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BASIC DIFFERENTIAL FORMS FOR ACTIONS OF LIE GROUPS

PETER W. MICHOR

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Abstract. A section of a Riemannian G-manifold M is a closed submanifold Σ which meets each orbit orthogonally. It is shown that the algebra of G-invariant differential forms on M which are horizontal in the sense that they kill every vector which is tangent to some orbit, is isomorphic to the algebra of those differential forms on Σ which are invariant with respect to the generalized Weyl group of Σ, under some condition.

1. Introduction

A section of a Riemannian G-manifold M is a closed submanifold Σ which meets each orbit orthogonally. This notion was introduced by Szenthe [26], [27], and in a slightly different form by Palais and Terng in [19], [20]. The case of linear representations was considered by Bott and Samelson [4] and Conlon [9], and then by Dadok [10] who called representations admitting sections polar representations and completely classified all polar representations of connected compact Lie groups. Conlon [8] considered Riemannian manifolds admitting flat sections. We follow here the notion of Palais and Terng.

If M is a Riemannian G-manifold which admits a section Σ, then the trace on Σ of the G-action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$. Palais and Terng [19] showed that then the algebras of invariant smooth functions coincide, $C^\infty(M,\mathbb{R})^G \cong C^\infty(\Sigma,\mathbb{R})^{W(\Sigma)}$.

In this paper we will extend this result to the algebras of differential forms. Our aim is to show that pullback along the embedding $\Sigma \to M$ induces an isomorphism $\Omega^p_{\text{hor}}(M)^G \cong \Omega^p(\Sigma)^{W(\Sigma)}$ for each p, where a differential form ω on M is called *horizontal* if it kills each vector tangent to some orbit. For each point x in M, the slice representation of the isotropy group G_x on the normal space $T_x(G.x)^\perp$ to the tangent space to the orbit through x is a polar representation. The first step is to show that the result holds for polar representations. This is done in Theorem 3.7 for polar representations whose generalized Weyl group is really a Coxeter group, i.e., is generated by reflections. Every polar representation of a connected compact Lie group has this property. The method used there is inspired by Solomon [25]. Then the general result is proved under the assumption that each slice representation

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has a Coxeter group as a generalized Weyl group. The last section gives some perspective to the result.

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2. Basic differential forms

2.1. Basic differential forms. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and multiplication $\mu : G \times G \to G$, and for $g \in G$ let $\mu_g, \mu^g : G \to G$ denote the left and right translations.

Let $\ell : G \times M \to M$ be a left action of the Lie group G on a smooth manifold M. We consider the partial mappings $\ell_g : M \to M$ for $g \in G$ and $\ell^x : G \to M$ for $x \in M$ and the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ given by $\zeta_X(x) = T_e(\ell^x)X$. Since ℓ is a left action, the negative $-\zeta$ is a Lie algebra homomorphism.

A differential form $\varphi \in \Omega^p(M)$ is called *G-invariant* if $(\ell_g)^*\varphi = \varphi$ for all $g \in G$ and *horizontal* if φ kills each vector tangent to a G-orbit: $i_{\zeta_X}\varphi = 0$ for all $X \in \mathfrak{g}$. We denote by $\Omega^p_{\text{hor}}(M)^G$ the space of all horizontal G-invariant p-forms on M. They are also called *basic forms*.

2.2. Lemma. *Under the exterior differential $\Omega_{\text{hor}}(M)^G$ is a subcomplex of $\Omega(M)$.*

Proof. If $\varphi \in \Omega_{\text{hor}}(M)^G$, then the exterior derivative $d\varphi$ is clearly G-invariant. For $X \in \mathfrak{g}$ we have

$$i_{\zeta_X} d\varphi = i_{\zeta_X} d\varphi + d i_{\zeta_X}\varphi = \mathcal{L}_{\zeta_X}\varphi = 0,$$

so $d\varphi$ is also horizontal. □

2.3. Sections. Let M be a connected complete Riemannian manifold, and let G be a Lie group which acts isometrically on M from the left. A connected closed smooth submanifold Σ of M is called a *section* for the G-action, if it meets all G-orbits orthogonally.

Equivalently we require that $G.\Sigma = M$ and that for each $x \in \Sigma$ and $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X(x)$ is orthogonal to $T_x\Sigma$.

We only remark here that each section is a totally geodesic submanifold and is given by $\exp(T_x(x.G)^\perp)$ if x lies in a principal orbit.

If we put $N_G(\Sigma) := \{g \in G : g.\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : g.s = s \text{ for all } s \in \Sigma\}$, then the quotient $W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$ turns out to be a discrete group acting properly on Σ. It is called the generalized Weyl group of the section Σ.

See [19] or [20] for more information on sections and their generalized Weyl groups.

2.4. Main Theorem. *Let $M \times G \to M$ be a proper isometric right action of a Lie group G on a smooth Riemannian manifold M, which admits a section Σ. Let us assume that*

(1) *For each $x \in \Sigma$ the slice representation $G_x \to O(T_x(G.x)^\perp)$ has a generalized Weyl group which is a reflection group (see section 3).*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G-invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ.

The proof of this theorem will take up the rest of this paper. According to Dadok [10], remark after Proposition 6, for any polar representation of a connected compact Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group, so condition (1) holds if we assume that:

(2) Each isotropy group G_x is connected.

Proof of injectivity. Let $i : \Sigma \to M$ be the embedding of the section. We claim that $i^* : \Omega^p_{\text{hor}}(M)^G \to \Omega^p(\Sigma)^{W(\Sigma)}$ is injective. Let $\omega \in \Omega^p_{\text{hor}}(M)^G$ with $i^*\omega = 0$. For $x \in \Sigma$ we have $i_X\omega_x = 0$ for $X \in T_x\Sigma$ since $i^*\omega = 0$, and also for $X \in T_x(G.x)$ since ω is horizontal. Let $x \in \Sigma \cap M_{\text{reg}}$ be a regular point; then $T_x\Sigma = (T_x(G.x))^\perp$ and so $\omega_x = 0$. This holds along the whole orbit through x since ω is G-invariant. Thus $\omega|M_{\text{reg}} = 0$, and since M_{reg} is dense in M, $\omega = 0$.

So it remains to show that i^* is surjective. This will be done in 4.2 below. □

3. Representations

3.1. Invariant functions. Let G be a reductive Lie group and let $\rho : G \to GL(V)$ be a representation in a finite dimensional real vector space V.

According to a classical theorem of Hilbert (as extended by Nagata [15], [16]), the algebra of G-invariant polynomials $\mathbb{R}[V]^G$ on V is finitely generated (in fact finitely presented), so there are G-invariant homogeneous polynomials $f_1, \dots, f_m$ on V such that each invariant polynomial $h \in \mathbb{R}[V]^G$ is of the form $h = q(f_1, \dots, f_m)$ for a polynomial $q \in \mathbb{R}[\mathbb{R}^m]$. Let $f = (f_1, \dots, f_m) : V \to \mathbb{R}^m$; then this means that the pullback homomorphism $f^* : \mathbb{R}[\mathbb{R}^m] \to \mathbb{R}[V]^G$ is surjective.

D. Luna proved in [14] that the pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \to C^\infty(V, \mathbb{R})^G$ is also surjective onto the space of all smooth functions on V which are constant on the fibers of f. Note that the polynomial mapping f in this case may not separate the G-orbits.

G. Schwarz proved already in [23] that if G is a compact Lie group, then the pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \to C^\infty(V, \mathbb{R})^G$ is actually surjective onto the space of G-invariant smooth functions. This result implies in particular that f separates the G-orbits.

3.2. Lemma. *Let $\ell \in V^*$ be a linear functional on a finite dimensional vector space V, and let $f \in C^\infty(V, \mathbb{R})$ be a smooth function which vanishes on the kernel of ℓ, so that $f|\ell^{-1}(0) = 0$. Then there is a unique smooth function g such that $f = \ell.g$*

Proof. Choose coordinates $x^1, \dots, x^n$ on V with $\ell = x^1$. Then $f(0, x^2, \dots, x^n) = 0$ and we have $f(x^1, \dots, x^n) = \int_0^1 \partial_1 f(tx^1, x^2, \dots, x^n)dt.x^1 = g(x^1, \dots, x^n).x^1$. □

3.3. Lemma. *Let W be a finite reflection group acting on a finite dimensional vector space Σ. Let $f = (f_1, \dots, f_n) : \Sigma \to \mathbb{R}^n$ be the polynomial map whose components $f_1, \dots, f_n$ are a minimal set of homogeneous generators of the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on Σ. Then the pullback homomorphism $f^* : \Omega^p(\mathbb{R}^n) \to \Omega^p(\Sigma)$ is surjective onto the space $\Omega^p(\Sigma)^W$ of W-invariant differential forms on Σ.*

For polynomial differential forms and more general reflection groups this is the main theorem of Solomon [25]. We adapt his proof to our needs.

Proof. The polynomial generators f_i form a set of algebraically independent polynomials, $n = \dim \Sigma$, and their degrees $d_1, \dots, d_n$ are uniquely determined up to order. We even have (see [12]):

$$d_1 \dots d_n = |W|, \quad \text{the order of } W. \tag{1}$$

$$d_1 + \dots + d_n = n + N, \quad \text{where } N \text{ is the number of reflections in } W. \tag{2}$$

Let us consider the mapping $f = (f_1, \dots, f_n) : \Sigma \to \mathbb{R}^n$ and its Jacobian $J(x) = \det(df(x))$. Let $x^1, \dots, x^n$ be coordinate functions in Σ. Then for each $\sigma \in W$ we have

$$\begin{aligned} J.dx^1 \wedge \dots \wedge dx^n &= df_1 \wedge \dots \wedge df_n = \sigma^*(df_1 \wedge \dots \wedge df_n) \\ &= (J \circ \sigma)\sigma^*(dx^1 \wedge \dots \wedge dx^n) = (J \circ \sigma)\det(\sigma)(dx^1 \wedge \dots \wedge dx^n), \end{aligned}$$

$$J \circ \sigma = \det(\sigma^{-1})J. \tag{3}$$

The generators $f_1, \dots, f_n$ are algebraically independent over $\mathbb{R}$, thus $J \neq 0$. Since J is a polynomial of degree $(d_1 - 1) + \dots + (d_n - 1) = N$ (see (2)), the W-invariant set $U = \Sigma \setminus J^{-1}(0)$ is open and dense in Σ; by the inverse function theorem f is a local diffeomorphism on U, thus the 1-forms $df_1, \dots, df_n$ are a coframe on U.

Now let $(\sigma_\alpha)_{\alpha=1,\dots,N}$ be the set of reflections in W, with reflection hyperplanes H_α. Let $\ell_\alpha \in \Sigma^*$ be linear functionals with $H_\alpha = \ell^{-1}(0)$. If $x \in H_\alpha$ we have $J(x) = \det(\sigma_\alpha)J(\sigma_\alpha.x) = -J(x)$, so that $J|H_\alpha = 0$ for each α, and by Lemma 3.2 we have

$$J = c.\ell_1 \dots \ell_N. \tag{4}$$

Since J is a polynomial of degree N, c must be a constant. Repeating the last argument for an arbitrary function g and using (4), we get:

(5) If $g \in C^\infty(\Sigma, \mathbb{R})$ satisfies $g \circ \sigma = \det(\sigma^{-1})g$ for each $\sigma \in W$, we have $g = J.h$ for $h \in C^\infty(\Sigma, \mathbb{R})^W$.

After these preparations we turn to the assertion of the lemma. Let $\omega \in \Omega^p(\Sigma)^W$. Since the 1-forms df_j form a coframe on U, we have

$$\omega|U = \sum_{j_1 < \dots < j_p} g_{j_1 \dots j_p} df_{j_1}|U \wedge \dots \wedge df_{j_p}|U$$

for $g_{j_1 \dots j_p} \in C^\infty(U, \mathbb{R})$. Since ω and all df_i are W-invariant, we may replace $g_{j_1 \dots j_p}$ by their averages over W, or assume without loss that $g_{j_1 \dots j_p} \in C^\infty(U, \mathbb{R})^W$.

Let us choose now a form index $i_1 < \dots < i_p$ with $\{i_{p+1} < \dots < i_n\} = \{1, \dots, n\} \setminus \{i_1 < \dots < i_p\}$. Then for some sign $\varepsilon = \pm 1$ we have

$$\omega|U \wedge df_{i_{p+1}} \wedge \dots \wedge df_{i_n} = \varepsilon.g_{i_1 \dots i_p}.df_1 \wedge \dots \wedge df_n = \varepsilon.g_{i_1 \dots i_p}.J.dx^1 \wedge \dots \wedge dx^n,$$

$$\omega \wedge df_{i_{p+1}} \wedge \dots \wedge df_{i_n} = \varepsilon.k_{i_1 \dots i_p} dx^1 \wedge \dots \wedge dx^n \tag{6}$$

for a function $k_{i_1 \dots i_p} \in C^\infty(\Sigma, \mathbb{R})$. Thus

$$k_{i_1 \dots i_p}|U = g_{i_1 \dots i_p}.J|U. \tag{7}$$

Since ω and each df_i is W-invariant, from (6) we get $k_{i_1 \dots i_p} \circ \sigma = \det(\sigma^{-1})k_{i_1 \dots i_p}$ for each $\sigma \in W$. But then by (5) we have $k_{i_1 \dots i_p} = \omega_{i_1 \dots i_p}.J$ for unique $\omega_{i_1 \dots i_p} \in C^\infty(\Sigma, \mathbb{R})^W$, and (7) then implies $\omega_{i_1 \dots i_p}|U = g_{i_1 \dots i_p}$, so that the lemma follows since U is dense. □

3.4. Question. *Let $\rho : G \to GL(V)$ be a representation of a compact Lie group in a finite dimensional vector space V. Let $f = (f_1, \dots, f_m) : V \to \mathbb{R}^m$ be the polynomial mapping whose components f_i are a minimal set of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials.*

We consider the pullback homomorphism $f^* : \Omega^p(\mathbb{R}^m) \to \Omega^p(V)$. Is it surjective onto the space $\Omega^p_{\text{hor}}(V)^G$ of G-invariant horizontal smooth p-forms on V?

The proof of Theorem 3.7 below will show that the answer is yes for polar representations of compact Lie groups if the corresponding generalized Weyl group is a reflection group.

In general the answer is no. A counterexample is the following: Let the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of order n, viewed as the group of n-th roots of unity, act on $\mathbb{C} = \mathbb{R}^2$ by complex multiplication. A generating system of polynomials consists of $f_1 = |z|^2$, $f_2 = \text{Re}(z^n)$, $f_3 = \text{Im}(z^n)$. But then each df_i vanishes at 0 and there is no chance to have the horizontal invariant volume form $dx \wedge dy$ in $f^*\Omega(\mathbb{R}^3)$.

3.5. Polar representations. Let G be a compact Lie group and let $\rho : G \to GL(V)$ be an orthogonal representation in a finite dimensional real vector space V which admits a section Σ. Then the section turns out to be a linear subspace and the representation is called a *polar representation*, following Dadok [10], who gave a complete classification of all polar representations of connected Lie groups. They were called variationally complete representations by Conlon [9] before.

3.6. Theorem (Terng [28], Theorem D or [19], 4.12). *Let $\rho : G \to GL(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Then the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on the section Σ, via the restriction mapping $f \mapsto f|\Sigma$.*

3.7. Theorem. *Let $\rho : G \to GL(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Let us suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Then the pullback to Σ of differential forms induces an isomorphism*

$$\Omega^p_{\text{hor}}(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}.$$

According to Dadok [10], remark after Proposition 6, for any polar representation of a connected compact Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group. This theorem is true for polynomial differential forms, and also for real analytic differential forms, by essentially the same proof.

Proof. Let $i : \Sigma \to V$ be the embedding. By the first part of the proof of Theorem 2.4 the pullback mapping $i^* : \Omega^p_{\text{hor}}(V)^G \to \Omega^p_{\text{hor}}(\Sigma)^W$ is injective, and we shall show that it is also surjective. Let $f_1, \dots, f_n$ be a minimal set of homogeneous generators of the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on Σ. Then by Lemma 3.3 each $\omega \in \Omega^p(\Sigma)^W$ is of the form

$$\omega = \sum_{j_1 < \dots < j_p} \omega_{j_1 \dots j_p} df_{j_1} \wedge \dots \wedge df_{j_p},$$

where $\omega_{j_1 \dots j_p} \in C^\infty(\Sigma, \mathbb{R})^W$. By Theorem 3.6 the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on the section Σ, via the restriction mapping i^*. Choose polynomials $\tilde{f}_1, \dots \tilde{f}_n \in \mathbb{R}[V]^G$ with $\tilde{f}_i \circ i = f_i$ for all i. Put $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) : V \to \mathbb{R}^n$. Then we use the theorem of G. Schwarz (see 3.1) to find $h_{i_1,\dots,i_p} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $h_{i_1,\dots,i_p} \circ f = \omega_{i_1,\dots,i_p}$ and consider

$$\tilde{\omega} = \sum_{j_1 < \dots < j_p} (h_{j_1 \dots j_p} \circ \tilde{f}) d\tilde{f}_{j_1} \wedge \dots \wedge d\tilde{f}_{j_p},$$

which is in $\Omega^p_{\text{hor}}(V)^G$ and satisfies $i^*\tilde{\omega} = \omega$. □

Sketch of another proof avoiding 3.3 (suggested by a referee). Let $R = C^\infty(V)^G = C^\infty(\Sigma)^W$ and let Ω^p_R be its module of Kähler p-forms (see Kunz [13] for the notion of Kähler forms). Also let $S = \mathbb{R}[V]^G = \mathbb{R}[\Sigma]^W$ (using 3.6). Then the canonical mapping $\Omega^p_R \to \Omega^p(\Sigma)^W$ is surjective. This follows for the canonical mapping from Ω^p_S into the space of forms with polynomial coefficients from the result of Solomon [25] by using 3.6 again as in the proof of 3.7; and it can be extended to smooth coefficients by Theorem 1.4 of Ronga [22], which says that equivariant stability and infinitesimal equivariant stability are equivalent, in a way which is similar to the argument of Proposition 6.8 of Schwarz [24]. So we see that the composition $\Omega^p_R \to \Omega^p(V)^G \to \Omega^p(\Sigma)^W$ is surjective, thus also the right hand side mapping has to be surjective. □

3.8. Corollary. *Let $\rho : G \to O(V, \langle \quad , \quad \rangle)$ be an orthogonal polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Let us suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Let $B \subset V$ be an open ball centered at 0.*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(B)^G \xrightarrow{\cong} \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$

Proof. Check the proof of 3.7 or use the following argument. Suppose that $B = \{v \in V : |v| < 1\}$ and consider a smooth diffeomorphism $f : [0, 1) \to [0, \infty)$ with $f(t) = t$ near 0. Then $g(v) := \frac{f(|v|)}{|v|} v$ is a G-equivariant diffeomorphism $B \to V$ and by 3.7 we get:

$$\Omega^p_{\text{hor}}(B)^G \xrightarrow{(g^{-1})^*} \Omega^p_{\text{hor}}(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)} \xrightarrow{g^*} \Omega^p(\Sigma \cap B)^{W(\Sigma)}. \qquad \square$$

4. Proof of the main theorem

Let us assume that we are in the situation of the main theorem 2.4, for the rest of this section.

4.1. For $x \in M$ let S_x be a (normal) slice and G_x the isotropy group, which acts on the slice. Then $G.S_x$ is open in M and G-equivariantly diffeomorphic to the associated bundle $G \to G/G_x$ via

$$\begin{array}{ccccc} G \times S_x & \xrightarrow{q} & G \times_{G_x} S_x & \xrightarrow{\cong} & G.S_x \\ & & \big\downarrow & & \big\downarrow r \\ & & G/G_x & \xrightarrow{\cong} & G.x \end{array}$$

where r is the projection of a tubular neighborhood. Since $q : G \times S_x \to G \times_{G_x} S_x$ is a principal G_x-bundle with principal right action $(g, s).h = (gh, h^{-1}.s)$, we have an isomorphism

$$q^* : \Omega(G \times_{G_x} S_x) \to \Omega_{G_x-\mathrm{hor}}(G \times S_x)^{G_x}.$$

Since q is also G-equivariant for the left G-actions, the isomorphism q^* maps the subalgebra $\Omega^p_{\mathrm{hor}}(G.S_x)^G \cong \Omega^p_{\mathrm{hor}}(G \times_{G_x} S_x)^G$ of $\Omega(G \times_{G_x} S_x)$ to the subalgebra $\Omega^p_{G_x-\mathrm{hor}}(S_x)^{G_x}$ of $\Omega_{G_x-\mathrm{hor}}(G \times S_x)^{G_x}$. So we have proved:

Lemma. *In this situation there is a canonical isomorphism*

$$\Omega^p_{\mathrm{hor}}(G.S_x)^G \xrightarrow{\cong} \Omega^p_{G_x-\mathrm{hor}}(S_x)^{G_x}$$

which is given by pullback along the embedding $S_x \to G.S_x$.

4.2. Rest of the proof of Theorem 2.4. Now let us consider $\omega \in \Omega^p(\Sigma)^{W(\Sigma)}$. We want to construct a form $\tilde{\omega} \in \Omega^p_{\mathrm{hor}}(M)^G$ with $i^*\tilde{\omega} = \omega$. This will finish the proof of Theorem 2.4.

Choose $x \in \Sigma$ and an open ball B_x with center 0 in T_xM such that the Riemannian exponential mapping $\exp_x : T_xM \to M$ is a diffeomorphism on B_x. We consider now the compact isotropy group G_x and the slice representation $\rho_x : G_x \to O(V_x)$, where $V_x = \mathrm{Nor}_x(G.x) = (T_x(G.x))^\perp \subset T_xM$ is the normal space to the orbit. This is a polar representation with section $T_x\Sigma$, and its generalized Weyl group is given by $W(T_x\Sigma) \cong N_G(\Sigma) \cap G_x/Z_G(\Sigma) = W(\Sigma)_x$ (see [19]); it is a Coxeter group by assumption (1) in 2.4. Then $\exp_x : B_x \cap V_x \to S_x$ is a diffeomorphism onto a slice and $\exp_x : B_x \cap T_x\Sigma \to \Sigma_x \subset \Sigma$ is a diffeomorphism onto an open neighborhood Σ_x of x in the section Σ.

Let us now consider the pullback $(\exp|B_x \cap T_x\Sigma)^*\omega \in \Omega^p(B_x \cap T_x\Sigma)^{W(T_x\Sigma)}$. By Corollary 3.8 there exists a unique form $\varphi^x \in \Omega^p_{G_x-\mathrm{hor}}(B_x \cap V_x)^{G_x}$ such that $i^*\varphi^x = (\exp|B_x \cap T_x\Sigma)^*\omega$, where i_x is the embedding. Then we have

$$((\exp|B_x \cap V_x)^{-1}) * \varphi^x \in \Omega^p_{G_x-\mathrm{hor}}(S_x)^{G_x}$$

and by Lemma 4.1 this form corresponds uniquely to a differential form $\omega^x \in \Omega^p_{\mathrm{hor}}(G.S_x)^G$ which satisfies $(i|\Sigma_x)^*\omega^x = \omega|\Sigma_x$, since the exponential mapping commutes with the respective restriction mappings. Now the intersection $G.S_x \cap \Sigma$ is the disjoint union of all the open sets $w_j(\Sigma_x)$ where we pick one w_j in each left coset of the subgroup $W(\Sigma)_x$ in $W(\Sigma)$. If we choose $g_j \in N_G(\Sigma)$ projecting on w_j for all j, then

$$\begin{aligned}(i|w_j(\Sigma_x))^*\omega^x &= (\ell_{g_j} \circ i|\Sigma_x \circ w_j^{-1})^*\omega^x = (w_j^{-1})^*(i|\Sigma_x)^*\ell^*_{g_j}\omega^x \\ &= (w_j^{-1})^*(i|\Sigma_x)^*\omega^x = (w_j^{-1})^*(\omega|\Sigma_x) = \omega|w_j(\Sigma_x),\end{aligned}$$

so that $(i|G.S_x \cap \Sigma)^*\omega^x = \omega|G.S_x \cap \Sigma$. We can do this for each point $x \in \Sigma$.

Using the method of Palais ([18], proof of 4.3.1) we may find a sequence of points $(x_n)_{n\in\mathbb{N}}$ in Σ such that the $\pi(\Sigma_{x_n})$ form a locally finite open cover of the orbit space $M/G \cong \Sigma/W(\Sigma)$, and a smooth partition of unity f_n consisting of G-invariant functions with $\mathrm{supp}(f_n) \subset G.S_{x_n}$. Then $\tilde{\omega} := \sum_n f_n\omega^{x_n} \in \Omega^p_{\mathrm{hor}}(M)^G$ has the required property $i^*\tilde{\omega} = \omega$. □

5. Basic versus equivariant cohomology

5.1. Basic cohomology. For a Lie group G and a smooth G-manifold M, by 2.2 we may consider the basic cohomology $H^p_{G-\text{basic}}(M) = H^p(\Omega^*_{\text{hor}}(M)^G, d)$.

The best known application of basic cohomology is the case of a compact connected Lie group G acting on itself by left translations; see e.g. [11] and papers cited therein: By homotopy invariance and integration we get $H(G) = H_{G-\text{basic}}(G) = H(\Lambda(\mathfrak{g}^*))$, and the latter space turns out as the space $\Lambda(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\text{ad}(\mathfrak{g})$-invariant forms, using the inversion. This is the theorem of Chevalley and Eilenberg. Moreover, $\Lambda(\mathfrak{g}^*)^{\mathfrak{g}} = \Lambda(P)$, where P is the graded subspace of primitive elements, using the Weil map and transgression, whose determination in all concrete cases by Borel and Hirzebruch is a beautiful part of modern mathematics.

In more general cases the determination of basic cohomology was more difficult. A replacement for it is equivariant cohomology, which comes in two guises:

5.2. Equivariant cohomology, Borel model. For a topological group and a topological G-space the equivariant cohomology was defined as follows; see [3]: Let $EG \to BG$ be the classifying G-bundle, and consider the associated bundle $EG \times_G M$ with standard fiber the G-space M. Then the equivariant cohomology is given by $H^p(EG \times_G M; \mathbb{R})$.

5.3. Equivariant cohomology, Cartan model. For a Lie group G and a smooth G-manifold M we consider the space

$$(S^k\mathfrak{g}^* \otimes \Omega^p(M))^G$$

of all homogeneous polynomial mappings $\alpha : \mathfrak{g} \to \Omega^p(M)$ of degree k from the Lie algebra $\mathfrak{g}$ of G to the space of p-forms, which are G-equivariant: $\alpha(\text{Ad}(g^{-1})X) = \ell_g^*\alpha(X)$ for all $g \in G$. The mapping

$$\begin{aligned}
d_{\mathfrak{g}} : A^q_G(M) &\to A^{q+1}_G(M),\\
A^q_G(M) &:= \bigoplus_{2k+p=q} (S^k\mathfrak{g}^* \otimes \Omega^p(M))^G,\\
(d_{\mathfrak{g}}\alpha)(X) &:= d(\alpha(X)) - i_{\zeta_X}\alpha(X)
\end{aligned}$$

satisfies $d_{\mathfrak{g}} \circ d_{\mathfrak{g}} = 0$ and the following result holds.

Theorem. *Let G be a compact connected Lie group and let M be a smooth G-manifold. Then*

$$H^p(EG \times_G M; \mathbb{R}) = H^p(A^*_G(M), d_{\mathfrak{g}}).$$

This result is stated in [1] together with some arguments, and it is attributed to [5], [6] in chapter 7 of [2]. I was unable to find a satisfactory published proof.

5.4. Let M be a smooth G-manifold. Then the obvious embedding $j(\omega) = 1 \otimes \omega$ gives a mapping of graded differential algebras

$$j : \Omega^p_{\text{hor}}(M)^G \to (S^0\mathfrak{g}^* \otimes \Omega^p(M))^G \to \bigoplus_k (S^k\mathfrak{g}^* \otimes \Omega^{p-2k}(M))^G = A^p_G(M).$$

On the other hand evaluation at $0 \in \mathfrak{g}$ defines a homomorphism of graded differential algebras $\mathrm{ev}_0 : A^*_G(M) \to \Omega^*(M)^G$, and $\mathrm{ev}_0 \circ j$ is the embedding $\Omega^*_{\mathrm{hor}}(M)^G \to \Omega^*(M)^G$. Thus we get canonical homomorphisms in cohomology

$$\begin{array}{ccccc} H^p(\Omega^*_{\mathrm{hor}}(M)^G) & \xrightarrow{J^*} & H^p(A^*_G(M), d_{\mathfrak{g}}) & \longrightarrow & H^p(\Omega^*(M)^G, d) \\ \| & & \| & & \| \\ H^p_{G-\mathrm{basic}}(M) & \longrightarrow & H^p_G(M) & \longrightarrow & H^p(M)^G. \end{array}$$

If G is compact and connected we have $H^p(M)^G = H^p(M)$, by integration and homotopy invariance.

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Erwin Schrödinger International Institute of Mathematical Physics, Wien, Austria, Institut für Mathematik, Universität Wien, Austria
E-mail address: `MICHOR@ESI.AC.AT`

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria