noncompact symmetric spaces, are bounded from L^p to $L^p, 1 < p < \infty$, if and only if $p = 2$.

The study of convergence properties of inverse spherical transforms of radial functions on noncompact symmetric spaces [1], [6], [7] requires estimates on singular integrals with exponential weights, as defined below, due to the exponential growth at infinity of the radial part $D(t)dt$ of the measure, where

$$D(t) = (Sht)^p \ (Sh\ 2t)^q,$$

with p and q suitable nonnegative integers that depend upon the geometry of the symmetric space.

In what follows we define a class of functions $w(t)$, that include exponentials, and prove boundedness of the operators $\frac{1}{w(t)}V(f(r)w(r))(t)$ from $L^p(w^2(t)dt)$ to $L^p(w^2(t)dt) + L^2(w^2(t)dt), 1 < p \leq 2$, where V is the classical Hardy-Littlewood maximal function, the maximal Hilbert transform or the maximal Carleson operator. Boundedness from L^p to $L^p, 1 < p < \infty$, w.r. to the measure $w^2(t)dt$, holds for $p = 2$ only.

Theorem 1. *We denote by $L^p = L^p([1,+\infty), w^2(t)dt)$ the space of functions defined on $[1,+\infty)$ that are L^p with respect to the density measure $w^2(t)dt$.*

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Let $\overline{V}$ *be one of the following operators*:

$$\overline{M}f(t) = \frac{1}{w(t)} \sup_{\sigma>0} \frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |f(r)|w(r)dr,$$
$$\overline{H}f(t) = \frac{1}{w(t)} \int_1^\infty \frac{f(r)}{t-r} w(r)dr,$$
$$\overline{C}f(t) = \frac{1}{w(t)} \sup_{R>1} \left| \int_1^\infty \frac{e^{iR(t-r)}}{t-r} f(r)w(r)dr \right|.$$

Then $\overline{V}$ *is a bounded operator from* L^p *to* $L^p + L^2$, $1 < p \leq 2$, *provided that*

i) $w(t) \geq a > 0$ *for all* $t \in [1, +\infty)$;

ii) $w(t)$ *satisfies the following "doubling condition"*

$$c_1 w(k) \leq w(t) \leq c_2 w(k) \quad \textit{for all } t \in [k-1, k+2],$$

where k *denotes any positive integer and the constants* $c_i = c_i(w), i = 1, 2,$ *are positive and independent of* k.

Proof. Let us observe that

$$\overline{V}f(t) = \frac{1}{w(t)} V(f(r)w(r))(t)$$

where V denotes respectively the classical Hardy-Littlewood maximal function, the Hilbert transform and Carleson operators [3], [4], [8].

Due to the $L^2(\mathbb{R}, dr)$ boundedness of these classical operators the theorem in the case $p = 2$ is trivial, for

$$||\overline{V}f||_2^2 = \int |\overline{V}f(t)|^2 w^2(t)dt = \int |V(f(r)w(r))(t)|^2 dt$$
$$\leq \int |f(r)|^2 w^2(r)dr = ||f||_2^2.$$

Next we decompose the operators as follows. For every fixed $k = 1, 2, 3, \ldots$ we write

$$g_k(r) = f(r)w(r)\chi_{[k,k+1)}(r),$$
$$\varphi_k(t) = \chi_{[k-1,k+2)}(t),$$
$$\varphi_k(t) + \psi_k(t) = 1 \quad \text{for every } t \geq 1$$

where χ_E denotes the characteristic function of the set E.

Suppose for instance $\overline{V} = \overline{M}$. Then observe that

$$\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |f(r)|w(r)dr = \sum_{k=1}^\infty \left(\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |g_k(r)|dr \right) [\varphi_k(t) + \psi_k(t)] \tag{1}$$

and

$$\begin{aligned} \overline{M}f(t) \leq{}& \frac{1}{w(t)} \sup_\sigma \sum_{k=1}^\infty \left(\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |g_k(r)|dr \right) \varphi_k(t) \\ &+ \frac{1}{w(t)} \sup_\sigma \sum_{k=1}^\infty \left(\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |g_k(r)|dr \right) \psi_k(t). \end{aligned} \tag{2}$$

We adopt a similar decomposition if $\overline{V} = \overline{H}$ or $\overline{C}$.

We shall prove that the first term on the right hand side of (2) maps boundedly L^p into L^p, while the second one maps boundedly L^p into $L^2, 1 < p \leq 2$. The theorem then follows.

We start with the first term which is dominated by $\frac{1}{w(t)} \sum\limits_k V g_k(t)\varphi_k(t)$. Since the φ_k's have essentially disjoint supports and V is bounded on $L^p(\mathbb{R}, dr), 1 < p < \infty$, we have

$$\begin{aligned}
&\left\| \frac{1}{w(t)} \sum_{k=1}^{\infty} V g_k(t)\varphi_k(t) \right\|_p^p \\
&\cong \sum_k w^{-p}(k) \int_{k-1}^{k+2} |V g_k(t)|^p w^2(t) dt \leq c \sum_k w^{2-p}(k) \int_1^{\infty} |V g_k(t)|^p dt \\
&\leq c \sum_{k=1}^{\infty} w^{2-p}(k) \int_1^{\infty} |g_k(t)|^p dt = c \sum_{k=1}^{\infty} w^{2-p}(k) \int_k^{k+1} |f(r)|^p w^p(r) dr \\
&\cong \sum_{k=1}^{\infty} \int_k^{k+1} |f(r)|^p w^2(r) dr = c||f||_p^p.
\end{aligned}$$

So we proved

$$\left\| \frac{1}{w(t)} \sum_{k=1}^{\infty} V g_k(t)\varphi_k(t) \right\|_p \leq c_{p,w}||f||_p, \quad 1 < p < \infty. \tag{3}$$

Next we deal with the second term on the right hand side of (2).

Suppose again $\overline{V} = \overline{M}$. Then from (1) we know that

$$\frac{1}{w(t)} \sup_{\sigma} \sum_{k=1}^{\infty} \left(\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |g_k(r)| dr \right) \psi_k(t) \leq \overline{M} f + \frac{1}{w(t)} \sum_{k=1}^{\infty} V g_k(t)\varphi_k(t)$$

is also bounded from L^2 to itself, due to the L^2 boundedness of $\overline{M} f$ and the estimate in (3) in the case $p = 2$.

By standard interpolation techniques to prove boundedness of $\overline{M}$ from L^p to L^2 it is then sufficient to show a restricted (L^p, L^2) estimate, $1 < p < 2$.

Similarly for $\overline{H}$ and $\overline{C}$.

Now let $f = \chi_E, \quad E \subset [1, +\infty)$ any measurable set. In the case $||\chi_E||_2 \geq 1$ the desired estimate is trivial and it is based on the L^2 boundedness of the operator we are dealing with, already established and the fact that $||\chi_E||_2 \leq ||\chi_E||_p, \quad 1 < p \leq 2$.

If instead $||\chi_E||_2 < 1$, then we observe that for t in the support of ψ_k

$$M g_k(t) = \sup_{\sigma > 0} \left(\frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |g_k(r)| dr \right) \leq \frac{2}{|t-k|} \int_k^{k+1} \chi_E(r) w(r) dr. \tag{4}$$

Indeed it has to be $\sigma \geq \frac{\text{dist } (t,k)}{2}$ to have a nonzero integral under the supremum. Therefore $\frac{1}{w(t)} \sum\limits_k M g_k(t)\psi_k(t) \leq \frac{2}{w(t)} \frac{w(k)}{|t-k|} |E_k| \psi_k(t)$, when $|E_k|$ denotes the Euclidean measure of $E \cap [k, k+1)$. Then by i)

$$\begin{aligned}
&\left\| \frac{1}{w(t)} \sum_k Mg_k(t)\psi_k(t) \right\|_2 \le \sum_k \left\| \frac{1}{w(t)} \sum_k Mg_k(t)\psi_k(t) \right\|_2 \\
&\le \sum_k w(k)|E_k| \left(\int \frac{1}{(t-k)^2} \psi_k(t)dt \right)^{\frac{1}{2}} \\
&\le c \sum_k w(k)|E_k| \le c_w \sum_k w^2(k)|E_k| \\
&\le c_w \sum_k \left(\int_k^{k+1} \chi_E(r)dr \right) w^2(k) \cong c_w \sum_k \int_k^{k+1} \chi_E(r) w^2(r)dr \\
&= c_w \int_1^\infty \chi_E(r) w^2(r)dr \le c_w ||\chi_E||_p.
\end{aligned}$$

In case $\overline{V} = \overline{H}$ or $\overline{C}$ the estimate in (4) can be obtained by taking the absolute value inside the sign of integral.

The theorem is then proved.

Corollary. *Let*

$$H^* f(t) = \frac{1}{w(t)} \sup_{\varepsilon > 0} \left| \int_{\substack{1 \\ |t-r|>\varepsilon}}^\infty \frac{f(r)}{t-r} w(r)dr \right|$$

and

$$C^* f(t) = \frac{1}{w(t)} \sup_{\substack{R>1 \\ \varepsilon>0}} \left| \int_{\substack{1 \\ |t-r|>\varepsilon}}^\infty \frac{e^{iR(t-r)}}{t-r} f(r)w(r)dr \right|.$$

Under the same assumptions of Theorem 1, H^* and C^* map boundedly L^p into $L^p + L^2$, $\quad 1 < p \le 2$.

Proof. The Corollary follows at once from Theorem 1 and the estimate ([9], [5])

$$H^* f(t) \le \frac{1}{w(t)} [M(f(r)w(r))(t) + MH(f(r)w(r))(t)]$$

and

$$C^* f(t) \le \frac{1}{w(t)} [M(f(r)w(r))(t) + MC(f(r)w(r))(t)].$$

Theorem 2. *Let $\overline{V}$ be as in Theorem 1. Then $\overline{V}$ is a bounded operator from L^p to L^p if and only if $p = 2$.*

Proof. We just have to prove unboundedness for $p \ne 2$. Suppose $1 < p < 2$ and take $f(r) = \chi_{[1,2]}(r)$, $w(t) = e^t$. Then

$$\overline{H} f(t) \cong \frac{1}{te^t}, \text{ for } t >> 1.$$

From this the unboundedness of $\overline{H}$ follows. Now suppose $p > 2$ and assume by contradiction that $\overline{H}f$ is a bounded operators from L^p to the space of tempered distributions. Then by duality $\overline{H}f : C_c^\infty \to L^{p'}, \quad p' < 2$, is a bounded operator. This is false as the preceding counterexample (slightly modified to have $f \in C^\infty$) shows.

Finally we observe that $w(t) = t^\alpha e^{\beta t}$ with $\alpha \in \mathbb{R}$ and $\beta > 0$ satisfies the conditions of Theorem 1. In the case $\overline{V} = \overline{C}, \alpha = 0$ and $\beta = \frac{p+2q}{2}$ (more precisely $w(t) = \sqrt{D(t)}$), Theorem 1 has been used to obtain sharp results on almost everywhere convergence of inverse pherical transforms on noncompact symmetric spaces [7]. In [2] estimates in the context of Lorentz spaces have been obtained for $\overline{V} = \overline{H}$ in the case $\alpha = 0$ and $\beta = 1$.

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