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A WHITNEY-STRATIFIED CURVE IN $\mathbf{R}^3$ WITH MULTIPLE-POINT PROJECTIONS

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ABSTRACT. We construct a compact $\mathcal{C}^\infty$ Whitney-stratified curve in $\mathbf{R}^3$, such that each of its plane projections has an infinite number of multiple points.

1. INTRODUCTION

Almost all hyperplane projections of a stratified set of large codimension in $\mathbf{R}^n$ are stratified embeddings. In [1], the authors showed that in general the projections do not preserve Whitney regularity, answering a question of Thom [2]. The aim of this paper is to show that in small codimensions, the image under a hyperplane projection of a $\mathcal{C}^\infty$ Whitney-stratified set is in general not stratifiable (i.e. cannot be partitioned into a locally finite union of submanifolds).

Let us first specify some terminology and notation. We say that a point M is a *multiple point* of a $\mathcal{C}^\infty$ curve $\psi : I \to \mathbf{R}^n$ (with I an interval of $\mathbf{R}$) if there exist $t_1, t_2 \in I$, $t_1 \neq t_2$ such that $\psi(t_1) = \psi(t_2) = M$ and $\psi'(t_1), \psi'(t_2)$ are not colinear. For a point x in the two-dimensional sphere $\mathbf{S}^2 \subset \mathbf{R}^3$, we denote by $\pi_x : \mathbf{R}^3 \to \mathbf{R}^3$ the orthogonal projection of $\mathbf{R}^3$ with kernel $\mathbf{R}x$, followed by the inclusion of the hyperplane $x^\perp$ (orthogonal to x) in $\mathbf{R}^3$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbf{R}^3$. The main result of the paper is the following

Example. We will construct a $\mathcal{C}^\infty$ curve $f : [0, +\infty[\to \mathbf{R}^3$, such that

1. $f'(t) \neq 0$ for all $t \in \mathbf{R}$,
2. $\|f(t)\|$ is strictly decreasing and $f(t) \to 0$, $t \to +\infty$,
3. $(\frac{f'(t)}{\|f'(t)\|} + \frac{f(t)}{\|f(t)\|}) \to 0$, $t \to +\infty$,
4. for *any* $x \in \mathbf{S}^2$, the curve $\pi_x \circ f$ has a sequence of multiple points (different from 0) converging to 0.

The first two conditions imply that $\{0\}, \{f(0)\}, f(]0, \infty[)$ is a stratification of the set $S = \overline{f(]0, +\infty[)}$. The third condition means precisely that the above stratification is Whitney b-regular at 0 (locally, near $f(0)$, the set S is a manifold with boundary). The fourth condition means that no hyperplane projection of the set S is stratifiable. Such a phenomenon does not occur for hyperplane projections of stratified curves in Euclidean n-space for $n \geq 4$ (see [1]), nor does it occur for smooth curves in $\mathbf{R}^3$. Indeed, almost all hyperplane projections of a $\mathcal{C}^\infty$ compact curve in $\mathbf{R}^3$ are injective outside a finite set of points. This is easily proved using

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Sard's theorem. More generally, if the set S is subanalytic, then all projections are stratifiable.

Our example will be a *rapid spiral*, i.e. a curve of the type $f(t) = e^{-h(t)}\phi(t)$. Here $h : [0, +\infty[\to \mathbf{R}$ is a $\mathcal{C}^\infty$ function, with $h'(t) > 0$ for all t and $h'(t) \to +\infty$, $t \to +\infty$, whereas $\phi : [0, +\infty[\to \mathbf{S}^2$ is a $\mathcal{C}^\infty$ curve, such that $\|\phi'(t)\| \leq 1$ for all t. We will say that the rapid spiral f has *base* ϕ and *speed* h. It is easily seen that f satisfies the first three conditions required for the example, given any base and speed. A particular choice of ϕ and h will ensure that the fourth condition is also fulfilled.

2. A curve with a multiple point

Let $N = (0, 0, 1)$ be the "North Pole" of the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$. Choose any $\mathcal{C}^\infty$ map $\psi : [0, 1] \to \mathbf{S}^2$, such that

1. $\|\psi'(t)\| \leq 1$, for all $t \in [0, 1]$,
2. $\psi(\frac{1}{3}) = \psi(\frac{2}{3}) = N$,
3. $\psi'(\frac{1}{3})$ and $\psi'(\frac{2}{3})$ are not colinear.

We have the following

Property ($\propto$). For any $k \in \mathbf{R}$, $k > 0$, there exists a neighbourhood U^k of N in $\mathbf{S}^2$, such that for all $x \in U^k - \{N\}$, the curve $\tilde{\psi}_x : [0, 1] \to x^\perp \hookrightarrow \mathbf{R}^3$; $\tilde{\psi}_x(t) = \pi_x(e^{-kt}\psi(t))$ has a multiple point different from 0.

Sketch of proof. For any $x \in (\mathbf{S}^2 - \mathbf{R}^2 \times \{0\})$, let $\hat{\pi}_x$ be the projection with kernel $\mathbf{R}x$, on the coordinate hyperplane $\mathbf{R}^2 \times \{0\}$. Let $\hat{\psi}_x : [0, 1] \to \mathbf{R}^2 \times \{0\}$ be the curve defined by $\hat{\psi}_x(t) = \hat{\pi}_x(e^{-kt}\psi(t))$. Of course it is equivalent to show Property ($\propto$) with $\tilde{\psi}_x$ replaced by $\hat{\psi}_x$.

That, in turn, will follow automatically from the existence of local diffeomorphisms δ and μ from a neighbourhood of 0 in $\mathbf{R}^2$, to respectively $\mathbf{S}^2$ and $\mathbf{R}^2 \times \{0\}$, with $\delta(0) = N$ and $\mu(0) = 0$, such that $\mu(s, t)$ is a multiple point of the curve $\hat{\psi}_{\delta(s,t)}$. To construct δ and μ, set

$$P(s) = e^{-k(s+\frac{1}{3})}\psi\left(s + \frac{1}{3}\right) \quad \text{and} \quad Q(t) = e^{-k(t+\frac{2}{3})}\psi\left(t + \frac{2}{3}\right).$$

Then define

$$\delta(s, t) = \frac{P(s) - Q(t)}{\|P(s) - Q(t)\|} \quad \text{and} \quad \mu(s, t) = \frac{Q_3(t)\pi_N(P(s)) - P_3(s)\pi_N(Q(t))}{Q_3(t) - P_3(s)}.$$

Here P_i and Q_i, are the ith coordinates of P and Q. One calculates:

$$(1 - e^{-\frac{k}{3}})\frac{\partial \delta}{\partial s}(0, 0) = (1 - e^{\frac{k}{3}})\frac{\partial \mu}{\partial s}(0, 0) = \psi'\left(\frac{1}{3}\right),$$
$$(1 - e^{\frac{k}{3}})\frac{\partial \delta}{\partial t}(0, 0) = (1 - e^{-\frac{k}{3}})\frac{\partial \mu}{\partial t}(0, 0) = \psi'\left(\frac{2}{3}\right),$$

which shows that both mappings are local diffeomorphisms. Furthermore $\mu(s, t) = \hat{\psi}_{\delta(s,t)}(s + \frac{1}{3}) = \hat{\psi}_{\delta(s,t)}(t + \frac{2}{3})$ is a multiple point of $\hat{\psi}_{\delta(s,t)}$ for (s, t) sufficiently close to 0. □

3. The construction

We are now ready to construct ϕ and h. For each point $y \in \mathbf{S}^2$, choose a rotation $\mathcal{R}_y$ of $\mathbf{R}^3$ around 0 which sends N to y. For every $k = 1, 2, \ldots$, choose $y_1^k, \ldots, y_{n_k}^k \in \mathbf{S}^2$ such that the punctured open sets $\mathcal{R}_{y_1^k}(U^k - \{N\}), \ldots, \mathcal{R}_{y_{n_k}^k}(U^k - \{N\})$ cover $\mathbf{S}^2$. Now ϕ can be chosen as any curve joining all the curves $\mathcal{R}_{y_i^k} \circ \psi$. To be precise, we take ϕ as any $\mathcal{C}^\infty$ map for which there exist, for each $k = 1, 2, \ldots$, sequences of reals $T_1^k, \ldots, T_{n_k}^k$, with $T_i^k + 1 < T_{i+1}^k$, for all k, i and $T_{n_k}^k + 1 < T_1^{k+1}$ for all k, and such that $\phi(t) = \mathcal{R}_{y_i^k}(\psi(t - T_i^k))$ for $t \in [T_i^k, T_i^k + 1]$, for all k, i. Furthermore, we require that $\|\phi'(t)\| \leq 1$ for all t. Such a curve is easily obtained by standard smoothing techniques.

As h, we can take any $\mathcal{C}^\infty$ function, with weakly increasing positive derivative, for which there exists a sequence of reals $\{t_k\}_{k=1}^\infty$, such that $h(t) = kt - t_k$ for $t \in [T_1^k, T_{n_k}^k]$, for all k. Such a function is also easily constructed.

We now show that the rapid spiral f with base ϕ and speed h satisfies property 4 of the example. Indeed, let $x \in \mathbf{S}^2$. For each k, choose $i_k \in \{1, \ldots, n_k\}$, such that $x \in \mathcal{R}_{y_{i_k}^k}(U^k - \{N\})$. Denote $\mathcal{R} = \mathcal{R}_{y_{i_k}^k}$. On $[T_{i_k}^k, T_{i_k}^k + 1]$ we have

$$\pi_x(f(t)) = \pi_x(e^{-kt+t_k}\mathcal{R}(\psi(t - T_{i_k}^k))) = e^{(t_k - kT_{i_k}^k)}\mathcal{R}(\tilde{\psi}_{\mathcal{R}^{-1}(x)}(t - T_{i_k}^k)).$$

Since $\mathcal{R}^{-1}(x) \in U^k - \{N\}$, by Property ($\propto$), the curve $\tilde{\psi}_{\mathcal{R}^{-1}(x)}$ has a multiple point different from 0 and thus, by the last equality, so has $\pi_x \circ f|_{[T_{i_k}^k, T_{i_k}^k + 1]}$. Since the above holds for any k, this means that for any $T > 0$, the curve $\pi_x \circ f$ has a multiple point $\pi_x \circ f(t)$, with $t > T$. Hence, $\pi_x \circ f$ has multiple points different from 0, arbitrarily close to 0. This finishes the proof that our rapid spiral satisfies the fourth condition of the example.

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