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THE LARGENESS OF SETS OF POINTS WITH NON-DENSE ORBIT IN BASIC SETS ON SURFACES

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ABSTRACT. We show that if f is a diffeomorphism of a closed surface and Λ is a basic set for f, then $HD(\{x \in \Lambda : \textit{the orbit of } x \textit{ by } f \textit{ is not dense in } \Lambda\}) = HD(\Lambda)$.

1. INTRODUCTION

We consider the largeness of the set of points with non-dense orbit for a given diffeomorphism of a compact manifold.

In general, if a homeomorphism defined on a compact metric space is topologically transitive, then such a set belongs to the first category. This fact means that it is a small set in the sense of general topology.

On the other hand, Urbański [6] has proved the following theorem: For a given transitive C^2-Anosov diffeomorphism of a compact manifold M, the Hausdorff dimension of the set of points with non-dense orbit is equal to $\dim M$. From this theorem we can say that it has the same 'fatness' as the manifold.

Let Λ be a basic set of a diffeomorphism $f : M \to M$ of a closed manifold M. There is no way to calculate the Hausdorff dimension of Λ in general. However when M is a surface, McCluskey and Manning [5] have shown the Hausdorff dimension of Λ can be represented by the topological pressures of two continuous functions.

Using this result, we shall estimate a certain probability measure on Λ and show that the set of points with non-dense orbit for the restriction $f\,|_\Lambda: \Lambda \to \Lambda$ has the same Hausdorff dimension as Λ.

2. DEFINITIONS AND THE RESULT

Let M be a closed manifold and $f : M \to M$ a diffeomorphism, and let $\Lambda \subset M$ be a compact invariant set, $f\Lambda = \Lambda$. We say that Λ is *hyperbolic* for f if: (a) the tangent bundle of M restricted to Λ decomposes as a continuous direct sum, $T_\Lambda M = E^s \oplus E^u$, which is invariant by the differential of f, Df; (b) there exists a Riemannian metric (adapted metric) and a number $0 < \gamma < 1$ such that $\|Df(v)\| \leq \gamma\|v\|$ and $\|Df^{-1}(u)\| \leq \gamma\|u\|$ for any $v \in E^s, u \in E^u$. Further we say that a hyperbolic set Λ for f is *isolated* if there is a neighborhood U of Λ such that $\bigcap_{n=-\infty}^{\infty} f^n U = \Lambda$. $\Lambda \subset M$ is called a *basic set* for f if Λ is an isolated hyperbolic

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set for f and $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive. We say that a basic set for f is *trivial* if it is a periodic orbit for f.

For each subset A of M and $\alpha > 0$ we define the *α-measure* of A by

$$m_\alpha(A) = \lim_{\varepsilon \downarrow 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\mathrm{diam} U)^\alpha$$

where the infimum is taken over all countable covers $\mathcal{U}$ of A by sets with diameter less than ε. The *Hausdorff dimension* of A is defined by

$$HD(A) = \inf\{\alpha : m_\alpha(A) = 0\}.$$

This definition is independent of the choice of a Riemannian metric. Remark that for each $A \subset B \subset M$, $0 \le HD(A) \le HD(B) \le HD(M) = \dim M$; also $HD(A) = 0$ if $A \subset M$ is at most countable.

Our purpose is to prove the following.

Theorem. *Let $f : M \to M$ be a C^1-Hölder diffeomorphism of a closed surface M and $\Lambda \subset M$ be a basic set for f. For each open set $V \subset M$ which intersects with Λ,*

$$HD(\{x \in V \cap \Lambda : \mathcal{O}_f(x) \ \textit{is not dense in} \ \Lambda\}) = HD(\Lambda)$$

where $\mathcal{O}_f(x)$ denotes the orbit of x by f.

Remark. When $\Lambda = M$, i.e. $f : M \to M$ is a transitive Anosov diffeomorphism, this result is the same as Urbański's theorem [6].

It is unknown that the theorem is extended on manifolds of higher dimension.

In order to prove the above theorem we need a few known results.

Let (Σ_A, σ) be a topologically mixing Markov subshift and $\eta : \Sigma_A \to \mathbb{R}$ a Hölder continuous function. Denote for each $\mathbf{a} = (a_i) \in \Sigma_A$ and $k, l \in \mathbb{Z}$ with $k \le l$, ${}_k[a_k a_{k+1} \dots a_l]_l = \{\mathbf{b} = (b_i) \in \Sigma_A : a_j = b_j \ \textit{for every} \ j = k, k+1, \dots, l\}$. Bowen [2] has proved that there is a unique Borel probability measure m called Gibbs measure on Σ_A for which one can find a constant $C \ge 1$ such that for any $\mathbf{a} = (a_i) \in \Sigma_A$, $n \in \mathbb{N}$,

$$C^{-1} \le \frac{m({}_0[a_0 a_1 \dots a_{n-1}]_{n-1})}{\exp\{-P(\Sigma_A, \sigma, \eta)n + \sum_{j=0}^{n-1} \eta(\sigma^j \mathbf{a})\}} \le C$$

where $P(\Sigma_A, \sigma, \eta)$ is the topological pressure of η for (Σ_A, σ).

Let μ be a Borel probability measure on a compact manifold M with distance d for which one can find constants $h > 0$, $D \ge 1$, and $r_0 > 0$ such that $\mu(B_r(x)) \le Dr^h$ for every $x \in \mathrm{supp}\mu$ and $0 < r < r_0$ where $B_r(x) = \{y \in M : d(x,y) \le r\}$. For any integer $k \ge 1$ let E_k denote a finite collection of compact subsets of $\mathrm{supp}\mu$ with positive measure μ and let $\bigcup E_k$ denote the union of all elements of E_k. We assume that the collection E_k satisfies the following conditions: $\bigcup E_1 = \mathrm{supp}\mu$, $\mu(F \cap G) = 0$ for $F, G \in E_k$ with $F \ne G$, and every set $H \in E_{k+1}$ is contained in a unique element $I \in E_k$. We write $\Delta_k = \inf\{\mathrm{density}(\bigcup E_{k+1}, F) : F \in E_{k+1}\}$ where $\mathrm{density}(\bigcup E_{k+1}, F) = \frac{\mu(\bigcup E_{k+1} \cap F)}{\mu(F)}$, and $d_k = \sup\{\mathrm{diam} F : F \in E_k\}$ for every $k \ge 1$. It is known that if $\Delta_k > 0$, $d_k < 1$ for every $k \ge 1$ and $\lim_{k\to\infty} d_k = 0$, then

$$HD(\bigcap_{k=1}^{\infty} \bigcup E_k) \ge h - \limsup_{k\to\infty} \frac{\sum_{j=1}^{k} \log \Delta_j}{\log d_k}.$$

This is a result by McMullen (see [6]).

In order to state the McCluskey-Manning theorem [5], let $f : M \to M$ be a diffeomorphism of a closed surface M and $\Lambda \subset M$ be a basic set for f with $\dim E^s = 1$. Define $\phi^{(u)}, \phi^{(s)} : \Lambda \to \mathbb{R}$ by $\phi^{(u)}(x) = -\log \|D_x f|_{E^u}\|$, $\phi^{(s)}(x) = -\log \|D_x f^{-1}|_{E^s}\|$. Then there exist unique $0 \le \delta^u, \delta^s \le 1$ such that

$$P(\Lambda, f|_\Lambda, \delta^u \phi^{(u)}) = 0, \quad P(\Lambda, f^{-1}|_\Lambda, \delta^s \phi^{(s)}) = 0 \text{ and } HD(\Lambda) = \delta^u + \delta^s.$$

3. Proof of the theorem

We shall prove that

$$HD(\{x \in V \cap \Lambda : \mathcal{O}_f(x) \textit{ is not dense in } \Lambda\}) \ge HD(\Lambda).$$

We can check the case $\dim E^s = 0$ or $\dim E^s = 2$, Λ is trivial. So, $HD(\Lambda) = 0$ and there is nothing to prove. Thus we consider $\dim E^s = 1$.

Taking an adapted metric $\|\cdot\|$, there is a constant $\beta > 1$ such that $\min_{x\in\Lambda}\{\|D_x f\|, \|D_x f^{-1}\|\} > \beta$. Remark that $\|D_x f\| = \|D_x f|_{E^u}\|$, $\|D_x f^{-1}\| = \|D_x f^{-1}|_{E^s}\|$. Let d be the distance on M induced by $\|\cdot\|$ and δ^u, δ^s be as above. If $\delta^u = 0$ or $\delta^s = 0$, then the topological entropy of $(\Lambda, f|_\Lambda)$ is zero. Thus Λ is trivial, and $HD(\Lambda) = 0$. Therefore it suffices to check with the case $\delta^u, \delta^s > 0$. For convenience we may assume that $f|_\Lambda : \Lambda \to \Lambda$ is topologically mixing. (Use the spectral decomposition [2] if necessary.)

Let $0 < c < 1$ be an expansive constant of $f|_\Lambda$. As Λ is a basic set of a C^1-Hölder diffeomorphism f, we can choose $0 < \varepsilon_0 < c/4$ so small as to satisfy $y \mapsto \|D_y f\|$, $\|D_y f^{-1}\|$ are Hölder continuous functions on $2\varepsilon_0$ neighborhood of Λ, and there is a $\lambda \in (0,1)$ such that if $x, z \in \Lambda$ and $d(f^k x, f^k z) \le \varepsilon_0$ for $k = -n, \dots, 0, \dots, n$, then $d(x,z) < \lambda^n$ (see [2]).

Let $\varepsilon_0 > 0$ be as above and pick a Markov partition $\mathcal{R} = \{R_1, \dots, R_s\}$ of Λ such that $\mathrm{diam}\mathcal{R} < \varepsilon_0$ and $\sharp\{1 \le q \le s : R_p \cap R_q = \emptyset\} > 0$ for every $p = 1, 2, \dots, s$ (see [2]).

Let $A = (A_{ij})_{1\le i,j\le s}$ be the structure matrix of $\mathcal{R}$ and (Σ_A, σ) be the corresponding Markov subshift. We denote the set of all words of length n of Σ_A by $\Sigma(n)$. Since (Σ_A, σ) is topologically mixing, there is a $K_0 \ge 2$ such that $A^K > 0$, i.e. $(A^K)_{ij} > 0$ for each $1 \le i, j \le s$ and $K \ge K_0$.

Let $h : \Sigma_A \to \Lambda$ be the continuous surjection which satisfies $h \circ \sigma = f \circ h$ and $\bigcap_{j=k}^{l} f^{-j} R_{a_j} = h({}_k[a_k a_{k+1} \dots a_l]_l)$ for every $(a_k \dots a_l) \in \Sigma(k+l+1)$. We denote $\mathcal{R}(k,l) = \{\bigcap_{j=k}^{l} f^{-j} R_{a_j} : (a_k \dots a_l) \in \Sigma(k+l+1)\}$ for all $k, l \in \mathbb{Z}$ with $k \le l$. Remark that h is Hölder continuous (see [4]) and that h is bounded finite to one (this is a Bowen's result, see [1]), so there is an integer $e_0 > 0$ such that $\sharp\{W \in \mathcal{R}(k,l) : x \in W\} \le e_0$ for every $x \in \Lambda$ and $k, l \in \mathbb{Z}$ with $k \le l$. (If Λ is totally disconnected, then the map h is bijective.)

By choosing $\varepsilon_0 > 0$ small enough and reordering if necessary we can assume that $R_1 \subset V \cap \Lambda$ holds.

Since $P(\Sigma_A, \sigma, \delta^u \phi^{(u)} \circ h) = P(\Lambda, f|_\Lambda, \delta^u \phi^{(u)}) = 0$, and $\delta^u \phi^{(u)} \circ h$ is Hölder continuous, there is a Borel probability measure (Gibbs measure) m_+ on Σ_A and a constant $C_3 \ge 1$ such that

$$C_3^{-1} \le \frac{m_+({}_0[a_0 \dots a_n]_n)}{\exp\{\sum_{j=0}^{n} \delta^u \phi^{(u)} \circ h(\sigma^j \mathbf{a})\}} \le C_3$$

for every $\mathbf{a} = (a_i) \in \Sigma_A$, $n \in \mathbb{Z}^+$.

Similarly, we can find a Borel probability measure m_- on Σ_A and a constant $C_4 \geq 1$ such that

$$C_4^{-1} \leq \frac{m_-({}_{-n}[a_{-n}\dots a_0]_0)}{\exp\{\sum_{j=0}^{n} \delta^s \phi^{(s)} \circ h(\sigma^{-j}\mathbf{a})\}} \leq C_4$$

for every $\mathbf{a} = (a_i) \in \Sigma_A,\ n \in \mathbb{Z}^+$.

Let m be a Borel probability measure on Σ_A such that

$$m({}_{-k}[a_{-k}\dots a_0 \dots a_l]_l) = \begin{cases} C_5 m_+({}_0[a_0\dots a_l]_l) m_-({}_{-k}[a_{-k}\dots a_0]_0) & (a_0 = 1), \\ 0 & (a_0 \neq 1) \end{cases}$$

for each $\mathbf{a} = (a_i) \in \Sigma_A,\ k, l \in \mathbb{Z}^+$, and $C_5 = m_+({}_0[1]_0)^{-1} m_-({}_0[1]_0)^{-1}$.

Define a Borel probability measure μ on M by

$$\mu(A) = m(h^{-1}(A \cap \Lambda))$$

for each Borel set A of M. Clearly $\operatorname{supp}\mu = R_1$ and $\mu(\bigcup_{i=-\infty}^{\infty} f^{-i}(\partial\mathcal{R})) = 0$ where $\partial\mathcal{R} = \{x \in \Lambda : x \in R_p \cap R_q \text{ for some } 1 \leq p < q \leq s\}$.

If $a_0 = 1$ and $(a_{-k}\dots a_0 \dots a_l) \in \Sigma(k+l+1),\ k, l \in \mathbb{Z}^+$, then we have that for $x \in \bigcap_{i=-k}^{l} f^{-i} R_{a_i}$,

$$C_1^{-1} \leq \frac{\mu(\bigcap_{i=-k}^{l} f^{-i} R_{a_i})}{\exp\{\sum_{i=0}^{l} \delta^u \phi^{(u)}(f^i x) + \sum_{j=0}^{k} \delta^s \phi^{(s)}(f^{-j}x)\}} \leq C_1$$

where $C_1 = C_3 C_4 C_5$. (Such a measure μ has been constructed by Mañé in [4].)

Lemma 1. *There are constants $C_2 \geq 1$ and $r_0 > 0$ such that*

$$\mu(B_r(x)) \leq C_2 r^{HD(\Lambda)}$$

for all $x \in R_1$ and $0 < r < r_0$.

Remark. The measure μ is true that for some constant $C_7 \leq 1$,

$$\mu(B_r(x)) \geq C_7 r^{HD(\Lambda)} \quad (x \in R_1,\ 0 < r < r_0).$$

Applying this μ to McMullen's result, our result is obtained as follows.

Take an integer $l_0 \geq 1$ so large as to satisfy $C_1^2 l \beta^{-\delta^u l},\ C_1^2 l \beta^{-\delta^s l} < 1/4$ for all $l \geq l_0$.

As (Σ_A, σ) is topologically mixing, for some $1 \leq p_0, p_1, p_2, q_0, q_1, q_2 \leq s$,

$$p_1 \neq p_2,\ q_1 \neq q_2, \quad \text{and} \quad A_{p_0 p_1} = A_{p_0 p_2} = A_{q_1 q_0} = A_{q_2 q_0} = 1.$$

Fix an integer $l > \max\{l_0, K_0\}$, and pick $(v_{-l}\dots v_0 \dots v_l) \in \Sigma(2l+1)$ with $v_0 \neq 1$ and so that $(p_0 p_1), (q_1 q_0)$ do not appear as any substring of $(v_0 \dots v_l)$, $(v_{-l}\dots v_0)$, respectively. Set $Y = \bigcap_{j=-l}^{l} f^{-j} R_{v_j}$ and define

$$\begin{aligned} E_1 &= \{R_1\}, \\ E_{k+1} &= E_{k+1}(l) \\ &= \left\{ \begin{array}{l} W = \bigcap_{j=-kl}^{kl} f^{-j} R_{a_j} : (a_{-kl}\dots a_0 \dots a_{kl}) \in \Sigma(2kl+1),\ a_0 = 1, \\ f^n W \cap \mathring{Y} = \emptyset \ \textit{for every}\ n = -kl+1, \dots, 0, \dots, kl-1 \end{array} \right\} \end{aligned}$$

where $\mathring{Y}$ is the interior of Y in Λ, $k \in \mathbb{N}$.

Then we can check that for each $k \geq 1$, $d_k < \lambda^{(k-1)l}$.

Lemma 2. (1) $\varDelta_1 > 0$, (2) $\varDelta_{k+1} \geq 1/2$ *for every* $k \geq 1$.

Therefore,

$$\frac{\sum_{j=1}^{k} \log \varDelta_j}{\log d_k} \leq \frac{\log \varDelta_1 + (k-1)\log 1/2}{(k-1)l \log \lambda}$$

for every $k \geq 2$. Other conditions of McMullen's result are obviously satisfied.

On the other hand, since

$$\bigcap_{k=1}^{\infty} \cup E_k \subset \{x \in V \cap \varLambda : \mathcal{O}_f(x) \textit{ is not dense in } \varLambda\},$$

we have

$$\begin{aligned} &HD(\{x \in V \cap \varLambda : \mathcal{O}_f(x) \textit{ is not dense in } \varLambda\}) \\ &\quad \geq HD(\bigcap_{k=1}^{\infty} \cup E_k) \\ &\quad \geq HD(\varLambda) - \lim_{k\to\infty} \frac{\log \varDelta_1 + (k-1)\log 1/2}{(k-1)l\log\lambda} \\ &\quad = HD(\varLambda) - \frac{\log 1/2}{l \log \lambda}. \end{aligned}$$

Since $l > \max\{l_0, K_0\}$ is arbitrary, our requirement is obtained.

4. Proof of Lemma 1

Mañé [4] has proved Lemma 1 when $\varLambda$ is totally disconnected. We give a proof of Lemma 1, for the general case, as follows.

Choose $C_6 \geq 1$ such that for any $n, m \in \mathbb{N}$,

$$C_6^{-1} \leq \frac{\prod_{j=0}^{n-1} \|D_{f^j z} f\|}{\prod_{j=0}^{n-1} \|D_{f^j x} f\|} \leq C_6$$

for all $x \in \varLambda$, $z \in M$ satisfying $\max_{0\leq k\leq n-1} d(f^k x, f^k z) \leq \varepsilon_0$, and

$$C_6^{-1} \leq \frac{\prod_{j=0}^{m-1} \|D_{f^{-j} w} f^{-1}\|}{\prod_{j=0}^{m-1} \|D_{f^{-j} y} f^{-1}\|} \leq C_6$$

for all $y \in \varLambda$, $w \in M$ satisfying $\max_{0\leq k\leq m-1} d(f^{-k} y, f^{-k} w) \leq \varepsilon_0$.

Put $\delta_0 = \inf\{d(x,y) : x \in R_p, y \in R_q, R_p \cap R_q = \emptyset, 1 \leq p < q \leq s\} > 0$ and $r_0 = \min\{\delta_0/2, \ \varepsilon_0/2\} > 0$.

Step 1. *For each* $x \in \varLambda$, $0 < r < r_0$, *and* $l, n \in \mathbb{N}$, *if* $C_6 r \exp\{-\sum_{i=0}^{n-1} \phi^{(u)}(f^i x)\} < r_0$ *and* $C_6 r \exp\{-\sum_{i=0}^{l-1} \phi^{(s)}(f^{-i} x)\} < r_0$, *then*

$$B_r(x) \subset B^f_{r_0}(x, -l, n)$$

where $B^f_{r_0}(x,-l,n) = \{z \in M : d(f^j x, f^j z) \leq r_0 \textit{ for } j = -l, \ldots, 0, \ldots, n\}$.

Proof. We first show that for $y \in B_r(x)$, $d(f^j x, f^j y) \le r_0$ for every $j = 1, \ldots, n$ if $C_6 r \exp\{-\sum_{i=0}^{n-1} \phi^{(u)}(f^i x)\} < r_0$.

To see this, we choose $\varepsilon > 0$ so small that $C_6(r+\varepsilon)\exp\{-\sum_{i=0}^{n-1} \phi^{(u)}(f^i x)\} < r_0$ and take a smooth curve $\xi : [0,1] \to M$ such that $\xi(0) = x$, $\xi(1) = y$, and $\int_0^1 \| \dot{\xi}(s) \| \, ds \le r + \varepsilon$.

Since

$$\begin{aligned} d(x, \xi(t)) &\le \int_0^t \|\dot{\xi}(s)\| ds \\ &\le r + \varepsilon \quad < \varepsilon_0 \quad (0 \le t \le 1), \end{aligned}$$

we have

$$\begin{aligned} d(fx, f\xi(t)) &\le \int_0^t \|Df(\dot{\xi}(s))\| ds \\ &\le \int_0^t \|D_{\xi(s)} f\| \cdot \|\dot{\xi}(s)\| ds \\ &\le C_6 \exp\{-\phi^{(u)}(x)\} \int_0^t \|\dot{\xi}(s)\| ds \quad (0 \le t \le 1). \end{aligned}$$

Assume that for each $1 \le k-1 < n$, $0 \le t \le 1$, and $j = 1, 2, \ldots, k-1$,

$$d(f^j x, f^j \xi(t)) \le C_6 \exp\{-\sum_{i=0}^{j-1} \phi^{(u)}(f^i x)\} \int_0^t \|\dot{\xi}(s)\| ds.$$

Then we have

$$\begin{aligned} d(f^j x, f^j \xi(t)) &\le C_6(r+\varepsilon) \exp\{-\sum_{i=0}^{n-1} \phi^{(u)}(f^i x)\} \\ &< r_0 \quad < \varepsilon_0. \end{aligned}$$

This implies that

$$\begin{aligned} d(f^k x, f^k \xi(t)) &\le \int_0^t \|Df^k(\dot{\xi}(s))\| ds \\ &\le \int_0^t \prod_{i=0}^{k-1} \|D_{f^i \xi(s)} f\| \cdot \|\dot{\xi}(s)\| ds \\ &\le C_6 \exp\{-\sum_{i=0}^{k-1} \phi^{(u)}(f^i x)\} \int_0^t \|\dot{\xi}(s)\| ds \end{aligned}$$

for each $0 \le t \le 1$.

Using the induction, for each $0 \le t \le 1$ and $j = 1, 2, \ldots, n$

$$\begin{aligned} d(f^j x, f^j \xi(t)) &\le C_6 \exp\{-\sum_{i=0}^{j-1} \phi^{(u)}(f^i x)\} \int_0^t \|\dot{\xi}(s)\| ds \\ &\le C_6(r+\varepsilon) \exp\{-\sum_{i=0}^{n-1} \phi^{(u)}(f^i x)\} \quad < r_0. \end{aligned}$$

Therefore, $d(f^j x, f^j y) < r_0$ $(1 \le j \le n)$ if $t = 1$.

Similarly, we can check that $d(f^{-j}x, f^{-j}y) < r_0$ for $j = 1, \dots, l$ if $C_6 r \exp\{-\sum_{i=0}^{l-1} \phi^{(s)}(f^{-i}x)\} < r_0$. □

Step 2. *For each* $x \in \Lambda$, $0 < r < r_0$, *let*

$$n_1 = \min\left\{ n \in \mathbb{Z}^+ : C_6 r \exp\{-\sum_{i=0}^{n} \phi^{(u)}(f^i x)\} \geq r_0 \right\},$$
$$n_2 = \min\left\{ n \in \mathbb{Z}^+ : C_6 r \exp\{-\sum_{i=0}^{n} \phi^{(s)}(f^{-i} x)\} \geq r_0 \right\}.$$

Then there exist $m = m(x, r_0, -n_2, n_1) \leq s^2 e_0$ *and* $W_1, \dots, W_m \in \mathcal{R}(-n_2, n_1)$ *such that*

$$W_k \cap B^f_{r_0}(x, -n_2, n_1) \neq \emptyset \quad \text{for every} \quad k = 1, \dots, m,$$

$$B_r(x) \cap \Lambda \subset \bigcup_{k=1}^{m} W_k.$$

Proof. For $y \in \Lambda$, define $\mathcal{R}_y = \{Q \in \mathcal{R} : R \cap Q \neq \emptyset \quad \textit{for some } y \in R \in \mathcal{R}\}$ and $P_y = \bigcup_{Q \in \mathcal{R}_y} Q$. Remark that $P_y \subset B_{2\epsilon_0}(y) \cap \Lambda$, and if $z \in \Lambda, d(y, z) < \delta_0$, then for any $Q \in \mathcal{R}$ containing z, $Q \in \mathcal{R}_y$.

Put

$$\mathcal{P}(x, -n_2, n_1) = \left\{ \begin{array}{l} W = \bigcap_{j=-n_2}^{n_1} f^{-j} R_{a_j} \in \mathcal{R}(-n_2, n_1) : \\ R_{a_j} \subset P_{f^j x} \quad \textit{for every } j = -n_2, \dots, 0, \dots, n_1 \end{array} \right\}.$$

By Step 1, we have

$$\begin{aligned} B_r(x) \cap \Lambda &\subset B^f_{r_0}(x, -n_2, n_1) \cap \Lambda \\ &\subset \bigcup_{W \in \mathcal{P}(x, -n_2, n_1)} W \\ &\subset \bigcap_{j=-n_2}^{n_1} f^{-j} P_{f^j x}. \end{aligned}$$

Let $\mathcal{P}(x, r_0, -n_2, n_1) = \{W \in \mathcal{P}(x, -n_2, n_1) : B^f_{r_0}(x, -n_2, n_1) \cap W \neq \emptyset\}$. Then we have that $m = \sharp\mathcal{P}(x, r_0, -n_2, n_1) \leq s^2 e_0$. To prove this, take $l_1 = \sharp\mathcal{R}_{f^{n_1}x}$, $l_2 = \sharp\mathcal{R}_{f^{-n_2}x}$, and $\alpha_1, \dots, \alpha_{l_1}, \beta_1, \dots, \beta_{l_2} \in \{1, 2, \dots, s\}$ such that $P_{f^{n_1}x} = \bigcup_{p=1}^{l_1} R_{\alpha_p}$, $P_{f^{-n_2}x} = \bigcup_{q=1}^{l_2} R_{\beta_q}$.

Then, $\mathcal{P}(x, r_0, -n_2, n_1) = \bigcup_{p=1}^{l_1} \bigcup_{q=1}^{l_2} \mathcal{Q}_{p,q}$ where $\mathcal{Q}_{p,q} = \{W \in \mathcal{P}(x, r_0, -n_2, n_1)$: $W \subset f^{n_2} R_{\beta_q} \cap f^{-n_1} R_{\alpha_p}\}$.

Fix $1 \leq p \leq l_1$, $1 \leq q \leq l_2$ such that $\mathcal{Q}_{p,q} \neq \emptyset$, and put $n = n(p, q) = \sharp\mathcal{Q}_{p,q}$. Let $W^1_{p,q}, W^2_{p,q}, \dots, W^n_{p,q}$ denote all elements belonging to $\mathcal{Q}_{p,q}$.

We can choose $(\alpha_p w_1 \dots w_{K_0-1}\beta_q) \in \Sigma(K_0+1)$ and take

$$z^i_{p,q} \in W^i_{p,q} \cap (\bigcap_{k=1}^{K_0-1} f^{-n_1-k} R_{w_k}) \quad \text{such that} \quad f^{n_1+n_2+K_0} z^i_{p,q} = z^i_{p,q}$$

for each $i = 1, 2, \dots, n$.

Then we have that for $1 \le i, j \le n$,

$$\begin{aligned} d(f^k z^i_{p,q}, f^k z^j_{p,q}) &\le d(f^k z^i_{p,q}, f^k x) + d(f^k x, f^k z^j_{p,q}) \\ &\le 2\varepsilon_0 + 2\varepsilon_0 \quad < c \\ &\qquad \text{for } k = -n_2, \dots, 0, \dots, n_1, \\ d(f^{n_1+k} z^i_{p,q}, f^{n_1+k} z^j_{p,q}) &\le \operatorname{diam} R_{w_k} \quad < c \\ &\qquad \text{for } k = 1, \dots, K_0 - 1. \end{aligned}$$

Thus, $z^i_{p,q} = z^j_{p,q}$ for each $1 \le i, j \le n$, and therefore

$$z^1_{p,q} \in \bigcap_{i=1}^{n} W^i_{p,q} = \bigcap_{W \in \mathcal{Q}_{p,q}} W.$$

Since $\sharp\{W \in \mathcal{R}(-n_2, n_1) : z^1_{p,q} \in W\} \le e_0$, we have $\sharp\mathcal{Q}_{p,q} \le e_0$.

Therefore,

$$\begin{aligned} m &= \sharp\mathcal{P}(x, r_0, -n_2, n_1) \\ &= \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sharp\mathcal{Q}_{p,q} \qquad \le s^2 e_0. \quad \square \end{aligned}$$

Step 3. *There exists $C_2 \ge 1$ such that*

$$\mu(B_r(x)) \le C_2 r^{HD(\Lambda)}$$

for all $x \in R_1$ and $0 < r < r_0$.

Proof. Fix $x \in R_1$ and $0 < r < r_0$. Take $n_1, n_2 \in \mathbb{Z}^+$, $m \le s^2 e_0$, and $W_1, \dots, W_m \in \mathcal{R}(-n_2, n_1)$ as Step 2. For $k = 1, \dots, m$, pick $y_k \in W_k \cap B^f_{r_0}(x, -n_2, n_1)$. Then we have

$$\exp \sum_{i=0}^{n_1} \phi^{(u)}(f^i y_k) \le C_6 \exp \sum_{i=0}^{n_1} \phi^{(u)}(f^i x),$$

$$\exp \sum_{j=0}^{n_2} \phi^{(s)}(f^{-j} y_k) \le C_6 \exp \sum_{j=0}^{n_2} \phi^{(s)}(f^{-j} x),$$

from which

$$\begin{aligned} \mu(W_k) &\le C_1 \exp\{\sum_{i=0}^{n_1} \delta^u \phi^{(u)}(f^i y_k) + \sum_{j=0}^{n_2} \delta^s \phi^{(s)}(f^{-j} y_k)\} \\ &\le C_1 C_6^{\delta^u+\delta^s} \exp\{\sum_{i=0}^{n_1} \delta^u \phi^{(u)}(f^i x) + \sum_{j=0}^{n_2} \delta^s \phi^{(s)}(f^{-j} x)\} \\ &\le C_1 C_6^{HD(\Lambda)} (\frac{C_6 r}{r_0})^{\delta^u+\delta^s}. \end{aligned}$$

Therefore,

$$\mu(B_r(x)) = \mu(B_r(x)\cap\Lambda) \le \sum_{k=1}^{m}\mu(W_k) \quad \le C_2 r^{HD(\Lambda)}$$

where $C_2 = s^2 e_0 r_0^{-HD(\Lambda)} C_1 C_6^{2HD(\Lambda)}$. □

5. Proof of Lemma 2

Proof of (1). As $l-1 \ge K_0$, we pick $(a_{-l}\dots a_0\dots a_l) \in \Sigma(2l+1)$ with $a_{-l} = q_1$, $a_{-l+1} = q_0$, $a_0 = 1$, $a_{l-1} = p_0$, and $a_l = p_1$.

Put $W_0 = \bigcap_{j=-l}^{l} f^{-j}R_{a_j}$; then $W_0 \in \bigcup E_2$ and so $\Delta_1 = \frac{\mu(R_1\cap\bigcup E_2)}{\mu(R_1)} \ge \mu(W_0) > 0$. □

Proof of (2). Fix $k\ge 1$ and $W = \bigcap_{j=-kl}^{kl} f^{-j}R_{a_j} \in E_{k+1}$, $(a_{-kl}\dots a_0\dots a_{kl}) \in \Sigma(2kl+1)$. By the definition of E_{k+1}, we have that $f^jW\cap\mathring{Y} = \emptyset$ for $j = -kl+1,\dots,0,\dots,kl-1$ and that

$$\begin{aligned}
f^{-kl}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{(k-1)l}\dots a_{kl}) = (v_{-l}\dots v_0),\\
f^{-kl-1}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{(k-1)l+1}\dots a_{kl}) = (v_{-l}\dots v_{-1}),\\
&\dots\\
f^{-(k+1)l+1}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{kl-1}a_{kl}) = (v_{-l}v_{-l+1}),\\
f^{kl}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{-kl}\dots a_{-(k-1)l}) = (v_0\dots v_l),\\
f^{kl+1}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{-kl}\dots a_{-(k-1)l-1}) = (v_1\dots v_l),\\
&\dots\\
f^{(k+1)l-1}W\cap\mathring{Y}\ne\emptyset \quad &\text{iff} \quad (a_{-kl}a_{-kl+1}) = (v_{l-1}v_l).
\end{aligned}$$

For $j = 0,1,\dots,l-1$, put

$$W_j^- = \begin{cases} h({}_{-kl}[a_{-kl}\dots a_0\dots a_{kl}v_{-j+1}\dots v_{-j+l}]_{(k+1)l}) & (f^{-kl-j}W\cap\mathring{Y}\ne\emptyset),\\ \emptyset & (otherwise),\end{cases}$$

$$W_j^+ = \begin{cases} h({}_{-(k+1)l}[v_{j-l}\dots v_{j-1}a_{-kl}\dots a_0\dots a_{kl}]_{kl}) & (f^{kl+j}W\cap\mathring{Y}\ne\emptyset),\\ \emptyset & (otherwise).\end{cases}$$

Then we have $W\setminus\bigcup E_{k+2} = \bigcup_{j=0}^{l-1}(W_j^-\cup W_j^+)$, and take $x\in W_j^-\subset W$ if $W_j^-\ne\emptyset$ for $j = 0,1,\dots,l-1$. Then,

$$\begin{aligned}
\frac{\mu(W_j^-)}{\mu(W)} &\le \frac{C_1\exp\{\sum_{i=0}^{(k+1)l}\delta^u\phi^{(u)}(f^ix) + \sum_{j=0}^{kl}\delta^s\phi^{(s)}(f^{-j}x)\}}{C_1^{-1}\exp\{\sum_{i=0}^{kl}\delta^u\phi^{(u)}(f^ix) + \sum_{j=0}^{kl}\delta^s\phi^{(s)}(f^{-j}x)\}}\\
&= C_1^2\exp\{\sum_{i=kl+1}^{(k+1)l}\delta^u\phi^{(u)}(f^ix)\}\\
&\le C_1^2\beta^{-\delta^u l} \quad \le \frac{1}{4l}.
\end{aligned}$$

Similarly, for every $j = 0, 1, \ldots, l-1$

$$\frac{\mu(W_j^+)}{\mu(W)} \leq C_1^2 \beta^{-\delta^s l} \leq \frac{1}{4l}.$$

Thus,

$$\begin{aligned} \frac{\mu(\bigcup E_{k+2} \cap W)}{\mu(W)} &= 1 - \frac{\mu(W \setminus \bigcup E_{k+2})}{\mu(W)} \\ &\geq 1 - \sum_{j=0}^{l-1} \left(\frac{\mu(W_j^-)}{\mu(W)} + \frac{\mu(W_j^+)}{\mu(W)} \right) \quad \geq 1/2. \end{aligned}$$

Therefore, $\Delta_{k+1} \geq 1/2$ for any $k \geq 1$. □

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