

## THE STRUCTURE OF MEASURABLE MAPPINGS WITH VALUES IN LOCALLY CONVEX SPACES

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ABSTRACT. The purpose of this paper is to show that a theorem of A. Wisniewski remains valid without the approximation property.

### 1. INTRODUCTION

Let  $X$  and  $Y$  be Hausdorff topological spaces. Denote by  $\mathcal{B}(X)$  and  $\mathcal{B}(Y)$  respectively the Borel  $\sigma$ -algebras on these spaces. Let  $\mu$  be a finite Borel measure on  $X$ . By  $\mathcal{B}_\mu(X)$  we denote the completion of the  $\sigma$ -algebra  $\mathcal{B}(X)$  with respect to the measure  $\mu$ . A mapping  $f$  from  $X$  into  $Y$  is said to be  $\mu$ -measurable if it is measurable with respect to  $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$ . Recall that a Suslin space is a Hausdorff topological space which is the continuous image of some Polish space. It is well known but not easy to prove that every finite Borel measure on a Suslin space is Radon, and hence it is tight (see, e.g., [4, Theorem 10 of Chapter II in Part I]).

In [5], Wisniewski extended a theorem of Gihman and Skorohod [2, p. 544] which clarifies the connection between continuous and  $\mu$ -measurable mappings on Hilbert spaces. Namely, he proved that if  $\mu$  is a finite Borel measure on a metric space  $X$ , and if  $Y$  is a separable Banach space *with the approximation property*, then every  $\mu$ -measurable mapping  $f$  from  $X$  into  $Y$  is the limit of a sequence of continuous mappings with respect to  $\mu$ -almost everywhere convergence.

In this paper, we show that the above result remains valid in the case that  $Y$  is a Suslin locally convex topological linear space which *does not* necessarily have the approximation property. Consequently it turns out that Theorem 2 of [5] remains valid without the approximation property.

### 2. MAIN RESULTS

**Theorem 1.** *Let  $\mu$  be a finite Borel measure on a metric space  $X$ , and let  $Y$  be a locally convex Hausdorff topological linear space. Assume that every finite Borel measure on  $Y$  is tight (this is satisfied, for instance, if  $Y$  is Suslin). If  $f$  is a  $\mu$ -measurable mapping from  $X$  into  $Y$ , then there exists a sequence  $\{f_n\}$  of continuous mappings from  $X$  into  $Y$  such that  $f_n \rightarrow f$   $\mu$ -a.e.*

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*Proof.* To prove the theorem it suffices to show that for any  $\varepsilon > 0$  and any closed neighborhood  $U$  of the origin of  $Y$ , there exists a continuous mapping  $g : X \rightarrow Y$  such that

$$\mu \{x : f(x) - g(x) \notin U\} < \varepsilon.$$

Since  $Y$  is locally convex, we may assume that  $U$  is absorbent and absolutely convex (see, e.g., [3, p. 12]). Put  $\nu(B) = \mu(f^{-1}(B))$  for every  $B \in \mathcal{B}(Y)$ . Then  $\nu$  is a finite Borel measure on  $Y$ , and hence it is tight by the assumption. Thus we can find a compact subset  $K$  of  $Y$  such that  $\nu(Y - K) < \varepsilon/2$ . Put  $K' = f^{-1}(K)$ . Then we have

$$(1) \quad \mu(X - K') < \frac{\varepsilon}{2}.$$

Since  $K$  is compact, there exist  $y_1, y_2, \dots, y_m \in K$  such that

$$K \subset \left(y_1 + \frac{1}{2}U\right) \cup \left(y_2 + \frac{1}{2}U\right) \cup \dots \cup \left(y_m + \frac{1}{2}U\right).$$

Consider sets  $B_1 = y_1 + \frac{1}{2}U, \dots, B_i = \left\{\bigcup_{k=1}^{i-1} \left(y_k + \frac{1}{2}U\right)\right\}^c \cap \left(y_i + \frac{1}{2}U\right), \dots, B_m = \left\{\bigcup_{k=1}^{m-1} \left(y_k + \frac{1}{2}U\right)\right\}^c \cap \left(y_m + \frac{1}{2}U\right)$ . Then the  $B_i$ 's are Borel subsets of  $Y$  with  $K \subset \bigcup_{i=1}^m B_i$  and  $B_i \cap B_j = \emptyset$  if  $i \neq j$ . Hence

$$f(x) - \sum_{i=1}^m 1_{B_i}(f(x))y_i \in \frac{1}{2}U \text{ for all } x \in K',$$

where  $1_B$  denotes the indicator function of a set  $B$ . Consequently, using (1), we infer that

$$(2) \quad \mu \left\{x : f(x) - \sum_{i=1}^m 1_{B_i}(f(x))y_i \notin \frac{1}{2}U\right\} \leq \mu(X - K') \leq \frac{\varepsilon}{2}.$$

For every  $i = 1, 2, \dots, m$ , we put  $f_i(x) = 1_{B_i}(f(x))$  for all  $x \in X$ . Then the  $f_i$ 's are  $\mu$ -measurable mapping from  $X$  into the real line  $R$ . On the other hand, since  $U$  is absorbent, there exists  $\lambda > 0$  such that  $y_i \in \lambda U$  for all  $i = 1, 2, \dots, m$ . Therefore, in view of Theorem 1 of [5], for every  $i = 1, 2, \dots, m$ , we can find continuous mappings  $g_i$  from  $X$  into  $R$  such that

$$(3) \quad \mu \left\{x : |f_i(x) - g_i(x)| > \frac{1}{2m\lambda}\right\} < \frac{\varepsilon}{2m}.$$

Put  $g(x) = \sum_{i=1}^m g_i(x)y_i$  for every  $x \in X$ . Then  $g$  is a continuous mapping from  $X$  into  $Y$ . Moreover, from (2) and (3), together with the fact that  $U$  is convex, we

have

$$\begin{aligned}
\mu \{x : f(x) - g(x) \notin U\} &= \mu \left\{ x : f(x) - \sum_{i=1}^m g_i(x)y_i \notin U \right\} \\
&\leq \mu \left\{ x : f(x) - \sum_{i=1}^m f_i(x)y_i \notin \frac{1}{2}U \right\} \\
&\quad + \mu \left\{ x : \sum_{i=1}^m f_i(x)y_i - \sum_{i=1}^m g_i(x)y_i \notin \frac{1}{2}U \right\} \\
&\leq \frac{\varepsilon}{2} + \sum_{i=1}^m \mu \left\{ x : |f_i(x) - g_i(x)| > \frac{1}{2m\lambda} \right\} \\
&\leq \varepsilon/2 + m \cdot \varepsilon/(2m) = \varepsilon.
\end{aligned}$$

This completes the proof of the theorem.  $\square$

*Remark.* We can show that Theorem 1 also holds in the case that  $X$  is a normal space and  $\mu$  is a finite Borel measure on  $X$  satisfying the condition that  $\mu(B) = \sup \{\mu(F) : F \subset B \text{ and } F \text{ is a closed subset of } X\}$  for all  $B \in \mathcal{B}(X)$ .

**Corollary.** *In the following cases, Theorem 1 holds:*

- (a)  $Y$  is a separable Banach space.
- (b)  $Y$  is a separable Fréchet space.
- (c)  $Y$  is the strict inductive limit of an increasing sequence of separable Fréchet spaces.
- (d)  $Y$  is the strong dual of a nuclear Fréchet space.
- (e)  $Y$  is the strong dual of the strict inductive limit of an increasing sequence of nuclear Fréchet spaces.

*Proof.* (a)–(e) are Suslin (in fact, Lusin) locally convex topological linear spaces by [1, Theorem I.5.1].  $\square$

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