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AMENABILITY AND WEAK AMENABILITY OF SECOND CONJUGATE BANACH ALGEBRAS

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Abstract. For a Banach algebra $\mathfrak{A}$, amenability of $\mathfrak{A}^{**}$ necessitates amenability of $\mathfrak{A}$, and similarly for weak amenability provided $\mathfrak{A}$ is a left ideal in $\mathfrak{A}^{**}$. For $\mathfrak{G}$ a locally compact group, indeed more generally, $L^1(\mathfrak{G})^{**}$ is amenable if and only if $\mathfrak{G}$ is finite. If $L^1(\mathfrak{G})^{**}$ is weakly amenable, then $M(\mathfrak{G})$ is weakly amenable.

0. Introduction

For a Banach algebra $\mathfrak{A}, \mathfrak{A}^{**}$ is a Banach algebra under two Arens products, of which we will always take the first, or left, product. For further details see the survey article [8]. This product can be characterized as the extension to $\mathfrak{A}^{**} \times \mathfrak{A}^{**}$ of the bilinear map $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} : (x, y) \mapsto xy$ with the following continuity properties: for fixed $y \in \mathfrak{A}^{**}$, $x \mapsto xy$ is weak*-continuous on $\mathfrak{A}^{**}$; for fixed $y \in \mathfrak{A}$, $x \mapsto yx$ is weak*-continuous on $\mathfrak{A}^{**}$. Here, as elsewhere, we identify $\mathfrak{A}$ with its canonical image in $\mathfrak{A}^{**}$.

In terms of the asymmetry here, define the *topological centre* of $\mathfrak{A}^{**}$ by

$$Z_t(\mathfrak{A}^{**}) = \{y \in \mathfrak{A}^{**} : x \mapsto yx \text{ is weak}^*\text{-continuous}\}.$$

Clearly, $Z_t(\mathfrak{A}^{**})$ contains the (algebraic) centre $Z(\mathfrak{A}^{**})$ of $\mathfrak{A}^{**}$; it also contains $\mathfrak{A}$. In the case that $Z_t(\mathfrak{A}^{**}) = \mathfrak{A}^{**}$, $\mathfrak{A}^{**}$ is said to be *Arens regular*. Any C^*-algebra is Arens regular [6, Theorem 7.1], but for a locally compact group $\mathfrak{G}$, $L^1(\mathfrak{G})$ is Arens regular if and only if $\mathfrak{G}$ is finite [25].

A Banach algebra $\mathfrak{A}$ is *amenable* if every derivation $D : \mathfrak{A} \to X^*$ is inner, for every Banach $\mathfrak{A}$-bimodule X. If one only considers the bimodule $X = \mathfrak{A}$, one has the notion of *weak amenability*.

There are many alternative formulations of the notion of amenability, of which we need the following two, first introduced in [18]. For further details see [3, 16, 7]. The Banach algebra $\mathfrak{A}$ is amenable if and only if either, and hence both, of the following hold:

(i) $\mathfrak{A}$ has an *approximate diagonal*, that is, a bounded net $(m_i) \subset \mathfrak{A}\hat{\otimes}\mathfrak{A}$ such that for each $x \in \mathfrak{A}$, $m_i x - x m_i \to 0, \pi(m_i)x \to x$;
(ii) $\mathfrak{A}$ has a *virtual diagonal*, that is, an element $M \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ such that for each $x \in \mathfrak{A}$, $xM = Mx, (\pi^{**}M)x = x$.

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Note that here $\pi : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is the natural product map.

The purpose of the present note is to investigate the relation between the amenability of $\mathfrak{A}$ and that of $\mathfrak{A}^{**}$. Some of the difficulties of this task were already apparent in [17], and were dealt with in some cases in [5]. Here we take a different viewpoint by generally putting the primary hypothesis on $\mathfrak{A}^{**}$ rather than $\mathfrak{A}$.

1. Amenability

Lemma 1.1. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{**}$ has a bounded approximate identity. Then $\mathfrak{A}^{**}$ has an identity.*

Proof. Let (e_i) be a bounded approximate identity in $\mathfrak{A}^{**}$. By passing to a subnet, if necessary, we may suppose that (e_i) converges weak* to some $E \in \mathfrak{A}^{**}$. For $x \in \mathfrak{A}$, we have, using weak*-continuity in the second variable for $x \in \mathfrak{A}$,

$$xe_i \xrightarrow{\|\cdot\|} x \quad \text{and} \quad xe_i \xrightarrow{\text{weak}^*} xE,$$

so that $x = xE$. But then by weak*-density of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ and weak*-continuity in the first variable, we conclude that $m = mE$ for $m \in \mathfrak{A}^{**}$. Further,

$$e_i m \xrightarrow{\text{weak}^*} Em \quad \text{and} \quad e_i m \xrightarrow{\|\cdot\|} m,$$

showing that E is the identity for $\mathfrak{A}^{**}$. □

Note that a standard argument using Goldstine's theorem shows consequently that $\mathfrak{A}$ has a bounded approximate identity, however this result is subsumed in Theorem 1.8 below.

We have defined the topological centre $Z_t(\mathfrak{A}^{**})$ above, and noted that $\mathfrak{A} \subseteq Z_t(\mathfrak{A}^{**})$ always holds. One important case of equality is the following. Recall that a semigroup $\mathfrak{G}$ is *weakly cancellative* if for any $a, b \in \mathfrak{G}$, $\{x \in \mathfrak{G} : xa = b\}, \{y \in \mathfrak{G} : ay = b\}$ are finite.

Theorem 1.2 ([21, 20])**.** *For $\mathfrak{G}$ a locally compact group, or a discrete weakly cancellative semigroup, $Z_t(L^1(\mathfrak{G})^{**}) = L^1(\mathfrak{G})$.*

This result is an ingredient in the following characterization.

Theorem 1.3. *Let $\mathfrak{G}$ be a locally compact group, or a discrete weakly cancellative semigroup. Then $L^1(\mathfrak{G})^{**}$ is amenable only if $\mathfrak{G}$ is finite (and conversely if $\mathfrak{G}$ is a group).*

Proof. The result is clear if $\mathfrak{G}$ is a finite group, so assume that $L^1(\mathfrak{G})^{**}$ is amenable. By Lemma 1.1, the hypothesis ensures that $L^1(\mathfrak{G})^{**}$ has an identity e. Clearly the (identity) map $x \mapsto ex$ on $L^1(\mathfrak{G})^{**}$ is weak* to weak*-continuous, so by Theorem 1.2 above, $e \in L^1(\mathfrak{G})$, so that $\mathfrak{G}$ must therefore be discrete. In the semigroup case we in fact have this by hypothesis. By [6, Theorem 3.2] for the group case and [8, Theorem 9] in the semigroup case, we thus have

$$L^1(\mathfrak{G})^{**} = L^1(\mathfrak{G}) \oplus C_0(\mathfrak{G})^{\perp},$$

a Banach space direct sum, with $C_0(\mathfrak{G})^{\perp}$ a weak*-closed two-sided ideal. Since $C_0(\mathfrak{G})^{\perp}$ is a complemented ideal in the amenable algebra $L^1(\mathfrak{G})^{**}$, it is itself amenable. In particular $C_0(\mathfrak{G})^{\perp}$ has a bounded approximate identity (e_i), and without loss of generality we may suppose that (e_i) converges weak* to some $E \in L^1(\mathfrak{G})^{**}$, necessarily $E \in C_0(\mathfrak{G})^{\perp}$. For $x \in C_0(\mathfrak{G})^{\perp}$ we have

$$e_i x \xrightarrow{\|\cdot\|} x \quad \text{and} \quad e_i x \xrightarrow{\text{weak}^*} Ex$$

by weak*-continuity in the first variable. Thus E is a left identity for $C_0(\mathfrak{G})^\perp$. But then for $x \in C_0(\mathfrak{G})^\perp$, $xE = \lim(xE)e_i = \lim x(Ee_i) = x$, so that E is a two-sided identity for $C_0(\mathfrak{G})^\perp$. Thus for $n \in L^1(\mathfrak{G})^{**}$, $En = (En)E = E(nE) = nE$, since $En, nE \in C_0(\mathfrak{G})^\perp$, and it follows that E is central in $L^1(\mathfrak{G})^{**}$. By Theorem 1.2, $E \in L^1(\mathfrak{G})$, hence $E = 0$, and $C_0(\mathfrak{G})^\perp = \{0\}$, showing that $\mathfrak{G}$ is finite. □

Corollary 1.4. *Let $\mathfrak{G}$ be a locally compact group, or a discrete weakly cancellative semigroup. Then $M(\mathfrak{G})^{**}$ is amenable only if $\mathfrak{G}$ is finite* (*and conversely if $\mathfrak{G}$ is a group*).

Proof. Since $L^1(\mathfrak{G})$ is complemented in $M(\mathfrak{G})$, $L^1(\mathfrak{G})^{**}$ is complemented in $M(\mathfrak{G})^{**}$. Further $L^1(\mathfrak{G})^{**}$ is an ideal in $M(\mathfrak{G})^{**}$, and so is itself amenable if $M(\mathfrak{G})^{**}$ is amenable [7]. The result is now an immediate consequence of Theorem 1.3. □

The latter part of the argument of the theorem in fact shows the following, though there is a paucity of examples where the basic hypothesis is satisfied.

Theorem 1.5. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{**} = Z_t(\mathfrak{A}^{**}) \oplus J$ for some weak*-closed ideal J. Then amenability of $\mathfrak{A}^{**}$ necessitates $J = 0$, so that $\mathfrak{A}^{**} = Z_t(\mathfrak{A}^{**})$ is Arens regular.* □

We need the following result.

Lemma 1.6 ([1, §3])**.** *Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces, $T : \mathfrak{X} \times \mathfrak{Y} \to \mathbb{C}$ a continuous bilinear form. Then T has a continuous extension $\overline{T} : \mathfrak{X}^{**} \times \mathfrak{Y}^{**} \to \mathbb{C}$ such that for $F \in \mathfrak{X}^{**}$, $G \in \mathfrak{Y}^{**}$ and nets $(a_i) \subset \mathfrak{X}, (b_j) \subset \mathfrak{Y}$ with $a_i \stackrel{\text{weak}^*}{\to} F, b_j \stackrel{\text{weak}^*}{\to} G$,*

$$\overline{T}(F, G) = \lim_i \lim_j T(a_i, b_j).$$

Lemma 1.7. *Let $\mathfrak{A}$ be a Banach algebra. Then there is a continuous linear mapping $\Psi : \mathfrak{A}^{**}\hat{\otimes}\mathfrak{A}^{**} \to (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ such that for $a, b, x \in \mathfrak{A}$ and $m \in \mathfrak{A}^{**}\hat{\otimes}\mathfrak{A}^{**}$ the following hold*:

(i) $\Psi(a \otimes b) = a \otimes b$;
(ii) $\Psi(m) \cdot x = \Psi(m \cdot x)$;
(iii) $x \cdot \Psi(m) = \Psi(x \cdot m)$;
(iv) $(\pi_{\mathfrak{A}})^{**}(\Psi(m)) = \pi_{\mathfrak{A}^{**}}(m)$.

Proof. First recall that for any Banach spaces X, Y there is an isometric isomorphism between the space of bilinear maps $X \times Y \to \mathbb{C}$ and $(X\hat{\otimes}Y)^*$, namely $T \mapsto \varphi_T$ where $\varphi_T(x \otimes y) = T(x, y)$. Define $\Psi : \mathfrak{A}^{**}\hat{\otimes}\mathfrak{A}^{**} \to (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ by

$$\langle \Psi(F \otimes G), \varphi_T \rangle = \overline{T}(F, G).$$

For $a, b \in \mathfrak{A}$,

$$\langle \Psi(a \otimes b), \varphi_T \rangle = \overline{T}(a, b) = T(a, b) = \langle \varphi_T, a \otimes b \rangle,$$

so (i) certainly holds.

For (ii), let $F \otimes G \in \mathfrak{A}^{**}\hat{\otimes}\mathfrak{A}^{**}$, $x \in \mathfrak{A}, T$ a bilinear map $X \times Y \to \mathbb{C}$, and take nets $(a_i), (b_j) \subset \mathfrak{A}$, with $a_i \overset{\text{weak}^*}{\to} F, b_j \overset{\text{weak}^*}{\to} G$. Then $b_j x \overset{\text{weak}^*}{\to} G \cdot x$, and so

$$\begin{aligned}\langle \Psi(F \otimes G) \cdot x, \varphi_T \rangle &= \langle \Psi(F \otimes G), x \cdot \varphi_T \rangle \\ &= \lim_i \lim_j (x \cdot \varphi_T)(a_i \otimes b_j) \\ &= \lim_i \lim_j \varphi_T(a_i \otimes b_j \cdot x) \\ &= \overline{T}(F, G \cdot x) \\ &= \langle \Psi(F \otimes G \cdot x), \varphi_T \rangle.\end{aligned}$$

The argument for (iii) is analogous.

Finally, for (iv), let $F \otimes G \in \mathfrak{A}^{**}\hat{\otimes}\mathfrak{A}^{**}$, $h \in \mathfrak{A}^*$, and take nets $(a_i), (b_j) \subset \mathfrak{A}$, with $a_i \overset{\text{weak}^*}{\to} F, b_j \overset{\text{weak}^*}{\to} G$. Then

$$\begin{aligned}\langle (\pi_{\mathfrak{A}})^{**}(\Psi(F \otimes G)), h \rangle &= \langle \Psi(F \otimes G), (\pi_{\mathfrak{A}})^* h \rangle \\ &= \overline{(\pi_{\mathfrak{A}})^* h}(F, G) \\ &= \lim_i \lim_j (\pi_{\mathfrak{A}})^* h(a_i, b_j) \\ &= \lim_i \lim_j \langle (\pi_{\mathfrak{A}})^* h, (a_i \otimes b_j) \rangle \\ &= \lim_i \lim_j \langle h, \pi_{\mathfrak{A}}(a_i \otimes b_j) \rangle \\ &= \lim_i \lim_j \langle h, a_i b_j \rangle \\ &= \langle FG, h \rangle = \langle \pi_{\mathfrak{A}^{**}}(F \otimes G), h \rangle. \quad \square\end{aligned}$$

Theorem 1.8 (Gourdeau)**.** *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{**}$ is amenable. Then $\mathfrak{A}$ is amenable.*

Proof. Let (m_i) be an approximate diagonal for $\mathfrak{A}^{**}$. By Lemma 1.7, for every $x \in \mathfrak{A}$, $\Psi(m_i) \cdot x - x \cdot \Psi(m_i) \to 0$ and $(\pi_{\mathfrak{A}})^{**}(\Psi(m_i)x) \to x$. If M is a weak*-cluster point of $(\Psi(m_i))$ in $(\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$, then necessarily $Mx = xM$ and $((\pi_{\mathfrak{A}})^{**}M)x = x$. Thus M is a virtual diagonal for $\mathfrak{A}$. $\square$

A variant on Theorem 1.3 is an immediate consequence.

Corollary 1.9. *For $\mathfrak{G}$ a discrete left* (*or right*) *cancellative semigroup with identity, $L^1(\mathfrak{G})^{**}$ is amenable if and only if $\mathfrak{G}$ is a finite group.*

Proof. If $L^1(\mathfrak{G})^{**}$ is amenable, then by Theorem 1.8, $L^1(\mathfrak{G})$ is amenable. But then by [15, Theorem 2.3] $\mathfrak{G}$ is a group. Now apply Theorem 1.3. $\square$

The example given at the end of [9] shows a discrete weakly cancellative semigroup $\mathfrak{G}$ with $L^1(\mathfrak{G})^{**}$ amenable need not be a group. So the conclusion of Corollary 1.9 that $\mathfrak{G}$ is a group cannot be added to Theorem 1.3. For a discrete semigroup $\mathfrak{G}$, amenability of $L^1(\mathfrak{G})$ is considered in [15, 9], amongst others. In particular, [9, Theorem 2] shows that if $L^1(\mathfrak{G})$ is amenable, then $\mathfrak{G}$ is regular with finitely many idempotents. Results on when $L^1(\mathfrak{G})^{**}$ is amenable for general discrete semigroups $\mathfrak{G}$ are given in the forthcoming paper [22].

For any locally compact space K, $C_0(K)$ is amenable, $C_0(K)^{**} = C(Y)$ for some compact space Y, and so is amenable. More generally, any C^*-algebra $\mathfrak{A}$ is Arens regular and weakly amenable, as is the von Neumann algebra $\mathfrak{A}^{**}$. However $\mathcal{K}(l_2)$

is amenable, yet $\mathcal{K}(l_2)^{**} = \mathcal{B}(l_2)$ is not. Indeed, amenable von Neumann algebras are precisely finite direct sums of $C(X) \otimes M_n(\mathbb{C})$ for some Stone space X, and some $n \geq 1$ [24, Corollary 1.9]. (This requires knowing that amenability and nuclearity are the same for C^*-algebras, a highly nontrivial result. A direct proof does not seem to be known.)

In view of this last example, it is natural to raise the question as to whether there is an infinite-dimensional commutative $\mathfrak{A}$ such that $\mathfrak{A}^{**}$ is amenable, but $\mathfrak{A}$ is not $C_0(X)$. If so, is there one with $\mathfrak{A}^{**}$ also commutative?

It is also of interest to look at the case when $\mathfrak{A}$ is amenable and reflexive. By Lemma 1.1 $\mathfrak{A}$ has an identity e. Then the same is true of the enveloping algebra $\mathfrak{A}^e = \mathfrak{A}\hat{\otimes}\mathfrak{A}^{\mathrm{op}}$. Consider $\mathfrak{A}$ as a left module over $\mathfrak{A}^e$. Now let M be a complemented ideal of $\mathfrak{A}$. Then M is reflexive as a Banach space, and is a $\mathfrak{A}^e$-submodule, so by [16, Proposition VII.2.30] there is a $\mathfrak{A}^e$-module projection $Q : \mathfrak{A} \to M$. But then if $x \in \mathfrak{A}$,

$$xQ(e) = xQ(e)e = Q(xee) = Q(x) = Q(eex) = eQ(e)x = Q(e)x.$$

It follows that $M = Q(e)\mathfrak{A}$ where $Q(e)$ is a central idempotent in $\mathfrak{A}$. This should be compared with [11, Lemma 2.2].

The same style of argument shows the following. In view of [11] it would be of considerable interest to know if the result holds for Banach algebras with reflexive underlying space.

Proposition 1.10. *An amenable Banach algebra $\mathfrak{A}$ which is a Hilbert space is finite dimensional.*

Proof. First note that such $\mathfrak{A}$ must be semisimple, for by the above any nontrivial closed ideal contains an idempotent. By Lemma 1.1, $\mathfrak{A}$ has an identity $\mathbf{1}$, and so it has maximal left ideals which are necessarily complemented, and so have right units. Similarly for right ideals, so that $\mathfrak{A}$ is an annihilator algebra and furthermore the annihilators always contain idempotents. So by [3, Theorem 32.3] any maximal left ideal has the form $\mathfrak{A}(\mathbf{1} - e)$ for some minimal idempotent e.

Let $\mathcal{E}$ be the set of minimal idempotents in $\mathfrak{A}$. Then the left ideal generated by $\mathcal{E}$ must be dense, for otherwise it lies in a maximal left ideal, whence there is a minimal idempotent annihilating it on the right, which is absurd. Thus in fact $\mathfrak{A} = \mathfrak{A}e_1 + \cdots + \mathfrak{A}e_k$ for some $e_1, \ldots, e_k \in \mathcal{E}$.

Similarly, $\mathfrak{A} = f_1\mathfrak{A} + \cdots + f_l\mathfrak{A}$ for some $f_1, \ldots, f_l \in \mathcal{E}$. In particular $\mathbf{1} = f_1a_1 + \cdots + f_la_l$ for some $a_1, \ldots, a_l \in \mathfrak{A}$, so that

$$\mathfrak{A} = \mathbf{1}\mathfrak{A} = (f_1a_1 + \cdots + f_ka_k)(\mathfrak{A}e_1 + \cdots + \mathfrak{A}e_k) = \bigoplus_{i,j} f_i\mathfrak{A}e_j,$$

which is finite dimensional by [3, Proposition 31.6(iii)]. □

2. Weak amenability

It is known that $L^1(\mathfrak{G})$ is weakly amenable for every locally compact group $\mathfrak{G}$; see [19], [10]. If $\mathfrak{G}$ is a compact abelian group, then by [6, Theorem 3.15, 3.18]

$$L^1(\mathfrak{G})^{**}/\mathfrak{S} \cong M(\mathfrak{G})$$

where $\mathfrak{S} = \{F \in L^1(\mathfrak{G})^{**} : L^1(\mathfrak{G})^{**}F = 0\}$ is a closed ideal having zero product. If $\mathfrak{G}$ is not discrete, then $M(\mathfrak{G})$ has nonzero point derivations [4], which lift to $L^1(\mathfrak{G})^{**}$, which is thus not weakly amenable. That this fact remains true without the compactness hypothesis follows from the next result.

Theorem 2.1. *Let $\mathfrak{G}$ be a locally compact group, and suppose that $L^1(\mathfrak{G})^{**}$ is weakly amenable. Then $M(\mathfrak{G})$ is weakly amenable.*

Proof. Recall that $LUC(\mathfrak{G})$ is the space of bounded left uniformly continuous functions on $\mathfrak{G}$, under the supremum norm. By Lemma 1.1 of [13], $LUC(\mathfrak{G})^* = M(\mathfrak{G}) \oplus C_0(\mathfrak{G})^{\perp}$, where the latter is a closed ideal in $LUC(\mathfrak{G})^*$. Thus for $f \in M(\mathfrak{G})^*$ define $T_f \in LUC(\mathfrak{G})^{**}$ by

$$\langle T_f, \mu + r \rangle = \langle f, \mu \rangle \qquad (\mu \in M(\mathfrak{G}), r \in C_0(\mathfrak{G})^{\perp}).$$

Now suppose that $M(\mathfrak{G})$ fails to be weakly amenable, and take a noninner derivation $D : M(\mathfrak{G}) \to M(\mathfrak{G})^*$. Define $\Delta : LUC(\mathfrak{G})^* \to LUC(\mathfrak{G})^{**}$ by

$$\Delta(\mu + r) = T_{D(\mu)}.$$

Since $C_0(\mathfrak{G})^{\perp}$ is an ideal in $LUC(\mathfrak{G})^*$, Δ is a derivation, which moreover cannot be inner. For if there was $\Phi \in LUC(\mathfrak{G})^{**}$ with $\Delta(n) = n\Phi - \Phi n$ for all $n \in LUC(\mathfrak{G})^*$, then restricting Φ to $M(\mathfrak{G})$ would give an element of $M(\mathfrak{G})^*$ which implemented D.

Now let E be a right identity of $L^1(\mathfrak{G})^{**}$ with $\|E\| = 1$. A characterization of such E is given in [13, Proposition 2.1]; what we need here is that $EL^1(\mathfrak{G})^{**}$ is isometrically isomorphic to $LUC(\mathfrak{G})^*$ [12], so that we may consider Δ above to be defined on $EL^1(\mathfrak{G})^{**}$. But since E is an idempotent in $L^1(\mathfrak{G})^{**}$, $L^1(\mathfrak{G})^{**} = EL^1(\mathfrak{G})^{**} + (1-E)L^1(\mathfrak{G})^{**}$ where the latter is a closed ideal having trivial product. Thus $n \mapsto \Delta(En) : L^1(\mathfrak{G})^{**} \to L^1(\mathfrak{G})^{***}$ is a noninner derivation. □

Corollary 2.2. *If $\mathfrak{G}$ is a nondiscrete abelian group, then $L^1(\mathfrak{G})^{**}$ is not weakly amenable.*

Proof. We need only observe as before that by [4] $M(\mathfrak{G})$ is not weakly amenable. □

In parallel to Theorem 1.8 we have the following result. For convenience we will write $x \mapsto \hat{x}$ for the canonical embedding of a Banach space into its second dual.

Theorem 2.3. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{**}$ is weakly amenable, and suppose that $\widehat{\mathfrak{A}}$ is a left ideal in $\mathfrak{A}^{**}$. Then $\mathfrak{A}$ is weakly amenable.*

Proof. Let $D : \mathfrak{A} \to \mathfrak{A}^*$ be a derivation, D^{**} its second adjoint. Take $F, G \in \mathfrak{A}^{**}$ and take bounded nets $(a_i), (b_j) \subset \mathfrak{A}$, with $a_i \overset{\text{weak}^*}{\to} F, b_j \overset{\text{weak}^*}{\to} G$. Then $b_j x \overset{\text{weak}^*}{\to} Gx$, and

$$\begin{aligned} D^{**}(FG) &= \text{weak}^* \lim_i \lim_j D^{**}(\widehat{a_i}\widehat{b_j}) \\ &= \text{weak}^* \lim_i \lim_j (D(a_i)b_j + a_i D(b_j))\widehat{\ } \\ &= D^{**}(F)G + \text{weak}^* \lim_i \widehat{a_i} D^{**}(G). \end{aligned}$$

Now let $R : \mathfrak{A}^{**} \to \mathfrak{A}^*$ be the "restriction": $\langle R(\Psi, a) \rangle = \langle \Psi, \hat{a} \rangle$ for $a \in \mathfrak{A}, \Psi \in \mathfrak{A}^{***}$, and Λ the subsequent "extension": $\Lambda(\Psi) = (R(\Psi))\widehat{\ }$. It is $\Lambda \circ D^{**}$ which we wish to consider. From above,

$$\Lambda \circ D^{**}(FG) = \Lambda(D^{**}(F)G) + \Lambda\biggl(\text{weak}^* \lim_i \widehat{a_i} D^{**}(G)\biggr).$$

But for $x \in \mathfrak{A}$,

$$\langle \Lambda(D^{**}(F)G), \hat{x} \rangle = \langle D^{**}(F)G, \hat{x} \rangle = \langle D^{**}(F), G\hat{x} \rangle.$$

Further, since $\widehat{\mathfrak{A}}$ is assumed to be a left ideal in $\mathfrak{A}^{**}$, $G\hat{x} \in \widehat{\mathfrak{A}}$. Thus

$$\langle D^{**}(F), G\hat{x} \rangle = \langle \Lambda(D^{**}(F)), G\hat{x} \rangle = \langle \Lambda \circ D^{**}(F)G, \hat{x} \rangle.$$

It follows that $\Lambda(D^{**}(F)G) = (\Lambda \circ D^{**})(F)G$.

Again, for $x \in \mathfrak{A}$,

$$\begin{aligned}
\langle \Lambda(\text{weak}^* \lim_i \widehat{a_i} D^{**}(G)), \hat{x} \rangle &= \langle \text{weak}^* \lim_i \widehat{a_i} D^{**}(G), \hat{x} \rangle \\
&= \lim_i \langle \hat{a}_i D^{**}(G), \hat{x} \rangle \\
&= \lim_i \langle D^{**}(G), \hat{x}\widehat{a_i} \rangle \\
&= \lim_i \langle \Lambda D^{**}(G), \hat{x}\widehat{a_i} \rangle \\
&= \lim_i \langle \hat{x}\widehat{a_i}, R(D^{**}(G)) \rangle \\
&= \langle \hat{x}F, R(D^{**}(G)) \rangle \\
&= \langle \Lambda(D^{**}(G)), \hat{x}F \rangle.
\end{aligned}$$

Since $F\Lambda(D^{**}(G)) = (F(R(D^{**}(G))))\widehat{\ }$, we have

$$\Lambda\left(\text{weak}^* \lim_i \widehat{a_i} D^{**}(G)\right) = F\Lambda(D^{**}(G)).$$

Thus we have that

$$\Lambda \circ D^{**}(FG) = \Lambda \circ D^{**}(F)G + F(\Lambda \circ D^{**}(G)),$$

showing that $\Lambda \circ D^{**}$ is a derivation of $\mathfrak{A}^{**}$ into $\mathfrak{A}^{***}$. But then by assumption there is $F_0 \in \mathfrak{A}^{***}$ such that $\Lambda \circ D^{**}(\Phi) = \Phi F_0 - F_0 \Phi$ for all $\Phi \in \mathfrak{A}^{**}$. Setting $f_0 = R(F_0)$ we have $D(a) = af_0 - f_0 a$ for $a \in \mathfrak{A}$ so that D is indeed inner. □

We remark that if derivations from $\mathfrak{A}$ into $\mathfrak{A}^*$ are known to be weakly compact, Theorem 2.3 is true without any hypothesis on $\widehat{\mathfrak{A}}$. For we then have

$$\text{weak}^* \lim_i \widehat{a_i} D^{**}(G) = FD^{**}(G),$$

whence

$$D^{**}(FG) = D^{**}(F)G + FD^{**}(G)$$

so that $D^{**} : \mathfrak{A}^{**} \to \mathfrak{A}^{***}$ is a derivation, and the argument finishes as before.

It is shown in [2] that for an infinite compact metric space K, $\mathrm{lip}_\alpha(K)$ is Arens regular and weakly amenable but not amenable for $0 < \alpha < \frac{1}{2}$. Since $\mathrm{lip}_\alpha(K)^{**} = \mathrm{Lip}_\alpha(K)$, the second conjugates are not weakly amenable [23, Proposition 9.2].

Theorem 2.3 clearly prompts the search for an example where $\mathfrak{A}$ is not weakly amenable, and not a left ideal in $\mathfrak{A}^{**}$, yet $\mathfrak{A}^{**}$ is weakly amenable. Perhaps $L^1(\mathfrak{G})$ for a discrete semigroup $\mathfrak{G}$ is a possible area to investigate for this—if $L^1(\mathfrak{G})$ is an ideal in $L^1(\mathfrak{G})^{**}$, [8, Theorem 9] shows that $\mathfrak{G}t \cup t\mathfrak{G}$ is finite for all $t \in \mathfrak{G}$.

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