PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 5, May 1996

POLYNOMIALS AND LIMITED SETS

PABLO GALINDO

(Communicated by Theodore Gamelin)

Abstract. We prove that scalar-valued polynomials are weakly continuous on limited sets and that, as in the case of linear mappings, every c_0-valued polynomial maps limited sets into relatively compact ones. We also show that a scalar-valued polynomial whose derivative is limited is weakly sequentially continuous.

Throughout this note E and F will mean infinite-dimensional complex Banach spaces. Recall that $L \subset E$ is said to be *limited* if every $w(E', E)$-null sequence is uniformly convergent to 0 on L. Equivalently, $L \subset E$ is limited if every linear operator $T\colon\ E \to c_0$ maps L into a relatively compact set. Attention to limited sets has focused in studying, for instance, whether they are *bounding* sets for holomorphic functions (see e.g. [12], [13] and [5]) or whether they are relatively compact sets (*Gelfand-Phillips* spaces). Every Banach space isomorphic to a subspace of $C(K)$, K a compact sequentially compact Hausdorff space, is a Gelfand-Phillips space ([5], 4.26). All weakly compactly generated spaces have this property as well as every weak Asplund space. Hence separable or reflexive Banach spaces are Gelfand-Phillips spaces. (See also [7] and [17].) Of course, their intrinsic properties have also been studied (see e.g. [3] and [17]). This note deals with the interplay between polynomials and limited sets and it turns out that it is quite similar to the interplay in the linear case. For instance, it is proved that every polynomial maps limited sets into limited sets. In particular, c_0-valued polynomials map limited sets into relatively compact ones. We also study those sets whose images for c_0-valued polynomials are weakly relatively compact, that is, a polynomial analogue of Grothendieck sets. For background on polynomials and holomorphic mappings we refer to [5]. All polynomials we consider are continuous for the norm topologies. $P(^kE; F)$ denotes the space of k-homogeneous F-valued polynomials defined on E. In the case $F = \mathbb{C}$ we will simply omit $\mathbb{C}$. U_E will denote the unit ball of E.

Lemma 1. *Let $E_1, E_2, \dots, E_k$ be Banach spaces.*

(i) *If $L_i \subset E_i$, $i = 1, \dots, k$, is limited, then $L_1 \otimes \cdots \otimes L_k$ is limited in $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_k$.*

(ii) *If $\{x_n^i\}$ is a limited weak Cauchy sequence in E_i, $i = 1, \dots, k$, and at least one of them is weakly null, then $\{x_n^1 \otimes \cdots \otimes x_n^k\}$ is a weakly null sequence in $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_k$.*

Received by the editors June 13, 1994 and, in revised form, October 19, 1994.
1991 *Mathematics Subject Classification.* Primary 46G20, Secondary 46B20.
Key words and phrases. Polynomial, limited set, weakly conditionally compact set.
The author was supported in part by DGICYT pr. 91-0326 and by grant 93-081.

Proof. (i) We argue inductively on k. For $k=1$ the result is trivial. Assume the result is true for $k-1$. Let $G = E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_{k-1}$. Let $\{T_n\} \subset (E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_k)' = (G\widehat{\otimes}_\pi E_k)'$ be a weak* null sequence and let $\{x_n^i\} \subset L_i$, $i=1,\dots,k$. If $x_n = x_n^1\otimes\cdots\otimes x_n^{k-1}$, the sequence $\{x_n\}$ is limited in G by the induction hypothesis. Every $T \in (G\widehat{\otimes}_\pi E_k)' = \mathcal{L}(G, E_k') = \mathcal{L}(E_k, G')$ is defined by $T(x\otimes y) = T(x)y = T(y)x$, where $x \in G$, $y \in E_k$. For each $y \in E_k$, the sequence $\{T_n(y)\}$ is weak* null in G'. Hence the limitedness of $\{x_n\}$ leads to $\lim_n T_n(x_n)y = \lim_n T_n(y)x_n = 0$. Thus, $\{T_n(x_n)\} \subset E_k'$ is a weak* null sequence and since $\{x_n^k\}$ is limited, we conclude that

$$\lim_n T_n(x_n\otimes x_n^k) = \lim_n T_n(x_n)x_n^k = 0.$$

Now, it follows that $\{T_n\}$ converges uniformly to 0 on $L_1\otimes\cdots\otimes L_k$. Thus $L_1\otimes\cdots\otimes L_k$ is limited in $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_k$.

(ii) Arguing by induction again, we may assume that the sequence $\{x_n\}$ where $x_n = x_n^1\otimes\cdots\otimes x_n^{k-1}$ is weakly null in G. For $T \in (E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_k)' = \mathcal{L}(G, E_k')$ as above, we obtain that $\{T(x_n)\}$ is weakly null in E_k'. Since $\{x_n^k\}$ is limited, $\lim_n T(x_n\otimes x_n^k) = \lim_n T(x_n)(x_n^k) = 0$, and we are done. $\square$

In the next corollary, let $\widehat{\bigotimes}_{k,s,\pi}E$ denote the symmetric k-fold completed projective tensor product of E with itself.

Corollary 2. *Let E be a Banach space.*

(i) *If $L \subset E$ is limited, then $\widehat{\bigotimes}_{k,s,\pi}L$ is limited in $\widehat{\bigotimes}_{k,s,\pi}E$.*

(ii) *If $\{x_n\}$ is a limited sequence weakly converging to $x \in E$, then $\{x_n\otimes\cdots\otimes x_n\}$ is weakly convergent to $x\otimes\cdots\otimes x$ in $\widehat{\bigotimes}_{k,s,\pi}E$.*

(iii) *If $\{x_n\}$ is a limited weakly Cauchy sequence in E, then $\{x_n\otimes\cdots\otimes x_n\}$ is a weakly Cauchy sequence in $\widehat{\bigotimes}_{k,s,\pi}E$.*

Proof. (i) Since $\widehat{\bigotimes}_{k,s,\pi}E$ is a complemented subspace of $\widehat{\bigotimes}_{k,\pi}E$, the statement follows from Lemma 1(i).

(ii) Let $T \in \left(\widehat{\bigotimes}_{k,s,\pi}E\right)'$ and $y_n = x_n - x$. If A is the symmetric k-linear form associated to T we get that

$$\begin{aligned} T(x_n\otimes\cdots\otimes x_n) &= A(x+y_n,\dots,x+y_n)\\ &= A(x,\dots,x) + A(x,\dots,x,y_n) + \cdots + A(y_n\dots,y_n). \end{aligned}$$

Since $\{y_n\}$ is weakly null, we may apply Lemma 1(ii) with $x \in E$ fixed to show that $T(x_n\otimes\cdots\otimes x_n) \to T(x\otimes\cdots\otimes x)$ as $n\to\infty$.

(iii) Let $T \in (\widehat{\bigotimes}_{k,s,\pi}E)'$. Assume that $\{T(x_n\otimes\cdots\otimes x_n)\}$ is not a Cauchy sequence in $\mathbb{C}$. Then there is $\varepsilon > 0$ and two subsequences $\{x_{n_j}\}$ and $\{x_{m_j}\}$ such that $|T(x_{n_j}\otimes\cdots\otimes x_{n_j}) - T(x_{m_j}\otimes\cdots\otimes x_{m_j})| > \varepsilon$ for all $j \in \mathbb{N}$. Put $b_j = x_{n_j} - x_{m_j}$. If A is the associated symmetric k-linear form we get that

$$\begin{aligned} &T(x_{n_j}\otimes\cdots\otimes x_{n_j}) - T(x_{m_j}\otimes\cdots\otimes x_{m_j})\\ &\qquad = A(x_{m_j},\dots,x_{m_j},b_j) + \cdots + A(x_{m_j},b_j,\dots,b_j). \end{aligned}$$

Since $\{x_n\}$ is weakly Cauchy, $\{b_j\}$ is a weakly null sequence in E, hence by Lemma 1(ii), $\{x_{m_j}\otimes \overset{i}{\cdots} x_{m_j}\otimes b_j \overset{k-i}{\cdots}\otimes b_j\}$ is weakly null in $E\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi E$, so

$$\lim_j A(x_{m_j}^i, b_j^{k-i}) = 0 \quad \text{if } 1 \le i < k.$$

Thus

$$\lim_j T(x_{n_j} \otimes \cdots \otimes x_{n_j}) - T(x_{m-j} \otimes \cdots \otimes x_{m_j}) = 0.$$

A contradiction. $\square$

We recall that *limited sets in Banach spaces are conditionally weakly compact* (i.e., any sequence has a weakly Cauchy subsequence) [3]. We need the following result (see Theorem 2.6 [11]): *If B is a conditionally weakly compact subset of E, then every point in the weak closure of B is the weak limit of a sequence of points in B.*

Theorem 3. *Every polynomial, $P\colon E \to \mathbb{C}$, is weakly continuous on limited sets.*

Proof. Let $L \subset E$ be a limited set. To prove that $P\colon\ L \to \mathbb{C}$ is weakly continuous, it suffices to check that $P(\mathrm{cl}_w(B)) \subset \mathrm{cl}_{\mathbb{C}} P(B)$ for each $B \subset L$. Since every $a \in L$ which is in the weak closure of B is the weak limit of a sequence $\{a_n\} \subset B$, we have by Corollary 2(ii) that $\lim_n P(a_n) = P(a)$ because there is $T \in (\widehat{\bigotimes}_{k,s,\pi} E)'$ such that $P(x) = T(x \otimes \cdots \otimes x)$. This completes the proof. $\square$

In general the assumption on limitedness can neither be dropped nor replaced by weak compactness. (Think for instance of $E = l^2$, $P(x) = \sum x_n^2$, and the unit basis $\{e_n\}$ in l^2.)

Remark 4. Let E be the dual of a Banach space Y. A subset $L \subset E$ is called *Y-limited* (see [4]) if every weakly null sequence in Y is uniformly convergent to 0 on L. Of course, every limited set in E is Y-limited. The ideas in the previous results allow us to show that any weak* separately continuous k-linear form on E is weak* sequentially continuous on Y-limited sets. Indeed, let $B\colon\ E \times \cdots \times E \to \mathbb{C}$ be a weak* separately continuous k-linear form on E, and let $\{a_j^1\}, \{a_j^2\}, \dots, \{a_j^k\}$ be Y-limited weakly* convergent sequences in E to $a^1, a^2, \dots, a^k \in E$. Put $b_j^i = a_j^i - a^i$; then

$$\begin{aligned} B(a_j^1, a_j^2, \dots, a_j^k) &- B(a^1, a^2, \dots, a^k) \\ &= B(b_j^1 + a^1, b_j^2 + a^2, \dots, b_j^k + a^k) - B(a^1, a^2, \dots, a^k) \\ &= B(b_j^1, b_j^2, \dots, b_j^k) + \sum B(b_j^1, \dots, a^i, \dots) \end{aligned}$$

where in the last sum every term has at least one of the variables equal to some a^i. If we argue by induction, each of the terms in the sum can be seen as a weak* separately continuous n-linear form with $n < k$, hence $\lim_j B(b_j^1, \dots, a^i, \dots) = 0$. For the remaining term, $B(b_j^1, b_j^2, \dots, b_j^k)$, the mapping $x \in E \rightsquigarrow B(x, b_j^2, \dots, b_j^k)$ is weak* continuous, so there is $y_j \in Y$ such that $\langle y_j, x\rangle = B(x, b_j^2, \dots, b_j^k)$ and again by the induction hypothesis, $\lim_j \langle y_j, x\rangle = \lim_j B(x, b_j^2, \dots, b_j^k) = 0$, because $B(x, \dots)$ is a weak* separately continuous $(k-1)$-linear form for each fixed $x \in E$. Therefore,

$$\lim_j B(a_j^1, a_j^2, \dots, a_j^k) = B(a^1, a^2, \dots, a^k).$$

This remark is related to the question (settled in [1]) of finding conditions which guarantee that a weak* separately continuous multilinear form on E is weak* continuous on bounded sets. So, if Y is a separable Schur space, U_E is a Y-limited and weak* metrizable bounded set in E and therefore every weak* separately continuous multilinear form on E is weak* continuous on bounded sets. Let us also mention that the same type of reasoning shows that if in (E, w^*) the bilinear separately continuous forms are continuous at 0 on bounded sets, then every separately continuous multilinear form continuous at 0 on bounded sets is continuous on bounded sets.

Theorem 5. *If $L \subset E$ is a limited set and $\{P_n\}$ is a sequence in $P^k(E)$ pointwise convergent to 0, then $\{P_n\}$ is uniformly convergent to 0 on L.*

Proof. Let $T_n \in (\widehat{\bigotimes}_{k,s,\pi} E)'$, $n \in \mathbb{N}$, be the associated linear maps to the sequence $\{P_n\}$. Since $T_n \to 0$ in $w((\widehat{\bigotimes}_{k,s,\pi} E)', \bigotimes_{k,s,\pi} E)$ and the set $\{T_n\}$ is equicontinuous, we have that $T_n \to 0$ in $w((\bigotimes_{k,s,\pi} E)', \widehat{\bigotimes}_{k,s,\pi} E)$. Now the result follows by using Corollary 2(i). □

Corollary 6. *If $L \subset E$ is limited and $P \in P({}^k E, F)$, then $P(L)$ is a limited set in F.*

Proof. Let $\{\phi_n\} \subset F'$ be a $w(F', F)$-null sequence. Since $\{\phi_n \circ P\} \subset P({}^k E)$ is pointwise convergent to 0, it follows from Theorem 5 that $\{\phi_n \circ P\}$ converges uniformly to 0 on L, hence $\{\phi_n\}$ converges uniformly to 0 on $P(L)$. Therefore $P(L)$ is limited. □

Proposition 7. *If $P \in P({}^k E, F)$ is weakly sequentially continuous on limited sets and $L \subset E$ is limited, then P is weakly continuous in L and $P(L)$ is a relatively compact set.*

Proof. The continuity statement follows in the same way as Theorem 3. Let us prove that $P(L)$ is a relatively compact set. Let A be the symmetric k-linear mapping associated to P. By the polarization formula and the fact that finite sums of limited sets are limited sets, we have that A is weakly continuous on limited sets and furthermore we can repeat the arguments given in ([2], Lemma 2.4) to show that if $\{a_j^1\}, \{a_j^2\}, \dots, \{a_j^k\}$ are limited weakly Cauchy sequences in E and at least one of them is weakly null, then the sequence $\{A(a_j^1, a_j^2, \dots, a_j^k)\}$ converges to 0 in F.

Let $\{a_n\}$ be an arbitrary sequence in L. We will prove that $\{P(a_n)\}$ has a Cauchy subsequence in F. Since L is weakly conditionally compact, $\{a_n\}$ has a weakly Cauchy subsequence $\{a_{n_j}\}$. The same argument as in Corollary 2(iii) shows that $\{P(a_{n_j})\}$ is a Cauchy sequence in F. □

Note that the above proposition applied to the identity map on E leads to the well-known fact [8] that if every limited weakly null sequence in E is norm null, then E is a Gelfand-Phillips space.

Recall the following definition due to Farmer and Johnson [9]: A Banach space E is said to be a P_N-*Schur* space ($N \in \mathbb{N}$) if any sequence $\{a_n\} \subset E$ is a null sequence in E provided that $\{P(a_n)\}$ converges to 0 in $\mathbb{C}$ for all polynomials $P \in P({}^N E)$.

Proposition 8. *Every P_N-Schur space is a Gelfand-Phillips space.*

Proof. It is enough to check that every limited weakly null sequence is convergent to 0, and this follows at once from Corollary 2(ii). □

A polynomial $P \in P({}^k E, F)$ is called *limited* (resp. *weakly compact, weakly conditionally compact*) if it maps the unit ball of E into a limited (resp. weakly relatively compact, weakly conditionally compact) set in F. It is known [15] that for every weakly compact (resp. weakly conditionally compact) polynomial $P \in P(^kE,F)$ there is a Banach space F_P reflexive (resp. non-containing copies of l^1) and a polynomial $P_1 \in P(^kE,F_P)$ such that $P = i \circ P_1$, where $i\colon F_P \to F$ is a linear operator. As a consequence of Corollary 6 and the fact that limited sets are relatively compact in reflexive Banach spaces and relatively weakly compact in spaces without copies of l^1 [3], we have the following.

Corollary 9. *If $P \in P(^kE,F)$ is weakly compact (resp. weakly conditionally compact) and $L \subset E$ is limited, then $P(L)$ is a relatively compact (resp. weakly relatively compact) subset of F. If, moreover, $Q \in P(^mF,G)$ is limited, then $Q \circ P$ is a compact (resp. weakly compact) polynomial.*

The second part of the above corollary slightly generalizes the result of Lindström stating that the composition of two limited operators is weakly compact ([16]).

Corollary 10. *If the derivative polynomial of $P\colon\ E \to \mathbb{C}, dP\colon\ E \to E'$, is a limited polynomial, then P is weakly sequentially continuous.*

Proof. We may suppose P is m-homogeneous. Recall that the derivative dP is given by $dP\colon\ E \to E'$, $dP(x)(u) = mA(u,x,\dots,x), x,u \in E$, where A is the symmetric m-linear mapping associated to P, and therefore, the symmetric $(m-1)$-linear mapping associated to dP, $\dot{A}$, is $\dot{A}(x_1,\dots,x_{m-1})(u) = mA(u,x_1,\dots,x_{m-1})$. We show the result by induction on the degree of P. It is obviously true for $m = 1$. Assume it is true for all $i \in \mathbb{N}$, $i < m$, and let us check it for m. Let $\{a_n\} \subset E$ be a weakly convergent sequence to $a \in E$. To prove that $P(a) = \lim_n P(a_n)$, it is enough to prove that $\lim_n A((a_n-a)^i, a^{m-i}) = 0$ for all $i \in \mathbb{N}, i \le m$. For $i \in \mathbb{N}, i \le m$, consider $P_i\colon\ E \to \mathbb{C}$ defined by $P_i(x) = A(x^i, a^{m-i})$, whose associated symmetric i-linear form is $A(x_1,\dots,x_i,a^{m-i})$ and whose derivative dP_i is given by $dP_i(x)(u) = iA(u,x^{i-1},a^{m-i})$. Since dP is limited, it maps any bounded set into a limited set in E' and since $\dot{A}$ shares with it this property, it follows that every dP_i is limited. By the induction hypothesis each P_i, $i < m$, is weakly sequentially continuous, hence $\lim_n P_i(a_n - a) = \lim_n A((a_n-a)^i, a^{m-i}) = 0$. Moreover, $\lim_n P(a_n - a) = \lim_n dP(a_n-a)(a_n-a) = 0$ because the set $\{dP(a_n - a)\colon\ n \in \mathbb{N}\} \subset E'$ is limited and the sequence $\{(a_n - a)\}$ is $w(E'', E')$-null. □

Remark 11. (1) The converse to the above proposition does not hold in general. For instance, consider $E = c_0 \times l_1$ and $A\colon\ E \times E \to \mathbb{C}$ defined by $A((x,\alpha),(y,\beta)) = \frac{\alpha(y)+\beta(x)}{2}$ where $x,y \in c_0$ and $\alpha,\beta \in l_1$. A is a symmetric bilinear form whose associated polynomial, P, is $P((x,\alpha)) = \alpha(x)$. P is weakly sequentially continuous: Indeed, let $\{(x_n,\alpha_n)\}$ be a sequence in E weakly convergent to (x,α); then $\{x_n\} \subset c_0$ is weakly convergent to x and $\{\alpha_n\} \subset l_1$ is weakly, hence norm, convergent to α. Now,

$$\begin{aligned}|P((x_n,\alpha_n)) - P((x,\alpha))| &= |\alpha_n(x_n) - \alpha(x)| = |\alpha_n(x_n) - \alpha(x_n) + \alpha(x_n) - \alpha(x)|\\ &\le |(\alpha_n-\alpha)(x_n)| + |\alpha(x_n - x)| \le \|\alpha_n - \alpha\| \|x_n\| + |\alpha(x_n - x)|\end{aligned}$$

and both last sequences converge to 0. On the other hand, $dP((x,\alpha)) = (\alpha, x)$, hence $dP(U_E) \supset U_{l_1} \times U_{c_0}$. Consequently, $dP(U_E)$ cannot be limited in $E' = l_1 \times l_\infty$, because then its projection into l_1 would be a limited set containing the unit ball of l_1.

(2) From the examples of limited nonbounding sets found by Josefson [13] and Schlumprecht [19], it follows that holomorphic mappings do not map limited sets into limited sets in general. In fact, every $f \in H(E,F)$ maps limited sets into limited sets if, and only if, every limited set in E is a bounding set.

As an application of our former results, we can show the following.

Proposition 12. *Let $f \in H(E)$. f is bounded on limited sets if, and only if, f maps limited weakly Cauchy sequences into convergent ones. Under these conditions f is weakly continuous on limited sets.*

Proof. ($\Rightarrow$) Since f is bounded on limited sets, its Taylor series expansion about 0 is uniformly convergent to f on limited sets as a consequence of Cauchy's inequalities. Therefore on any weakly Cauchy limited sequence, $\{a_n\}$, given $\varepsilon > 0$, there is a polynomial, P, such that $\|f - P\|_L < \varepsilon$, where $L = \{a_n | n \in \mathbb{N}\} \subset E$. Since $\{a_n\}$ is weakly Cauchy, by Corollary 2(iii), there is $p \in \mathbb{N}$ such that $|P(a_n) - P(a_m)| < \varepsilon$ if $n, m > p$. Thus if $n, m > p$, then $|f(a_n) - f(a_m)| < 3\varepsilon$. Hence $\{f(a_n)\}$ is a Cauchy sequence. Moreover, since f can be uniformly approximated by polynomials on limited sets, applying Theorem 3, it follows that f is weakly continuous on limited sets.

($\Leftarrow$) Just recall that limited sets are weakly conditionally compact sets. □

Remark 13. (i) As already mentioned, a limited set is not necessarily a bounding set. Nevertheless, the limited sets are the (balanced) bounding sets for a smaller class of entire functions, namely, those characterized in the above proposition. Note that if we replace limited by compact, the (balanced) bounding sets for $H(E)$ do not necessarily coincide with the compact sets. Let's prove now our statement concerning limited sets: Let B be a balanced bounding set in E for holomorphic functions bounded on limited sets. To see that B is limited, let $\{f_n\} \subset E'$ be a sequence pointwise convergent to 0. For any $\varepsilon > 0$, the function $f\colon E \to \mathbb{C}$ defined by $f(x) := \sum_{n=0}^{\infty} (\frac{f_n}{\varepsilon})^n(x)$ is easily seen to be bounded on limited sets, hence it is bounded on B. Therefore by Cauchy's inequality, $\|(\frac{f_n}{\varepsilon})^n\|_B \le \|f\|_B$, hence $\|\frac{f_n}{\varepsilon}\|_B \le \|f\|_B^{\frac{1}{n}}$. Consequently, $\|f_n\|_B \le 2\varepsilon$ for sufficiently large n.

(ii) Whenever the limited sets are relatively weakly compact, for instance when E does not contain a copy of l_1, the holomorphic functions which are bounded on limited sets are those which are weakly continuous on limited sets.

A related notion to that of limited set is the one of Grothendieck set [14]: $A \subset E$ is called a *Grothendieck* set if for every operator $T\colon E \to c_0$, $T(A)$ is a weakly relatively compact set in c_0. In view of Corollary 6, we may ask if polynomials map Grothendieck sets into Grothendieck sets. The negative answer to this question is provided by González and Gutiérrez in [10] who show the existence of a polynomial $P\colon l^\infty \to c_0$ which maps the unit ball of l^∞ into a non-weakly relatively compact set. Our next proposition is a refinement of their result and its proof goes back to Proposition 4 of [3]. We will call a subset A of E a *P-Grothendieck* set if $P(A) \subset c_0$ is weakly relatively compact for every polynomial $P\colon E \to c_0$.

Proposition 14. *Every P-Grothendieck set is weakly conditionally compact.*

Proof. Let $A \subset E$ be a P-Grothendieck set and suppose it is not weakly conditionally compact. Then there is a sequence $\{a_n\} \subset A$ equivalent to the unit basis of l^1. As in ([6], p. 223) we can construct an operator $N\colon E \to L_\infty[0,1]$ such that $N(a_n) = r_n$ where $\{r_n\}$ is the sequence of Rademacher functions. Let

$B\colon L_\infty[0,1] \times L_\infty[0,1] \to c_0$ be defined by

$$B(f,g) = \sum_{n=1}^{\infty} \left(\int f r_n \right) \left(\int g r_n \right) u_n$$

where $u_n = (1, 1, \overset{n}{\ldots}, 1, 0, 0, \ldots)$, $f, g \in L_\infty[0,1]$. Since $\{r_n\}$ is an orthonormal set, applying Bessel's inequality, we have $(\int h r_n)_n \in l^2$ and $\|(\int h r_n)_n\|_2 \le \|h\|_2 \le \|h\|_\infty$ for all $h \in L_\infty[0,1]$. Thus $\|B(f,g)\| \le \|f\|_\infty \|g\|_\infty$ and B is a continuous bilinear map. Therefore $Q(f) = B(f,f)$ is a 2-homogeneous polynomial and $Q \circ N$ is a 2-homogeneous polynomial on E. By assumption $(Q \circ N)(A)$ is a weakly relatively compact set in c_0 which contains the sequence $\{u_n\}$. A contradiction. □

Although every weakly relatively compact set is Grothendieck, it is not necessarily P-Grothendieck as it follows from considering $P\colon l^2 \to c_0$ defined by $P(x) = \sum x_n^2 u_n$ where $x = (x_n) \in l^2$, $u_n = (1, 1, \overset{n}{\ldots}, 1, 0, 0, \ldots)$ and the unit basis $\{e_n\}$ in l^2, for which $\{P(e_n)\}$ is not a weakly relatively compact set. On the other hand, since any scalar valued polynomial on c_0 is weakly sequentially continuous, the weakly relatively compact sets in c_0 are P-Grothendieck. Obviously from Corollary 6, every limited set is P-Grothendieck. It follows from linearization techniques (see [18]) that whenever $P({}^m E)$ is reflexive, for instance if E is Tsirelson's space, the unit ball is a (non-limited) P-Grothendieck set. We remark that any continuous polynomial maps P-Grothendieck sets into P-Grothendieck sets.

Proposition 15. *Let $f\colon E \to F$ be a holomorphic mapping which is bounded on bounded sets in E. If $A \subset E$ is P-Grothendieck* (*resp. limited*), *then $f(A)$ is a P-Grothendieck* (*resp. limited*) *set.*

Proof. We begin with the case $F = c_0$. Given $\varepsilon > 0$ we can obtain from the Taylor series expansion of f about $0 \in E$ a polynomial P such that $\|f - P\|_A < \varepsilon$, i.e., $f(A) \subset P(A) + \varepsilon U_{c_0} \subset \mathrm{cl}_w(P(A)) + \varepsilon U_{c_0}$, where U_{c_0} denotes the closed unit ball of c_0. Since A is P-Grothendieck, $P(A)$ is P-Grothendieck, hence weakly relatively compact. Therefore, $\mathrm{cl}_w(P(A)) + \varepsilon U_{c_0}$ is a weakly closed set, so it follows that $\mathrm{cl}_w(f(A)) \subset \mathrm{cl}_w(P(A)) + \varepsilon U_{c_0}$. Finally, we apply a lemma of Grothendieck (see [6], Lemma 2, p. 227) to prove that $\mathrm{cl}_w(f(A))$ is weakly compact.

For the general case, consider $P\colon F \to c_0$. By the above case, $(P \circ f)(A)$ is weakly relatively compact. Thus $f(A)$ is P-Grothendieck.

A similar argument holds for limited sets. □

Acknowledgment

This paper was written during the academic year 1993–94 while I was visiting the Department of Mathematics of Kent State University. Thanks are given to the people there for their hospitality. I am especially grateful to Richard M. Aron for the helpful conversations we had during the preparation of this note. I also thank the referee for suggesting the tensor product approach.

Added in proof

J. Gutiérrez pointed out that Theorem 3 also follows from Theorem 3.5 in his joint article with M. González, *Weakly continuous mappings on Banach spaces with Dunford-Pettis property*, J. Math. Anal. Appl. **173** (1993), 470–482.

References

1. R. Aron, B. Cole, and T. Gamelin, *Weak-star continuous analytic functions*, Canad. J. Math. **47** (1995), 673–683.
2. R. Aron, C. Hervés, and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. **52**, (1983), 189–204. MR **84g:**46066
3. J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. **11**, (1984), 55–58. MR **86d:**47024
4. J. Dieudonné, *Sur les espaces de Köthe*, J. Analyse Math. **1**, (1951), 81–115. MR **12:**834b
5. S. Dineen, *Complex analysis in locally convex spaces*, North-Holland Math. Stud., vol. 57, North-Holland, Amsterdam and New York, 1980. MR **84b:**46050
6. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Math., vol. 92, Springer-Verlag, Berlin and New York, 1984. MR **85i:**46020
7. L. Drewnowski, *On Banach spaces with the Gelfand-Phillips property* I, Math. Z. **19**, (1986), 405–411. MR **88d:**46024
8. L. Drenowski and G. Emmanuele, *On Banach spaces with the Gelfand-Phillips property* II, Rend. Cir. Mat. Palermo Ser. II **38**, (1989), 377–391. MR **92a:**46016
9. J. Farmer and W. Johnson, *Polynomial Schur and polynomial Dunford-Pettis properties*, Contemp. Math., vol. 144, Amer. Math. Soc., Providence, RI, 1993, pp. 95–105. MR **94e:**46078
10. M. González and J. Gutiérrez, *Unconditionally converging polynomials on Banach spaces*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 321–331. CMP95:05
11. ———, *The compact weak topology on a Banach space*, Proc. Roy. Soc. Edinburgh Sect. A **12**, (1992), 367–379. MR **93c:**46022
12. B. Josefson, *Bounding sets in* $l^\infty(A)$, J. Math. Pures Appl. **57**, (1975), 397–421. MR **81a:**46019
13. ———, *A Banach space containing non-trivial limited sets but no non-trivial bounding set*, Israel J. Math. **71**, (1990), 321–327. MR **92g:**46018
14. D. Leung, *A Gelfand-Phillips property with respect to the weak topology*, Math. Nachr. **14**, (1990), 177–181. MR92j:46032
15. M. Lindström, *On compact and bounding holomorphic mappings*, Proc. Amer. Math. Soc. **10**, (1989), 356–361. MR **89f:**46103
16. ———, *A characterization of Schwartz spaces*, Math. Z. **19**, (1988), 423–430. MR **89f:**46009
17. M. Lindström and T. Schlumprecht, *On limitedness on locally convex spaces*, Arch. Math. **53**, (1988), 65–74. MR **90f:**46003
18. J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **32**, (1991), 867–887. MR **91h:**46088
19. T. Schlumprecht, *A limited set that is not bounding*, Proc. Roy. Irish Acad. Sect. A **90**, (1990), 125–129. MR **93b:**46032

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

E-mail address: galindo@vm.ci.uv.es