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ON HANKEL OPERATORS NOT IN THE TOEPLITZ ALGEBRA

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ABSTRACT. In this paper we exhibit a class of Hankel operators, which is contained in the essential commutant of the unilateral shift, but disjoint from the Toeplitz algebra.

In [2] it is proved that if $\mathbf{E}$ is the essential commutant of the unilateral shift and $\mathbf{T}$ is the Toeplitz algebra, then the set $\mathbf{E}\backslash\mathbf{T}$ contains Hankel operators (the definitions of $\mathbf{E}$ and $\mathbf{T}$ are given below). The proof given in [2] makes use of the theory of maximal ideals of function algebras; since this method is not constructive, it does not yield concrete operators. The purpose of this paper is to exhibit a concrete class of Hankel operators which is contained in $\mathbf{E}\backslash\mathbf{T}$. The result in [2], and ours, answers one of the questions raised in [1] concerning membership in the Toeplitz algebra.

The underlying Hilbert space is $\mathbf{H}^2$ of the unit circle. Let ϕ be in $\mathbf{L}^\infty$ of the unit circle. The Toeplitz operator T_ϕ on $\mathbf{H}^2$ is defined by $T_\varphi f = P(\phi f)$, where P is the orthogonal projection from $\mathbf{L}^2$ to $\mathbf{H}^2$. The Hankel operator H_ϕ on $\mathbf{H}^2$ is defined by the Hankel matrix $(c_{-i-j-1})_{i,j=0}^\infty$, where $\{c_n\}_{-\infty}^\infty$ is the sequence of Fourier coefficients of ϕ. The unilateral shift S is the Toeplitz operator T_ϕ with $\phi(z) = z$. The essential commutant of S is the set $\mathbf{E}$ of all operators T on $\mathbf{H}^2$ for which $TS - ST \in \mathbf{K}$, where $\mathbf{K}$ is the ideal of all compact operators on $\mathbf{H}^2$. The Toeplitz algebra $\mathbf{T}$ is the C^*-algebra generated by $\{T_\phi\colon\ \phi \in \mathbf{L}^\infty\}$. The following result describes a class of Hankel operators in $\mathbf{E}\backslash\mathbf{T}$.

Theorem. *Let b be an infinite Blaschke product whose zero set Z has its cluster points in $\{-1, 1\}$. We further assume that there exists a sequence $\{a_n\}_{n=1}^\infty$ in Z such that*

$$\text{(i)} \qquad \lim_{n\to\infty} a_n = \lambda;$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{|\lambda - a_n|}{1 - |a_n|} = +\infty.$$

Then $H_{\bar b}$ is in the essential commutant of S, but $H_{\bar b}$ does not belong to the Toeplitz algebra.

The first assertion of the theorem follows from the result [2, Proposition 3.5], which we state below as a proposition. This proposition is proved in [2] using the theory of maximal ideals; the proof given below is based on function theory. First

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we introduce notation and facts which are needed. For $\phi \in \mathbf{L}^\infty$, we write $\tilde{\phi}$ for the function defined by $\tilde{\phi}(z) = \phi(\bar{z})$. For ϕ and ψ in $\mathbf{L}^\infty$ the following identities hold [1]:

$$T_{\phi\psi} = T_\phi T_\psi + H_{\tilde{\bar{\phi}}} H_\psi; \tag{1}$$

$$H_{\phi\psi} = H_\phi T_\psi + T_{\tilde{\bar{\phi}}} H_\psi. \tag{2}$$

For f in $\mathbf{L}^1$ and z in the open unit disk D, we write $P_z(f)$ for the Poisson integral of f:

$$P_z(f) = \frac{1}{2\pi}\int_{\partial D} f P_z \, d\theta,$$

where P_z denotes the Poisson kernel for the point z. We end this introduction with the algebra $\mathbf{H}^\infty + \mathbf{C}$, whose properties are important in our proof. Here $\mathbf{H}^\infty$ is the algebra of boundary functions for bounded analytic functions in D, and $\mathbf{C}$ is the algebra of continuous complex valued functions on ∂D.

Proposition. *Let b be a Blaschke product with zero set Z. Then $H_{\bar{b}}$ is in the essential commutant of S if and only if Z is finite or Z has its cluster points in $\{-1, 1\}$.*

Proof. Let $A = H_{\bar{b}}S - SH_{\bar{b}}$. Applying (2) twice we obtain

$$H_{\bar{b}z} = H_{\bar{b}}T_z + T_{\tilde{b}}H_z = H_{\bar{b}}T_z = H_{\bar{b}}S \quad \text{and} \quad H_{\bar{z}\bar{b}} = H_{\bar{z}}T_{\bar{b}} + T_z H_{\bar{b}} = H_{\bar{z}}T_{\bar{b}} + SH_{\bar{b}},$$

therefore $A = H_{(z-\bar{z})\bar{b}} + H_{\bar{z}}T_{\bar{b}}$. Since $H_{\bar{z}}$ is of finite rank, A is compact if and only if $(z - \bar{z})\bar{b} \in \mathbf{H}^\infty + \mathbf{C}$ [7, p. 101].

If Z is finite, then $b \in \mathbf{C}$ and therefore $(\bar{z} - z)b \in \mathbf{C}$. If Z has its cluster points in $\{-1, 1\}$, then b is continuous in $\partial D \backslash \{-1, 1\}$ [5, p. 68], and because b is bounded it follows that $(\bar{z} - z)b \in \mathbf{C}$. In either case we have $(z - \bar{z})\bar{b} \in \mathbf{H}^\infty + \mathbf{C}$.

For the converse we assume that $(z-\bar{z})\bar{b} = f$ for some f in $\mathbf{H}^\infty + \mathbf{C}$. We further assume that Z is infinite. Let λ be a cluster point of Z. Then $|\lambda| = 1$. Let $a_n \in Z$ such that $a_n \to \lambda$. Since the Poisson integral is asymptotically multiplicative on $\mathbf{H}^\infty + \mathbf{C}$ [3, p. 169], given $\varepsilon > 0$ there exists $\delta > 0$ such that $|P_z(bf) - P_z(b)P_z(f)| < \varepsilon$ for $1 - |z| < \delta$. But $P_z(b) = b(z)$ for z in D, and from $bf = z - \bar{z}$ on ∂D we have $P_z(bf) = 2i\Im(z)$ for z in D. Then $2|\Im(a_n)| < \varepsilon$ for $1 - |a_n| < \delta$. This shows that $\Im(a_n) \to 0$ and therefore $\Re(a_n) \to \lambda$. Hence λ is real and $\lambda \in \{-1, 1\}$. □

Lemma. *Let $g \in \mathbf{H}^\infty + \mathbf{C}$ such that its conjugate $\bar{g}$ is also in $\mathbf{H}^\infty + \mathbf{C}$. Then $TT_g - T_gT \in \mathbf{K}$ for all T in the Toeplitz algebra.*

Proof. For $f \in \mathbf{L}^\infty$, let $\mathcal{A}_f$ be the set of all operators T on $\mathbf{H}^2$ for which $TT_f - T_fT \in \mathbf{K}$. Clearly $\mathcal{A}_f$ is a Banach space, and from $T_1T_2T_f - T_fT_1T_2 = T_1(T_2T_f - T_fT_2) + (T_1T_f - T_fT_1)T_2$ it follows that $\mathcal{A}_f$ is a Banach algebra. Now we consider a function g satisfying the hypothesis of the lemma. Since H_g is compact [7, p. 101], from (1) we conclude that $T_{\phi g} - T_\phi T_g \in \mathbf{K}$ for all ϕ in $\mathbf{L}^\infty$. The same argument applied to $\bar{\phi}$ and $\bar{g}$ gives that $T_{\bar{\phi}\bar{g}} - T_{\bar{\phi}}T_{\bar{g}} \in \mathbf{K}$ and by taking the adjoint we conclude that $T_{\phi g} - T_gT_\phi \in \mathbf{K}$. Therefore $T_\phi T_g - T_gT_\phi \in \mathbf{K}$ for all ϕ in $\mathbf{L}^\infty$. Since we can interchange the roles of g and $\bar{g}$, the set $\{T_\phi\colon\ \phi \in \mathbf{L}^\infty\}$ is contained in $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$. Because $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$ is closed under the adjoint operation, $\mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$ is a C^*-algebra, and therefore $\mathbf{T} \subset \mathcal{A}_g \cap \mathcal{A}_{\bar{g}}$. □

Now to prove the second part of the theorem it is enough to exhibit a function g satisfying the hypothesis of the lemma, for which the operator $H_{\bar{b}}T_g - T_gH_{\bar{b}}$ is not compact. The construction of g will be carried out in the Banach space BMO of functions of bounded mean oscillation. For the definition and properties of this space we refer the reader to [7]. A key part in the construction of g is played by the following result [4, Lemma 1].

Proposition. *Let I be a subarc of ∂D and J a subarc of I with the same center. There is a continuous function u with values in $[0,1]$ such that $u=1$ on J, $u=0$ off I, and $\|u\|_{\mathrm{BMO}} \le \mathrm{const}/\log(|I|/|J|)$.*

Here, $|I|$ denotes the length of I, and $\| \, \|_{\mathrm{BMO}}$ is the norm in the space BMO.

Construction of a function. Now we are ready to construct a particular real-valued function g in $\mathbf{H}^\infty + \mathbf{C}$. By hypothesis we have the sequence $\{a_n\}_{n=1}^\infty$ in Z satisfying conditions (i) and (ii), and $\lambda \in \{-1,1\}$. We write $a_n = |a_n|e^{i\theta_n}$, with $-\pi < \theta_n \le \pi$. By (ii) we may assume that $\theta_n \ne 0$ and $\theta_n \ne \pi$ for all n. For $a = |a|e^{i\theta}$ in D we have the identity

$$\frac{|\lambda - a|^2}{(1-|a|)^2} = 1 + 2|a| \cdot \frac{1-\lambda\cos\theta}{(1-|a|)^2}.$$

Then from (ii) it follows that

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{1-\lambda\cos\theta_n}{(1-|a_n|)^2} = +\infty.$$

For the rest of the construction we assume that the set $\{n\colon\ \theta_n > 0\}$ is infinite. (If $\theta_n < 0$ for all n, then the modifications that are necessary are obvious.) Then from (i) it follows that $\theta_n \to 0$ (if $\lambda = 1$) or $\theta_n \to \pi$ (if $\lambda = -1$). First we consider the case $\lambda = 1$. Since $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$, from (iii) it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{1-|a_n|} = +\infty.$$

Then we can choose a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that

$$\text{(iv)} \qquad 2\pi\exp(2^k) < \frac{\theta_{n_k}}{1-|a_{n_k}|} \quad \text{and} \quad 3\theta_{n_{k+1}} < \theta_{n_k} \quad \text{for } k = 1, \dots, \infty.$$

Let I_k be the subarc of ∂D with center $e^{i\theta_{n_k}}$ and length θ_{n_k}. From the second inequality it follows that $I_k \cap I_l = \emptyset$ for $k \ne l$. Let J_k be the subarc of I_k with the same center as I_k and whose length is $2\pi(1-|a_{n_k}|)$. Then $|J_k|\exp(2^k) < |I_k|$, and therefore $1/\log(|I_k|/|J_k|) < 2^{-k}$ for $k \ge 1$. Now we apply the proposition above to I_k and J_k to obtain a continuous function u_k with values in $[0,1]$ such that $u_k = 1$ on J_k, $u_k = 0$ off I_k, and $\|u_k\|_{\mathrm{BMO}} \le (\mathrm{const})2^{-k}$. Let $u = \sum_{k=1}^\infty u_k$. The series converges in BMO and the terms are continuous.

Then from [7, p. 49], u can be written as $u = f + v$ where f is in $\mathbf{C}$ and v is the harmonic conjugate of a function h in $\mathbf{C}$. Then $h + iv$ is analytic in D, and since u is bounded ($\|u\|_\infty = 1$), it follows that $h + iv \in \mathbf{H}^\infty$. Since $f + ih \in \mathbf{C}$, the equality $u = -i(h+iv) + (f+ih)$ shows that $u \in \mathbf{H}^\infty + \mathbf{C}$. Now we define $g = u - \tilde{u}$. Then g is a real-valued function and $g \in \mathbf{H}^\infty + \mathbf{C}$. So g satisfies the hypothesis of the lemma.

Finally, we consider the case $\lambda = -1$. Then (iii) becomes

$$\lim_{n\to\infty} \frac{1-\cos(\pi-\theta_n)}{(1-|a_n|)^2} = +\infty,$$

and from this it follows that

$$\lim_{n\to\infty} \frac{\pi-\theta_n}{1-|a_n|} = +\infty.$$

Now one gets inequalities similar to (iv), where θ_{n_k} is replaced by $\pi-\theta_{n_k}$. Also, in the definition of the subarc I_k we need to change the length (but not the center) to $\pi-\theta_{n_k}$. Then the construction of g proceeds as above.

Last part of the theorem. For the function g constructed above we show that $H_{\bar b}T_g - T_g H_{\bar b}$ is not compact. From this and the lemma it will follow that $H_{\bar b}$ is not in the Toeplitz algebra. Let $A = H_{\bar b}T_g - T_g H_{\bar b}$. Applying (2) to $H_{\bar b g}$ and $H_{\tilde g \bar b}$ (as we did in the proof of the first proposition) we obtain

$$A = H_{(g-\tilde g)\bar b} + H_{\tilde g} T_{\bar b} - T_{\tilde{\bar b}} H_g.$$

Since g and $\tilde g$ are in $\mathbf{H}^\infty + \mathbf{C}$, H_g and $H_{\tilde g}$ are compact, and therefore A is not compact if and only if $(g-\tilde g)\bar b$ does not belong to $\mathbf{H}^\infty + \mathbf{C}$. To arrive at a contradiction, let us assume that $(g-\tilde g)\bar b = f$ for some $f \in \mathbf{H}^\infty + \mathbf{C}$. Then $g - \tilde g = bf$. From the definition of g it follows that $\tilde g = -g$, so $bf = 2g$. Since the Poisson integral is asymptotically multiplicative on $\mathbf{H}^\infty + \mathbf{C}$, there exists $\delta > 0$ such that

$$|P_z(bf) - P_z(b)P_z(f)| < 1 \quad \text{for } 1-|z| < \delta.$$

Therefore, $|P_{a_{n_k}}(g)| < \frac{1}{2}$ for all k's in the complement of a finite set. Now a contradiction will be obtained by showing that $P_{a_{n_k}}(g) \to 1$ as $k \to \infty$. For this we use a result in [6] concerning the Poisson integral and certain averaging functionals. For $z \neq 0$ in D, we let I_z denote the closed subarc of ∂D whose center is $z/|z|$ and whose length is $2\pi(1-|z|)$. Then from [6, Lemma 5] applied to g, we conclude that given $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\left| P_z(g) - \frac{1}{|I_z|}\int_{I_z} g(e^{i\theta})\,d\theta \right| < \varepsilon \quad \text{for } 1-|z| < \delta'.$$

But from the definition of J_k and I_z, with $z = a_{n_k}$, we have $I_{a_{n_k}} = J_k$. Also, from the definition of g, because the functions u_k have disjoint supports, we have

$$\int_{J_k} g(e^{i\theta})\,d\theta = \int_{J_k} u_k(e^{i\theta})\,d\theta = \int_{J_k} 1\,d\theta = |J_k|.$$

Then, from the above inequality, $|P_{a_{n_k}}(g) - 1| < \varepsilon$ for $1-|a_{n_k}| < \delta'$. Now we have obtained the desired contradiction, and therefore A is not compact. The proof of the theorem is complete. □

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