PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

ENDOMORPHISM RINGS OF SIMPLE MODULES OVER GROUP RINGS

ROBERT L. SNIDER

(Communicated by Ken Goodearl)

ABSTRACT. If N is a finitely generated nilpotent group which is not abelian-by-finite, k a field, and D a finite dimensional separable division algebra over k, then there exists a simple module M for the group ring $k[G]$ with endomorphism ring D. An example is given to show that this cannot be extended to polycyclic groups.

Let N be a finite nilpotent group and k a field. The Schur index of every irreducible representation of N is at most two. This means that for every irreducible module M of the group ring $k[N]$, $End_{k[N]}(M)$ is either a field or the quaternion algebra over a finite field extension of k [6, p.564]. The purpose of this paper is to examine the situation when N is an infinite finitely generated nilpotent group. In this case, the endomorphism rings of simple modules are still finite dimensional [5, p.337]. However the situation is much different and in fact we prove

Theorem 1. *Let k be a field, N a finitely generated nilpotent group which is not abelian-by-finite, and D a separable division algebra finite dimensional over k, then there exists a simple $k[N]$-module M with $End_{k[N]}(M) = D$.*

I do not know if D in Theorem 1 can be any inseparable division algebra. The calculations seem quite difficult for large inseparable division algebras although the calculations below would show that small inseparable algebras (where the center is a simple extension) do occur.

On the other hand, if N is abelian-by-finite, then the group ring $k[N]$ satisfies a polynomial identity of some degree n. Since endomorphism rings of simple modules are homomorphic images of subrings of $k[N]$, each endomorphism ring of a simple module must satisfy the same identity of degree n and hence must be a division ring of degree at most $n/2$.

If G is a polycyclic group which is not nilpotent-by-finite, one expects that somehow G will be "more noncommutative" and hence that it should be easier to get noncommutative division rings as endomorphism rings. However we will give an example of a polycyclic group G, not nilpotent-by-finite, and a field k such that there are finite dimensional separable division algebras over k which are not the endomorphism ring of any simple $k[G]$-module. (The endomorphism rings of simple modules are still finite dimensional [5, p. 337].)

Received by the editors October 17, 1994.
1991 *Mathematics Subject Classification.* Primary 16S34, 20C05; Secondary 16K20, 16S50.
Key words and phrases. Group ring, endomorphism ring, division ring.

To prove Theorem 1, we first consider some rings which are sometimes called multiplicative Weyl algebras. Let k be a field which is not algebraic over a finite field and let λ be a nonzero element of k which is not a root of unity. Our ring is $R(k,\lambda) = k[x,x^{-1},y,y^{-1}]$ subject to the conditions that $yx = \lambda xy$. We will usually shorten this to R when k and λ are clear. R is a simple noncommutative Noetherian ring [2]. The importance of R for us is that it is a homomorphic image of many group rings of nilpotent groups.

Theorem 2. *If D is a finite dimensional central (i.e. the center of D is k) division algebra over k, then there exists a simple R-module M with $End_R(M) = D$.*

We denote the reduced trace of an element d in a finite dimensional division algebra D over a field k by $tr(d)$. We recall that if L is a maximal subfield of D the reduced trace restricted to L is the same as the ordinary field trace of L over k [1, p. 150]. We will denote the centralizer of an element d by $C(d)$.

Lemma 1. *If D is a finite dimensional central division algebra over k, there exist elements d_1 and d_2 in D with $C(d_1) \cap C(d_2) = k$ and $tr(d_1) = 0$ and $tr(d_2) \neq 0$.*

Proof. If $D = k$, let $d_1 = 0$ and $d_2 = 1$. Suppose that $D \neq k$. D has a maximal separable subfield K. Since K is separable, the trace is nonzero on K. The primitive elements are the complement of the union of the proper subfields. As such they are a Zariski-open and hence dense set. Therefore there is a basis of primitive elements. Hence there is a primitive element d_2 with $tr(d_2) \neq 0$. Let $x \in D$. If $C(x) \cap K \neq k$, then x is in the centralizer of some subfield of K different from k. There are only finitely many such subfields. The trace 0 elements form a k-subspace of dimension $n^2 - 1$ where $dim(D) = n^2$. Each of these centralizers of proper subfields of K has smaller dimension, whence the finite union cannot cover the trace 0 elements. Therefore there exists d_1 with $tr(d_1) = 0$ and $C(d_1) \cap C(d_2) = k$. □

Let D be finite dimensional central simple. Let $S = R \otimes_k D = D[x, x^{-1}, y, y^{-1}]$. Consider the matrix ring $M_2(S)$. Let

$$A = \begin{bmatrix} d_1 x & 1 \\ d_2 & x^{-1} \end{bmatrix}$$

where d_1 and d_2 are as in the lemma. Now

$$\begin{bmatrix} d_1 x & 1 \\ d_2 & x^{-1} \end{bmatrix} \begin{bmatrix} x^{-1} & -1 \\ -d_2 & d_1 x \end{bmatrix} = \begin{bmatrix} d_1 - d_2 & 0 \\ 0 & d_1 - d_2 \end{bmatrix}.$$

Therefore A is invertible. Let $\rho = M_2(S)(y - A)$. ρ is a left ideal of $M_2(S)$. Now $y \cdot (B + \rho) = yB + \rho = yBy^{-1}y + \rho = B^y A + \rho$. Hence $y + \rho = A + \rho$, $y^2 + \rho = y \cdot (y + \rho) = yA + \rho = A^y A + \rho$, and inductively $y^n + \rho = A^{y^{n-1}} \cdots A^y A + \rho$. Also $y^{-1}(y - A) = 1 - y^{-1}A = 1 - A^{y^{-1}} y^{-1}$. Therefore $(A^{-1})^{y^{-1}} - y^{-1}$ is in ρ. This implies $y^{-1} + \rho = (A^{-1})^{y^{-1}} + \rho$, $y^{-2} + \rho = (A^{-1})^{y^{-2}} (A^{-1})^{y^{-1}} + \rho$, and inductively, $y^{-n} + \rho = (A^{-1})^{y^{-n}} \cdots (A^{-1})^{y^{-1}} + \rho$. We see then that every element of $M_2(S)/\rho$ can be written as $f(x) + \rho$ where $f(x) \in M_2(D[x, x^{-1}])$. Furthermore ρ contains no polynomials in x only since A is invertible. Therefore $M_2(S)/\rho \cong M_2(D[x, x^{-1}])$ with action $y \cdot f(x) = yf(x)y^{-1}y = f(\lambda x)A$. Let $V = M_2(D[x, x^{-1}])$.

Lemma 2. *V is a simple $M_2(S)$ module.*

Proof. Suppose W is a nonzero $M_2(S)$ submodule.

Case 1. W has uniform dimension 2 as an $M_2(D[x,x^{-1}])$ module. Then W is essential as a left ideal of $M_2(D[x,x^{-1}])$ and hence contains a two-sided ideal. Let I be the unique largest such two-sided ideal. Now I is generated by a polynomial $f(x) \in k[x,x^{-1}]$. Now $W = A^{-1}yW = A^{-1}yWy^{-1}A$, hence W and therefore I is invariant under conjugation by $A^{-1}y$. But $A^{-1}yf(x)y^{-1}A = f(\lambda x)$. Since $f(x)$ and $f(\lambda x)$ generate the same ideal, they have the same nonzero roots in some extension field K of k. But if $\{r_1, \ldots, r_n\}$ are the nonzero roots of $f(x)$, then $\{\lambda^{-1}r_1, \ldots, \lambda^{-1}r_n\}$ are the nonzero roots of $f(\lambda x)$. Since λ is not a root of unity, these sets must be different. Therefore $f(x)$ has no nonzero roots and hence $f(x)$ is a unit and $W = V$.

Case 2. W has uniform dimension 1 as an $M_2(D[x,x^{-1}])$ module. By the Morita theory, submodules (i.e. left ideals) of $M_2(D[x,x^{-1}])$ correspond to $D[x,x^{-1}]$ submodules of $D[x,x^{-1}]^{(2)}$. The correspondence takes a submodule U of $D[x,x^{-1}]^{(2)}$ to the left ideal $\begin{bmatrix} U \\ U \end{bmatrix}$ of $M_2(D[x,x^{-1}])$. Suppose T corresponds to W. As T must be uniform, T is cyclic generated by say $(a(x), b(x))$. Therefore W must be generated by $\begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix}$. Now

$$y\begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a(\lambda x) & b(\lambda x) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d_1 x & 1 \\ d_2 & x^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} a(\lambda x)d_1 x + b(\lambda x)d_2 & a(\lambda x) + b(\lambda x)x^{-1} \\ 0 & 0 \end{bmatrix}.$$

This must equal $c(x)\begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix}$. Since y is invertible, $y\begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix}$ must also generate W. Therefore $c(x)$ is a unit, say $c(x) = cx^n$. We have two equations

$$cx^n a(x) = a(\lambda x)d_1 x + b(\lambda x)d_2 \tag{1}$$

and

$$cx^n b(x) = a(\lambda x) + b(\lambda x)x^{-1}. \tag{2}$$

We will denote the degree of $e(x) \in D[x,x^{-1}]$ by $d(e(x))$ and the "lower" degree by $l(e(x))$.

Case i. $d(a(x)) \geq d(b(x))$. From equation (1) we have $n + d(a(x)) = 1 + d(a(x))$, whence $n = 1$. From equation (2), $n + d(b(x)) = d(a(x))$, whence $1 + d(b(x)) = d(a(x))$. Let $a(x) = \sum_{i=j}^{k} a_i x^i$ where $a_j \neq 0$ and $a_k \neq 0$. Let $b(x) = \sum_{i=l}^{m} b_i x^i$ where $b_l \neq 0$ and $b_m \neq 0$. (It can be easily verified that $a(x) \neq 0$ and $b(x) \neq 0$.) By multiplying by a_k^{-1} we may assume that $a_k = 1$. Equation (1) implies that $b_i = 0$ for $i \leq j$. Comparing the coefficients of x^{k+1} in equation (1) we have $c = \lambda^k d_1$. Using the coefficient of x^{j+1}

$$ca_j = a_j \lambda^j d_1 + b_{j+1}\lambda^{j+1} d_2.$$

From equation (2), we look at the coefficient of x^j

$$0 = \lambda^j a_j + \lambda^{j+1} b_{j+1}.$$

Since $c = \lambda^k d_1$, $ca_j = \lambda^k d_1 a_j = a_j \lambda^j d_1 + (-a_j \lambda^{-1})\lambda^{j+1} d_2$. Hence $a_j^{-1}\lambda^k d_1 a_j = \lambda^j d_1 - \lambda^j d_2$. Taking traces, we get $0 = -\lambda^j tr(d_2)$. But $tr(d_2) \neq 0$, a contradiction.

Case ii. $d(a(x))+1 < d(b(x))$. From the equation (1), $n+d(a(x)) = d(b(x))$, and from equation (2), $n+d(b(x)) = d(b(x))-1$. Therefore $n = -1$ and $-1+d(a(x)) = d(b(x))$, a contradiction.

Case iii. $d(a(x)) + 1 = d(b(x))$. In this case, using the notation from Case i, $m = k + 1$. Now $n + d(b(x)) \leq d(b(x)) - 1$, whence $n \leq -1$. Now from equation (1), $n + l(a(x)) = l(b(x))$, whence $l(b(x)) \leq l(a(x)) - 1$. From equation (2), $n + l(b(x)) = l(b(x)) - 1$, whence $n = -1$. Now from the coefficient of x^{j-1} in equation (1), we have $ca_j = b_{j-1}\lambda^{j-1}d_2$. Since $a_j \neq 0$, $b_{j-1} \neq 0$. From the coefficient of x^{j-2} in equation (2), we have $cb_{j-1} = b_{j-1}\lambda^{j-1}$, whence $c = \lambda^{j-1}$. The coefficient of x^{k+1} in equation (1) gives

$$0 = a_k\lambda^k d_1 + b_{k+1}\lambda^{k+1}d_2.$$

From x^k in equation (2), we have

$$cb_{k+1} = \lambda^k a_k + b_{k+1}\lambda^{k+1}.$$

Hence

$$cb_{k+1}d_1 = \lambda^k a_k d_1 + b_{k+1}\lambda^{k+1}d_1.$$

Substituting, we have

$$\lambda^{j-1}b_{k+1}d_1 = -\lambda^{k+1}b_{k+1}d_2 + b_{k+1}\lambda^{k+1}d_1.$$

Now $b_{k+1} \neq 0$. Cancelling and solving for d_2, we get

$$d_2 = (1 - \lambda^{j-k-2})d_1.$$

Taking traces gives a contradiction. □

Lemma 3. $End_{M_2(S)}(V) \cong k$.

Proof. $End_{M_2(S)}(V)$ is finite dimensional over k [5, p. 337]. Let ϕ be an endomorphism. As an $M_2(D[x,x^{-1}])$ endomorphism, ϕ is given by right multiplication by a matrix $B \in M_2(D[x,x^{-1}])$. Let $C \in V$. Applying ϕ, $\phi(y \cdot C) = y \cdot \phi(C)$, hence $(y \cdot C)B = y \cdot (CB)$. Therefore $C^y AB = C^y B^y A$. Hence

$$End_{M_2(S)}(V) = \{B : AB = B^y A\}.$$

Choose $B \in End_{M_2(S)}(V)$ of largest degree. Such a B exists by finite dimensionality. B can be written as $B = \sum_{i=m}^n B_i x^i$ where $B_i \in M_2(D)$ and $B_n \neq 0$ and $B_m \neq 0$. Suppose $n \geq 1$. Since B^2 is also an endomorphism, it follows that $B_n^2 = 0$ by the maximality of the degree of B. Now

$$AB = \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix} B_n x^{n+1} + (\begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix} B_{n-1} + \begin{bmatrix} 0 & 1 \\ d_2 & 0 \end{bmatrix} B_n)x^n$$
$$+(\begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix} B_{n-2} + \begin{bmatrix} 0 & 1 \\ d_2 & 0 \end{bmatrix} B_{n-1} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_n)x^{n-1} + \cdots$$

and

$$B^y A = \lambda^n B_n \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix} x^{n+1} + (\lambda^n B_n \begin{bmatrix} 0 & 1 \\ d_2 & 0 \end{bmatrix} + \lambda^{n-1} B_{n-1} \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix})x^n$$
$$+(\lambda^n B_n \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda^{n-1} B_{n-1} \begin{bmatrix} 0 & 1 \\ d_2 & 0 \end{bmatrix} + \lambda^{n-2} B_{n-2} \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix})x^{n-1} + \cdots$$

Case i. $D \neq k$. In this case $d_1 \neq 0$. Denoting the i,j entry of B_k by $b_{k,i,j}$, we have $d_1 b_{n,1,2} = 0$, whence $b_{n,1,2} = 0$ and $0 = \lambda^n b_{n,2,1} d_1$, whence $b_{n,2,1} = 0$. Therefore

$B_n = \begin{bmatrix} b_{n,1,1} & 0 \\ 0 & b_{n,2,2} \end{bmatrix}$. Since $B_n^2 = 0$, it follows that $b_{n,1,1}^2 = 0$ and $b_{n,2,2}^2 = 0$. Therefore $b_{n,1,1} = 0$ and similarly $b_{n,2,2} = 0$. Therefore $B_n = 0$ and hence the endomorphism ring contains no elements of positive degree. A similar calculation shows that the endomorphism ring can contain no elements with negative powers of x. Therefore $End_{M_2(S)}(V) \subseteq M_2(D)$.

Case ii. $D = k$. In this case $d_1 = 0$ and $d_2 = 1$. From the coefficient of x^n, we have

$$\begin{aligned} b_{n,2,1} &= \lambda^n b_{b,n,1,2}, \\ b_{n,2,2} &= \lambda^n b_{n,1,1}, \\ b_{n,1,1} &= \lambda^n b_{n,2,2}, \\ b_{n,1,2} &= \lambda^n b_{n,2,1}. \end{aligned}$$

Therefore $b_{n,1,1} = \lambda^n b_{b,2,2} = \lambda^{2n} b_{n,1,1}$. Hence $b_{n,1,1} = 0$ and $b_{n,2,2} = 0$. Similarly the other entries of B_n are 0, a contradiction. Similarly the endomorphism ring can have no element with negative powers of x.

Now a simple calculation yields $B = \begin{bmatrix} b_{1,1} & 0 \\ 0 & b_{2,2} \end{bmatrix}$. Now $AB = BA$; and $AB = \begin{bmatrix} d_1 b_{1,1} x & b_{2,2} \\ d_2 b_{1,1} & b_{2,2} x^{-1} \end{bmatrix}$. $BA = \begin{bmatrix} b_{1,1} d_1 x & b_{1,1} \\ b_{2,2} d_2 & b_{2,2} x^{-1} \end{bmatrix}$. Therefore $b_{1,1} = b_{2,2}$ and $b_{1,1} \in C(d_1) \cap C(d_2) = k$. □

Theorem 3. *V is a simple $M_2(R)$-module and $End_{M_2(R)}(V) = D$.*

Proof. Now multiplication by D and $M_2(R)$ commute whence $D \subseteq End(V)$. Now D is a finite dimensional central simple k-algebra. Therefore $End(V) = D \otimes_k C$ where C is the centralizer of D in $End(V)$. C commutes with D and $M_2(R)$ and hence with $M_2(S)$, whence $C \subseteq End_{M_2(S)}(V) = k$. It follows that $End_{M_2(R)}(V) = D$. Since D commutes with $M_2(R)$, it follows that $M_2(S)$ is a finite centralizing extension of $M_2(R)$. Since V is a simple $M_2(S)$- module, V is a semisimple $M_2(R)$-module [5, p.346]. Since the endomorphism ring is D, V must be a simple $M_2(R)$-module. □

Proof of Theorem 2. By the Morita theory, $Mod - R$ and $Mod - M_2(R)$ are equivalent categories. Therefore since $M_2(R)$ has such a module, R must also. □

Lemma 4. *Let A and B be k-algebras. Let V and W be simple modules for A and B respectively. If $End_A(V) = k$ and $End_B(W) = E$, then $V \otimes_k W$ is a simple $A \otimes_k B$- module with endomorphism ring E.*

Proof. Let $x = \sum_1^n v_i \otimes w_i$ be a nonzero element of $V \otimes W$. Let $y = \sum_1^m c_i \otimes d_i$ be any other element of $V \otimes W$. We must show that y is in the submodule generated by x. Let V_1 be the k-subspace of V spanned by $\{v_1, \dots, v_n, c_1, \dots, c_m\}$ and W_1 be the E-subspace of W spanned by $\{w_1, \dots, w_n, d_1, \dots, d_m\}$. Now $V_1 \otimes_k W_1$ is a simple module over $T = End_k(V_1) \otimes_k End_E(W_1)$. Therefore there is an $h \in T$ with $h(x) = y$. Now $h = \sum_1^l f_i \otimes g_i$. By the density theorem, there exist $F_i \in A$ and $G_i \in B$ such that F_i restricted to V_1 is f_i and G_i restricted to W_1 is g_i. Let $H = \sum_1^l F_i \otimes G_i \in A \otimes_k B$. Clearly $Hx = y$ and $V \otimes W$ is simple. That $End(V \otimes W) = E$ is well known. □

Let $R_n = R \otimes_k \cdots \otimes_k R$ be the tensor product of n copies of R.

Theorem 4. *If D be a finite dimensional central division ring over k, then there exists a simple R_n-module V with $End_{R_n}(V) = D$.*

Proof. This follows immediately form the previous lemma and theorem. □

Proof of Theorem 1. If any homomorphic image of N has the required module, then N does also. Since N has maximal condition on subgroups, we may assume that all proper quotients of N are abelian-by-finite. Let Z denote the center of N. If Z is not cyclic, there are nonzero subgroups K_1 and K_2 with $K_1 \cap K_2 = 1$. In this case N embeds in $N/K_1 \times N/K_2$. As a proper quotient of N, each N/K_i must be abelian-by-finite and hence N is, a contradiction. Therefore Z must be cyclic and in fact infinite cyclic as a finitely generated nilpotent group with finite center is itself finite. Therefore N/Z is torsion-free. There exist nonzero integers $n, k_1, \ldots, k_n$ with $n > 0$, such that the N is isomorphic to the class two nilpotent group with generators $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ where $[x_i, y_i] = z^{k_i}$, $[x_i, y_j] = 1$ if $i \neq j$, and $[x_i, z] = [y_i, z] = 1$ [3, p. 241]. We may replace N with this group. Let K be the center of D. Since D is separable, K is separable. Therefore there exists $a \in K$, such that $K = k(a)$, and furthermore we may assume that a is not a root of unity. We define a homomorphism ϕ of $k[N]$ onto R_n defined over K. Every element of N can be written as $x_1^{c_1} \cdots x_n^{c_n} y_1^{d_1} \cdots y_n^{d_n} z^t$. Define

$$\phi(x_1^{c_1} \cdots x_n^{c_n} y_1^{d_1} \cdots y^{d_n} z^t) = x^{c_1} y^{d_1} \otimes \ldots \otimes x^{c_n} y^{d_n} a^t$$

ϕ is clearly a homomorphism of N into the unit group of R_n and hence extends to a ring homomorphism of $k[N]$ onto R_n. Therefore R_n and hence $k[N]$ has the desired module by the previous theorem. □

Let G be a polycyclic group and A a normal torsion-free subgroup of G. We recall that A is a plinth if A contains no normal subgroups of any subgroup H of finite index in G except 0 and subgroups of finite index in A. A is a centric plinth if it has rank 1; otherwise we say that A is an eccentric plinth [8, p. 396].

Example 1. We now give an example of a polycyclic group G which is not nilpotent-by-finite and a field k such that not every finite dimensional separable k-division ring is the endomorphism ring of a simple $k[G]$-module. Let k be a field of characteristic 0 containing an algebraically closed subfield. Furthermore we shall assume that there is a finite dimensional noncommutative division algebra over k. Let $A = Z(\sqrt{2})$. We define an automorphism ϕ by $\phi(a) = (1+\sqrt{2})a$. Let G be the split extension $A \times_\phi \langle z \rangle$ where $\langle z \rangle$ is an infinite cyclic group. Changing to multiplicative notation, we let x be 1 and y be $\sqrt{2}$. A is a plinth in G. The eccentric plinth length of G is 1. Therefore the primitive ideals of $k[G]$ are the maximal ideals and 0 [8]. Let V be a simple module.

Case 1. $Ann(V) = 0$. In this case V must be induced from a subgroup H of lower Hirsch number [4], say $V = k[G] \otimes_{k[H]} W$. Groups of Hirsch number 1 must be cyclic since G is torsion free. Suppose $H = \langle az^i \rangle$ with $i \neq 0$. Consider $T = AH$. Note that W can be extended to a $k[T]$ module by allowing A to act trivially. Define ϕ mapping $k[T] \otimes_{k[H]} W$ onto W by $\phi(a \otimes w) = aw$. Since ϕ is a $k[T]$ homomorphism, this implies that $k[T] \otimes_{k[H]} W$ is not simple. Therefore $V = k[G] \otimes_{k[T]} k[T] \bigotimes W$ is not simple, a contradiction. Therefore if H has Hirsch number 1, then $H \subseteq A$. On the other hand, all subgroups of G of Hirsch number 2 are contained in A. We may replace H by A and W by $k[A] \otimes W$. Since H is normal and W is simple, W and $z \otimes W$ are not isomorphic $k[H]$-modules. For suppose that W and $z^i \otimes W$ are isomorphic as $k[H]$ modules. Now $W \cong k[H]/M$ for some maximal ideal M. It follows that $M^{z^i} = M$. Let $T = \{x \in G : M^x = M\}$. T is a subgroup and $W \otimes_{k[H]} k[T] \cong (k[H]/M)[T/A]$. But this last module is not simple, a contradiction. Therefore W is not isomorphic to $z^i \otimes W$ for any $i \neq 0$. Now $V = \sum_{i=-\infty}^{\infty} z^i \otimes W$. Now since H is abelian, $L = End_{k[H]}(W)$ is commutative. Now $End_{k[H]}(V) = \prod_{i=-\infty}^{\infty} L$. Therefore $End_{k[G]}(V)$ is commutative.

Case 2. $Ann(V) \neq 0$. Let $P = Ann(V)$. 0 is the only faithful prime. Therefore $P^+ = \{g : 1-g \in P\} \neq 1$. Since P^+ is a normal subgroup, P^+ contains a subgroup B of A with $[A : B] < \infty$ and B normal in G. Therefore, $k[G]/P$ is a homomorphic image of $k[G/B]$. G/B is an abelian-by-finite group and hence satisfies a polynomial identity. Therefore $k[G]/P$ is a finite dimensional simple algebra over the field k. In particular $k[G]/P$ is isomorphic to a matrix ring $M_n(D)$ over a division ring D where D is the endomorphism ring of V. Let K be the center of D. Now $P \cap k[A]$ is a semiprime ideal of $k[A]$. Since $B \subseteq P^+$, $dim k[A]/(k[A] \cap P) < \infty$. Therefore $k[A]/(k[A] \cap P) \cong L_1 \oplus \cdots \oplus L_m$, where each L_i is a field. Since k contains an algebraically closed subfield and A/P^+ is a finite group, each $L_i \cong k$. Let e be the identity of L_1. Now $(e+P)k[G]/P(e+P)$ is a matrix ring over D of smaller size. Furthermore this ring is a homomorphic image of the ring $ek[G]/k[G](P \cap k[A])e$. Now $\langle z \rangle$ permutes the identities of the L_i's by conjugation. Let $H = Stab(e)$. $k[G]/k[G](P \cap k[A])$ is a crossed product $k[A]/(P \cap k[A]) * \langle z \rangle$. Now $ez^ie = 0$ for all z^i not in H. It follows that $ek[A]/(P \cap k[A]) * \langle z \rangle e$ is a crossed product $L_1 * H$. Suppose $H = \langle x \rangle$. Since $L_i \cong k$, x must centralize L_1 and hence x is central. Moving to the image $R = (e+P)k[G]/P(e+P)$, we see that R is generated by a field L_1 and an element x centralizing L_1. Therefore R and hence D are fields. Since there are finite dimensional noncommutative division algebras over k, not all finite dimensional division algebras over k can be endomorphism rings of simple modules.

References

1. P. Draxl, *Skew fields*, Cambridge University Press, Cambridge 1983. MR **85a:**16022
2. V.A. Jategaonkar, *A multiplicative analog of the Weyl algebra*, Comm. Algebra **12** (1984), 1669-1688. MR **85k:**16013
3. D.L. Harper,*Primitive irreducible representations of nilpotent groups*, Math. Proc. Camb. Phil. Soc. **82** (1977), 241-247. MR **56:**5698
4. D.L. Harper,*Primitivity in representations of polycyclic groups*, Math. Proc. Camb. Phil. Soc. **88**, (1980), 15-31. MR **81g:**20009
5. J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley, New York, 1987. MR **89j:**16023
6. D.S. Passman, *The Algebraic Structure of Group Rings*, John Wiley, New York, 1977. MR **86j:**16001
7. R.S. Pierce *Associative Algebras*, Springer- Verlag, New York, 1982. MR **84c:**16001
8. J.E. Roseblade, *Prime ideals in group rings of polycyclic groups*, Proc. London Math. Soc. (3) **36** (1978), 385-477. MR **58:**10996b

Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061-0123
E-mail address: `snider@math.vt.edu`