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$2k$-REGULAR MAPS ON SMOOTH MANIFOLDS

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ABSTRACT. A continuous map $f : X \to \mathbf{R}^N$ is said to be *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent over $\mathbf{R}$. For smooth manifolds M we obtain new lower bounds on the minimum N for which a $2k$-regular map $M \to \mathbf{R}^N$ can exist in terms of the dual Stiefel-Whitney classes of M.

1. INTRODUCTION

The interest in k-regular maps, apart from their intrinsic geometric appeal, arose in the theory of Čebyšev approximation. We refer the reader to [14, pp. 237–242] and [7] for the latter connections. Work on existence and non-existence of k-regular maps by non-algebraic topological methods appears in [1], [2], [10], [12] and [13]. In [4], [5], [6], [7], [8] and [9], cohomological methods using configuration spaces are used to obtain non-existence results for k-regular maps. The present paper uses the basic strategy of [5], [6] and [8] involving cohomology and group actions on configuration spaces, together with new geometric arguments, to obtain a new non-existence result. All our manifolds are assumed to be smooth, positive-dimensional, non-empty, paracompact, and without boundary.

In [1] a variant of the following is proved:

Theorem 1.1 (Boltjanskiĭ-Ryškov-Šaškin)**.** *If a $2k$-regular map of $\mathbf{R}^n$ into $\mathbf{R}^N$ exists, then $N \geq (n+1)k$.*

See [5, Theorem 1.3] for a one sentence elementary proof. In the positive direction I am grateful to the referee for the following theorem:

Theorem 1.2. *Every smooth n-manifold admits a k-regular map into $\mathbf{R}^{(n+1)k+1}$.*

In fact, the referee observes that by a dimension-counting argument using Sard's Theorem, it follows that "most" smooth maps (in an appropriate sense) from an n-manifold into $\mathbf{R}^N$ are k-regular if $N > (n+1)k$; the argument is much like Whitney's proof that every smooth n-manifold admits a smooth injective map into $\mathbf{R}^{2n+1}$.

Our new result in the negative direction is the following extension of Theorem 1.1:

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Theorem 1.3. *Let M be an n-dimensional manifold and let q be the largest integer such that the q^{th} dual Stiefel-Whitney class of M is non-zero. If a $2k$-regular map of M into $\mathbf{R}^N$ exists, then $N \geq k(n+1+q)+\varepsilon$ where ε is 1 if M is compact and 0 if M is non-compact.*

For particular manifolds M, Theorem 1.3 generally does not give the best possible result. For example, better results for k-regular maps on $\mathbf{R}^2$ are obtained in [5]. In the latter, equivariance arguments with respect to the full symmetric group are used, while in the present paper, in the interests of dealing with a general manifold M and being able to make the relevant cohomology calculations, equivariance arguments with respect to a rather small subgroup of the symmetric group are used. (See §§2 and 3.) Hopefully, further efforts will result in successful calculations using larger subgroups of the symmetric group, yielding improvements of Theorem 1.3. Examples are given in §4.

2. Symmetric group actions and related vector bundles

In this section we recall from [5] and [8] some basic machinery for proving non-existence of k-regular maps, as well as classical results of Wu [15] on Stiefel-Whitney classes which will be used in §3 to prove Theorem 1.3.

Let G be a subgroup of the k^{th} symmetric group Σ_k and X be a Hausdorff space on which G acts freely. G also acts $\mathbf{R}$-linearly on $\mathbf{R}^k$ via permutation of factors, and so we obtain a real k-plane bundle $X \times_G \mathbf{R}^k \to X/G$ which we denote by $\xi(X,G)$. If G also acts freely on another Hausdorff space Y and $f: X \to Y$ is a continuous G-equivariant map, then f induces a map of k-plane bundles $\xi(X,G) \to \xi(Y,G)$.

Let X be a Hausdorff space and let $F(X,k)$ denote the k^{th} configuration space of X, i.e. the subspace of X^k consisting of all ordered k-tuples of *distinct* points of X. Σ_k acts freely on $F(X,k)$ by permuting factors, and so we obtain, for each subgroup G of Σ_k, a real k-plane bundle $\xi\big(F(X,k),G\big)$. We will use the following result ([8, Proposition 2.1]) whose proof is easy:

Proposition 2.1. *If a k-regular map $X \to \mathbf{R}^N$ exists, then for each subgroup G of Σ_k, the k-plane bundle $\xi\big(F(X,k),G\big)$ admits a real $(N-k)$-plane bundle inverse.*

Suppose Σ_2 acts freely on X. Write $[x,s,t] \in X \times_{\Sigma_2} \mathbf{R}^2$ for the point determined by $(x,s,t) \in X \times \mathbf{R}^2$. The real 2-plane bundle $\xi(X,\Sigma_2)$ splits as the Whitney sum of a trivial line bundle (whose total space consists of all the $[x,t,t]$) and a line bundle λ_X (whose total space consists of all the $[x,t,-t]$). λ_X is isomorphic to the line bundle $X \times_{\Sigma_2} \mathbf{R} \to X/\Sigma_2$ where Σ_2 acts on $\mathbf{R}$ via multiplication by -1. Thus the total Stiefel-Whitney class of $\xi(X,\Sigma_2)$ is given by

$$w\big(\xi(X,\Sigma_2)\big) = 1 + w_1\big(\lambda_X\big). \tag{2.2}$$

Let M be a manifold. Write TM for the total space of the tangent bundle of M, and T^0M for the complement of the 0-section in TM. Σ_2 acts on TM via multiplication by -1 on fibres. This action restricted to T^0M is free and so we obtain real line bundles λ_{T^0M} and $\lambda_{F(M,2)}$. Let Σ_2 act on $M \times M$ by switching factors. The following is well known (e.g. [11, Theorem 11.1 and Lemma 11.5]):

Theorem 2.3. *There is a Σ_2-equivariant embedding $i_M : TM \to M \times M$ which carries the 0-section of TM onto the diagonal in $M \times M$. Consequently there is a map of line bundles $\lambda_{T^0M} \to \lambda_{F(M,2)}$.*

We recall the following result from [15, §5]:

Theorem 2.4 (Wu)**.** *Let M be a connected n-dimensional manifold and q the largest integer such that the q^{th} dual Stiefel-Whitney class of M is non-zero. Let $u = w_1(\lambda_{T^0M})$ and $v = w_1(\lambda_{F(M,2)})$. Then in the cohomology rings of T^0M/Σ_2 and $F(M,2)/\Sigma_2$, respectively, with $\mathbf{Z}/2\mathbf{Z}$ coefficients:*
(a) $u^{n+q-1} \neq 0$, $u^{n+q} = 0$.
(b) *If M is compact, then* $v^{n+q} \neq 0$, $v^{n+q+1} = 0$.

3. Proof of Theorem 1.3

Using the notation and hypotheses of Theorem 1.3, suppose a $2k$-regular map $M \to \mathbf{R}^N$ exists. Let

$$G = \underbrace{\Sigma_2 \times \cdots \times \Sigma_2}_{k} \subset \Sigma_{2k}$$

where the generator of the i^{th} factor is the transposition which interchanges $2i-1$ and $2i$, $1 \leq i \leq k$. By Proposition 2.1, the proof of Theorem 1.3 will be complete if we show that the top non-zero dual Stiefel-Whitney class of $\xi(F(M,2k),G)$ occurs in dimension at least $k(n+q-1)+\varepsilon$.

Choose a Riemannian metric on M and let $\| \ \|$ denote the resulting norm on the fibres of TM. Identify TM with a subspace of $M \times M$ via the embedding i_M of Theorem 2.3. For $j > 0$ let

$$T_jM = \{v \in TM \mid j < \|v\| < j+1\}$$

and

$$F_j(M,2) = F(M,2) - \{v \in T^0M \mid \|v\| \geq j\}.$$

The inclusions $T_jM \subset T^0M$ and $F_j(M,2) \subset F(M,2)$ are Σ_2-equivariant homotopy equivalences. Thus, by Theorems 2.3 and 2.4,

$$\begin{aligned} w_1\big(\lambda_{T_jM}\big)^{n+q-1} &\neq 0, \\ w_1\big(\lambda_{F_j(M,2)}\big)^{n+q-1+\epsilon} &\neq 0. \end{aligned} \tag{3.1}$$

Because of the pairwise disjointness of $T_1M, \ldots, T_{k-1}M, F_k(M,2)$, the product of the inclusions of these spaces into $F(M,2)$ factors through $F(M,2k)$, yielding a continuous map

$$f : T_1M \times \cdots \times T_{k-1}M \times F_k(M,2) \to F(M,2k).$$

The product of the Σ_2-actions on the T_iM and $F_k(M,2)$ yields a free G-action on $T_1M \times \cdots \times T_{k-1}M \times F_k(M,2)$ and f is G-equivariant. Thus there is a map of $2k$-plane bundles

$$\xi\big(T_1M \times \cdots \times T_{k-1}M \times F_k(M,2), G\big) \to \xi\big(F(M,2k), G\big)$$

and so it suffices to show that the top non-zero dual Stiefel class of the bundle on the left occurs in dimension at least $k(n+q-1)+\varepsilon$. Note that

$$\begin{aligned} &\xi\big(T_1M \times \cdots \times T_{k-1}M \times F_k(M,2), G\big) \\ &\qquad \cong \xi(T_1M, \Sigma_2) \times \cdots \times \xi(T_{k-1}M, \Sigma_2) \times \xi\big(F_k(M,2), \Sigma_2\big) \end{aligned}$$

and so

$$\overline{w}\Big(\xi\big(T_1M \times \cdots \times T_{k-1}M \times F_k(M,2), G\big)\Big) = \overline{w}\big(\xi(T_1M, \Sigma_2)\big) \times \cdots \times \overline{w}\big(\xi(T_{k-1}M, \Sigma_2)\big) \times \overline{w}\big(\xi(F_k(M,2), \Sigma_2)\big). \tag{3.2}$$

By (2.2), for any Hausdorff space X on which Σ_2 acts freely,

$$\overline{w}\big(\xi(X, \Sigma_2)\big) = \big(1 + w_1(\lambda_X)\big)^{-1} = \sum_{j \geq 0} w_1(\lambda_X)^j.$$

Thus by (3.1) the top non-zero dual Stiefel-Whitney class of each of the $\xi(T_iM, \Sigma_2)$ occurs in dimension at least (in fact exactly) $n+q-1$, and that of $\xi\big(F_k(M,2), \Sigma_2\big)$ in dimension at least $n+q-1+\varepsilon$. By (3.2) it now follows that the top non-zero dual Stiefel-Whitney class of $\xi\big(T_1M \times \cdots \times T_{k-1}M \times F_k(M,2), G\big)$ occurs in dimension at least $k(n+q-1)+\varepsilon$, completing the proof.

4. Examples

Example 4.1. Regard S^1 as the space of complex numbers of absolute value 1. It is well known (see, e.g. [3, p. 78, Exercise 20]) that the map

$$f : S^1 \to \mathbf{R} \times \mathbf{C}^k = \mathbf{R}^{2k+1}$$

given by $f(z) = (1, z, z^2, \ldots, z^k)$ is $(2k+1)$-regular, and hence $2k$-regular. Applying Theorem 1.3 to S^1, there does not exist a $2k$-regular map of S^1 into $\mathbf{R}^{2k}$, and so Theorem 1.3 is best-possible in the case of S^1.

Example 4.2. If M is a compact n-dimensional manifold, it follows from [9, Theorem 2.1] that a 2-regular map $M \to \mathbf{R}^N$ exists if and only if M is topologically embeddable in $\mathbf{R}^{N-1}$. Thus Theorem 1.3 implies the well-known result that M is not topologically embeddable in $\mathbf{R}^{n+q}$ if $\overline{w}_q(M) \neq 0$ (see [15]).

Example 4.3. Let M be a 2-manifold. By Theorem 1.2 there exists a $2k$-regular map of M into $\mathbf{R}^{6k+1}$. By Theorem 1.3 there does not exist a $2k$-regular map of M into $\mathbf{R}^{(3+q)k+\varepsilon-1}$ where q is 1 if M is non-orientable and 0 if M is orientable, and ε is as in the statement of Theorem 1.3. Thus a considerable gap still exists between the best-known positive and negative results for $2k$-regular maps on surfaces.

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