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HNN BASES AND HIGH-DIMENSIONAL KNOTS

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ABSTRACT. There exists a 3-knot group having HNN bases of two types: bases that are arbitrarily large finitely presented and bases that are arbitrarily large finitely generated but not finitely presented. Any n-knot with such a group has a Seifert manifold that can be converted to a minimal one by a finite sequence of ambient 0- and 1-surgeries, but cannot be converted by 1-surgeries alone.

1. INTRODUCTION

Let G be a group and $h : G \twoheadrightarrow \mathbf{Z}$ an epimorphism. We can describe G as an HNN extension $\langle x, B|\ xsx^{-1} = \phi(s)\ \forall s \in S\rangle$, where $h(x) = 1$, $B \leq \ker h$, $S \leq B$, and ϕ is an isomorphism $\phi : S \tilde{\to} T \leq B$. The group B is called an *HNN base* (or *base*) for G relative to h, while S and T are called *associated subgroups*. When G is finitely presented, we can find such an extension in which B, S, and T are finitely generated [**3**].

Section 2 of this paper is concerned with the case that G is an n-knot group. A *base* for an n-knot group will mean an HNN base relative to the abelianization homomorphism. We recall that an *n-knot*, for $n \geq 1$, is a smoothly embedded n-sphere $\mathcal{K} \subset \mathcal{S}^{n+2}$. The group $\pi_1(\mathcal{S}^{n+2} - \mathcal{K})$ is called an n-knot group or the *group* of $\mathcal{K}$. For $n \geq 1$, any n-knot group is also an $(n+1)$-knot group (see [**7**]). A *Seifert manifold* for $\mathcal{K}$ is a compact, connected, oriented $(n+1)$-manifold $\mathcal{V} \subset \mathcal{S}^{n+2}$ with boundary equal to $\mathcal{K}$. By [**8**] any n-knot possesses a Seifert manifold. A Seifert manifold $\mathcal{V}$ is *minimal* if the inclusion map ι: *int* $\mathcal{V} \hookrightarrow \mathcal{S}^{n+2} - \mathcal{K}$ induces a monomorphism of fundamental groups.

If an n-knot $\mathcal{K}$ has a minimal Seifert manifold, then the group of $\mathcal{K}$ has a finitely presented base. Such a base can be realized as the fundamental group of the compact manifold obtained by "splitting" $\mathcal{S}^{n+2}$ along the Seifert manifold [**8**]. This fact was used in [**13**] in order to produce examples of n-knots, for $n > 2$, with no minimal Seifert manifolds. Whether or not there exists a 2-knot with no minimal Seifert manifold is an open question. (See [**9**] for an example of a knotted torus $\mathcal{T}$ in $\mathcal{S}^4$ such that $\pi_1(\mathcal{S}^4 - \mathcal{T})$ has no HNN decomposition with finitely presented base.) In higher dimensions the situation is clearer. We proved in [**14**] that any n-knot $\mathcal{K}$, for $n > 2$, has a minimal Seifert manifold if and only if its group has a finitely presented base. We showed moreover that in this case any Seifert manifold for $\mathcal{K}$ can be converted to a minimal one by a finite sequence of ambient 0- and 1-surgeries. (Ambient i-surgery is also called $(i+1)$-handle exchange [**8**].)

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In view of the above-mentioned results it is natural to ask whether a finitely presented group can have an HNN base (relative to some h) that is finitely presented and another that is finitely generated but not finitely presented. Can an n-knot group have this property? We begin with a simple construction that answers the first question affirmatively.

Let H be any finitely presented group with a subgroup H_0 that is finitely generated but not finitely presented. For definiteness one can use the example of J. Stallings [**16**] in which

$$H = \langle t, a_1, a_2, b \mid ta_1t^{-1} = a_1,\ ta_2t^{-1} = a_2,\ tbt^{-1} = a_1ba_1^{-1},\ a_1ba_1^{-1} = a_2ba_2^{-1}\rangle$$

while H_0 is the subgroup $gp(a_1, a_2, b)$ generated by $a_1,\ a_2$, and b. Given any such pair (H, H_0), we define

$$G = \langle x, H \mid x^2hx^{-2} = h\ \forall h \in H_0\rangle.$$

Since H_0 is finitely generated, G has a finite presentation. Let $\hat{H}_0$ be an isomorphic copy of H_0 with isomorphism $\hat{h} \mapsto h$. Then

$$G \cong \langle x, H * \hat{H}_0 \mid x^2hx^{-2} = h\ \forall h \in H_0,\ \hat{h} = x^{-1}hx\ \forall \hat{h} \in \hat{H}_0\rangle$$

$$\cong \langle x, H * \hat{H}_0 \mid xhx^{-1} = \hat{h}\ \forall h \in H_0,\ x\hat{h}x^{-1} = h\ \forall \hat{h} \in \hat{H}_0\rangle.$$

This last presentation exhibits G as an HNN extension with base $B = H * \hat{H}_0$ relative to the homomorphism $h : G \twoheadrightarrow \mathbf{Z}$ that maps x to 1 and $H * \hat{H}_0$ to 0. The associated subgroups S and T are the same, namely $H_0 * \hat{H}_0$, while $\phi : S \to T$ is the isomorphism that interchanges factors. Since H and H_0 are finitely generated, so is the base B. However, since H_0 is not finitely presented, neither is B.

In order to show that G has a finitely presented HNN base, we introduce some notation. For any subgroup $A \leq B$ and integer ν, we let A_ν denote the subgroup $x^\nu Ax^{-\nu} \leq \ker h$. With this notation, $\ker h$ can be described as the infinite amalgamated free product of the groups B_ν, $\nu \in \mathbf{Z}$, in which $S_{\nu+1}$ is identified with T_ν by the mapping $x^{\nu+1}sx^{-\nu-1} \mapsto x^\nu\phi(s)x^{-\nu}\ \forall s \in S$. For integers $i \leq j$ let $B_{i,j}$ denote the subgroup of $\ker h$ generated by $\bigcup_{i\leq\nu\leq j} B_\nu$. Whenever $i < j$ we can describe G as an HNN extension with base $B_{i,j}$, associated subgroups $B_{i,j-1}, B_{i+1,j}$ and isomorphism $B_{i,j-1} \to B_{i+1,j}$ described by the map $\sigma : x^\nu gx^{-\nu} \mapsto x^{\nu+1}gx^{-\nu-1}$. In particular, the base $B_{0,1}$ is the amalgamated free product of two copies of $H * \hat{H}_0$ in which $H_0 * \hat{H}_0$ in the first factor is identified with $H_0 * \hat{H}_0$ in the second by an isomorphism that interchanges the factors. Clearly, this base is isomorphic to $H * H$, a finitely presented group.

A byproduct of our construction is a finitely presented group $H * H$ with a nontrivial amalgamated free product decomposition in which the factors, both isomorphic to $H * \hat{H}_0$, and the amalgamated subgroups, isomorphic to $H_0 * \hat{H}_0$, are finitely generated but not finitely presented. The first such example was given by G. Baumslag and P. Shalen in [**2**] using wreath products of groups. Our example is relatively elementary, and it suggests the following.

Proposition 1.1. *Let H be a finitely presented group. Then H has a subgroup that is finitely generated but not finitely presented if and only if $H * H$ has an amalgamated free product decomposition in which some factor or amalgamated subgroup is finitely generated but not finitely presented.*

Proof. We have seen the forward implication. In order to prove the converse, assume that $H * H$ has an amalgamated free product decompostion $A *_C B$ in which at least one of A, B or C is finitely generated but not finitely presented. By the Kurosh Subgroup Theorem that subgroup is a free product of a free group and certain subgroups of conjugates of A and/or B. Each of the factors is finitely generated, but at least one must fail to be finitely presented. Hence H contains a subgroup that is finitely generated but not finitely presented. □

In the construction above we showed that the base $B_{0,1}$ is finitely presented. More generally, one sees that $B_{i,j}$ is finitely presented whenever $j - i > 0$. This follows also from the next result.

Proposition 1.2. *Assume that B is an HNN base for a finitely presented group G. If B is finitely presented, then for all integers $i < j$, the base $B_{i,j}$ is also finitely presented.*

Proof. Assume that S and T are the associated subgroups corresponding to B. For any integers $i < j$, the base $B_{i,j}$ is the free product of $j - i + 1$ copies of B amalgamated along copies of S, and by [**1**] such a group is finitely presented if and only if S is finitely generated. Associated subgroups S, T of an HNN decomposition for G need not be finitely generated. However, they must be finitely generated if the base B is finitely presented. To see this, consider the presentation

$$\langle x,\ B \mid xsx^{-1} = \phi(s)\ \forall s \in S\rangle.$$

Since G is finitely presented, only finitely many relators are needed [**11**], say those of a finite presentation of B together with $xs_1x^{-1} = \phi(s_1), \ldots, xs_nx^{-1} = \phi(s_n)$. Let $\tilde{S}$ be the subgroup of S generated by $s_1, \ldots, s_n$, and let $\tilde{T} = \phi(\tilde{S})$. Replacing S and T with their respective subgroups $\tilde{S}$ and $\tilde{T}$, known to be finitely generated, we can conclude that the base $B_{i,j}$ is finitely presented. Of course, we can also conclude that the original associated subgroups S and T were finitely generated, since $B_{i,j}$ cannot be both finitely presented and nonfinitely presented. □

2. High-dimensional knot groups

Proposition 1.2 implies that if a finitely presented group G has an HNN base that is finitely presented, then it has finitely presented bases that are "arbitrarily large" in the following sense.

Definition 2.1. A group G has arbitrarily large finitely presented (resp. finitely generated but not finitely presented) HNN bases relative to $h : G \twoheadrightarrow \mathbf{Z}$ if G has finitely presented (resp. finitely generated but not finitely presented) HNN bases $B(k),\ k = 1, 2 \ldots$, such that $\bigcup_k B(k) = \ker h$.

Theorem 2.2. *There exists a 3-knot group G with arbitrarily large finitely presented HNN bases and arbitrarily large finitely generated but not finitely presented*

HNN bases. Any n-knot with such a group has a Seifert manifold that can be converted to a minimal one by a finite sequence of ambient 0- and 1-surgeries, but cannot be converted by ambient 1-surgeries alone.

Remark. The last statement of Theorem 2.2 says that certain Seifert manifolds can be made minimal by ambient surgery only if their fundamental groups are first enlarged by attaching "hollow 1-handles" to the Seifert manifolds.

Lemma 2.3. *Let $G = \langle x, B |\ xsx^{-1} = \phi(s)\ \forall s \in S\rangle$ be a group with HNN base B relative to $h : G \twoheadrightarrow \mathbf{Z}$ and associated subgroups S and T, and let A be a subgroup of B. Assume that A and T are free factors of the subgroup that they generate. Then $x^{-1}Ax$ and B are free factors of $gp(x^{-1}Ax, B) \le \ker h$, and $gp(x^{-1}Ax, B)$ is a base for G relative to h.*

Proof. Let $\hat{A}$ be an isomorphic copy of A with isomorphism $\hat{a} \mapsto a$, and consider the presentation

$$\langle x, \hat{A} * B |\ x\hat{a}x^{-1} = a\ \forall \hat{a} \in \hat{A},\ xgx^{-1} = \phi(g)\ \forall g \in S\rangle. \tag{1.1}$$

The relations $x\hat{a}x^{-1} = a$ can be rewritten as $\hat{a} = x^{-1}ax$, thereby identifying $\hat{A}$ with $x^{-1}Ax \le G$; consequently, (1.1) is a presentation for G. Since $gp(A, T) \cong A*T$, the mappings $\hat{a} \mapsto a\ \forall a \in \hat{A}$ and $g \mapsto \phi(g)\ \forall g \in S$ determine an isomorphism between $\hat{A} * S \le \hat{A} * B$ and $gp(A, T)$. It follows that (1.1) presents G as an HNN extension with base $\hat{A} * B$. Since $\hat{A}$ is identical to $x^{-1}Ax$ in G, the lemma is proved. □

Proof of Theorem 2.2. J. Hillman showed in [**6**] that the following presentation describes the group of a 2-knot $\mathcal{K}$:

$$\langle x, a, b |\ a^2 = (ab)^3 = b^5,\ xax^{-1} = a,\ xbx^{-1} = b^{-1}a^{-1}b^2ab\rangle.$$

The commutator subgroup of this group is the binary icosahedral group I, a perfect group of order 120. Observe that the element $a \in I$ commutes with the meridianal generator x. Define U to be the group of the product 2-knot $\mathcal{K}\sharp\mathcal{K}$ with presentation

$$\langle x_U, a_1, b_1, a_2, b_2 |\ a_i^2 = (a_ib_i)^3 = b_i^5,\ x_U a_i x_U^{-1} = a_i,$$
$$x_U b_i x_U^{-1} = b_i^{-1}a_i^{-1}b_i^2a_ib_i, i = 1, 2\rangle.$$

The commutator subgroup U' of U is the free product $I * I$, a finitely generated perfect group. The element $l = a_1a_2$ has infinite order and commutes with the meridianal generator x_U. The subgroup $gp(l, x_U)$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

Let V be the group of any 3-knot for which no minimal Seifert manifold exists. The commutator subgroup of V must contain a finitely generated subgroup W that is not finitely presented. In fact, arguments of [**14**, p. 106] show that W can be chosen to be $\iota_*\pi_1(int\ \mathcal{V})$, where $\mathcal{V}$ is any Seifert manifold for the 3-knot. The main idea is that if $\iota_*\pi_1(int\ \mathcal{V})$ were finitely presented, then the kernel of ι_* would be the normal closure of only finitely many elements of $\pi_1(int\ \mathcal{V})$, elements that can be represented by embedded loops in $\mathcal{V}$ and then killed by ambient 1-surgery.

Let $\tilde{U}$ be an isomorphic copy of U with isomorphism $\tilde{u} \mapsto u$. Define

$$B \cong \langle V * U * \tilde{U} |\ x_V = x_U = \tilde{l}, l = \tilde{x}_{\tilde{U}}\rangle.$$

The group B is the amalgamated free product of the 3-knot groups $\langle V * U | x_V = x_U \rangle$ and $\tilde{U}$ in which the free abelian subgroup generated by x_U and l in the first factor is identified with the free abelian subgroup generated by $\tilde{l}$ and $\tilde{x}_{\tilde{U}}$ in the second factor. We record the observations for later use that the subgroup of B generated by V' and U' is the commutator subgroup of $\langle V * U | x_V = x_U \rangle$, and that this group is simply the free product of V' and U'.

Clearly $H_1 B$ is trivial. Using the Mayer-Vietoris sequence [**4**] (or see [**16**]) and the fact that every n-knot group has vanishing second homology [**7**], one checks that B is *superperfect*; i.e., $H_2 B$ is also trivial. Finally define

$$G \cong \langle x, B |\ xgx^{-1} = x_U g x_U^{-1}\ \forall g \in U' \rangle. \tag{2.1}$$

Since B is finitely presented and U' is finitely generated, G is finitely presented. Presentation (2.1) displays G as an HNN extension with base B, associated subgroups S, T equal to U', and isomorphism $\phi : S \tilde{\to} T$ given by the meridianal automorphism $g \mapsto x_U g x_U^{-1}$ in U. Since B is perfect, $H_1 G$ is an infinite cyclic group generated by x. It is a straightforward exercise to check that x normally generates G. One begins by killing x, thereby introducing the relations $g = x_U g x_U^{-1} \forall\ g \in U'$ which kill U' (since U is normally generated by x_U) thereby killing l, etc. Also, the exact sequence for HNN extensions [**4**] (or see [**5**]) combined with the facts that $H_2 B = 0$ and $H_1 U' = 0$ enable one to check that $H_2 G = 0$. Hence by Kervaire's theorem [**7**] the group G is a 3-knot group.

Using the notation established in Section 1, we can describe G as an HNN extension with base $B_{i,j}$, associated subgroups $B_{i,j-1}, B_{i+1,j}$ and isomorphism $B_{i,j-1} \to B_{i+1,j}$ determined by $\sigma : x^\nu g x^{-\nu} \mapsto x^{\nu+1} g x^{-\nu-1}$. In this way we obtain arbitrarily large finitely presented bases $B(k) = B_{-k,k}, k > 0$, for G.

We now produce arbitrarily large bases $B^*(k)$ for G that are finitely generated by not finitely presented. Recall that V' contains a subgroup W that is finitely generated but not finitely presented. For any integers $i < j$, the base $B_{i-1,j}$ is the amalgamated free product of B_{i-1} and $B_{i,j}$ in which U'_{i-1} is identified with U'_i. Earlier we observed that $gp(V', U')$ is the free product $V' * U'$. Hence $gp(W, U')$ is $W * U'$. It follows that the subgroup of $B_{i-1,j}$ generated by W_{i-1}, U'_{i-1}, and $B_{i,j}$ is the amalgamated free product of $W_{i-1} * U'_{i-1}$ and $B_{i,j}$ in which U'_{i-1} is identified with $U'_i \leq B_{i,j}$, and this group is clearly $W_{i-1} * B_{i,j}$. Hence W_{i-1} and $B_{i,j}$ are free factors of the subgroup of $B_{i-1,j}$ that they generate. By Lemma 2.3 the subgroup $W_{i-1} * B_{i,j}$ is a base $B^*_{i,j}$ for G. Of course, $B^*_{i,j}$ is finitely generated since each of its factors is, but it is not finitely presented since W_{i-1} is not. For $k > 0$, we define $B^*(k)$ to be $W_{-k-1} * B_{-k,k}$. Then $\bigcup_k B^*(k)$ is equal to G', and the first statement of Theorem 2.2 is proved.

Assume that $\mathcal{K}$ is any 3-knot with group G, and let $\mathcal{V}$ be a Seifert manifold for $\mathcal{K}$. The image $\iota_* \pi_1(int\ \mathcal{V})$ in $\pi_1(\mathcal{S}^3 - \mathcal{K})$ is a finitely generated subgroup of G'. Choosing $k > 0$ sufficiently large, we can assume that the image is contained in the base $B^*(k)$, and since $B^*(k)$ is finitely generated, we can perform ambient 0-surgeries on $\mathcal{V}$ until the image coincides with $B^*(k)$ (see [**14**] pp. 106 – 107). The new Seifert manifold $\mathcal{V}'$ that we obtain cannot be converted to a minimal one by any finite sequence of ambient 1- surgeries alone because $\iota_* \pi_1(int\ \mathcal{V}')$ is not finitely related. □

3. A conjecture about knot-like groups

Following Rapaport [**12**] we will say that a group G is *knot-like* if $G/G' \cong \mathbf{Z}$ and G has a finite presentation in which the number of generators exceeds the number of relators by one. Any 1-knot group is a knot-like group as are the groups of many n-knots for $n > 1$. By a *base* for a knot-like group we will mean an HNN base with respect to the abelianization homomorphism. By [**3**] every knot-like group has a finitely generated base.

Conjecture 3.1. *Every finitely generated base for a knot-like group is finitely presented.*

It is well known and not difficult to prove that if the commutator subgroup G' of a knot-like group is finitely generated, then any finitely generated base for G is equal to G'. A conjecture of Rapaport [**12**] asserts that in this case any finitely generated base for G is in fact a free group. Partial results about this conjecture were obtained in [**12**] and [**15**] (see also [**10**]).

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