PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 5, May 1996

A NOTE ON FUCHS' PROBLEM 34

U. F. ALBRECHT AND H. P. GOETERS

(Communicated by Ronald M. Solomon)

Abstract. We investigate to what extent an abelian group G is determined by the homomorphism groups $\text{Hom}(G, B)$ where B is chosen from a set $\mathcal{X}$ of abelian groups. In particular, we address Problem 34 in Professor Fuchs' book which asks if $\mathcal{X}$ can be chosen in such a way that the homomorphism groups determine G up to isomorphism. We show that there is a negative answer to this question. On the other hand, there is a set $\mathcal{X}$ which determines the torsion-free groups of finite rank up to quasi-isomorphism.

1. Introduction

Of fundamental importance in the study of abelian groups is the structure of the homomorphism group $\text{Hom}(A, G)$. Naturally one would like to know the extent to which A is determined by the structure of the groups $\text{Hom}(A, G)$ where G is chosen from a given set $\mathcal{X}$ of abelian groups. In particular, is it possible to choose a set $\mathcal{X}$ in such a way that A is determined up to isomorphism by the groups $\text{Hom}(A, G)$ for $G \in \mathcal{X}$, as has been asked in Problem 34 of [4]? The subject of this paper is to address these questions for p-groups and torsion-free groups of finite rank.

The first result of Section 2 shows that it is not possible to give a positive answer to [4, Problem 34]: For any set $\mathcal{X}$ of abelian groups, there are non-isomorphic reduced abelian groups A and C such that $\text{Hom}(A, G) \cong \text{Hom}(C, G)$ for all $G \in \mathcal{X}$. We, therefore, turn to the problem of describing properties shared by abelian groups A and C for which $\text{Hom}(A, G) \cong \text{Hom}(C, G)$ for all G in a given set of abelian groups. In Theorem 2.2, we characterize those collections $\mathcal{X}$ of abelian p-groups such that the groups $\text{Hom}(A, G)$ for $G \in \mathcal{X}$ determine the finite Ulm-Kaplansky invariants $\{f_n(A) | n < \omega\}$. Unfortunately, the characterization is valid only under the additional assumption that the weak Continuum Hypothesis ($2^\kappa = 2^\lambda \Rightarrow \kappa = \lambda$) holds. We show that there exists a model of ZFC in which this characterization fails (Corollary 2.6). As a consequence of the proof of Corollary 2.6, we obtain that there are non-isomorphic abelian groups A and C such that, for any set $\mathcal{X}$ of abelian groups, there is a model of ZFC in which $\text{Hom}(A, G) \cong \text{Hom}(C, G)$ for all $G \in \mathcal{X}$.

Section 3 addresses Problem 34 for torsion-free groups of finite rank. The restriction to finite rank is necessary to avoid set-theoretic problems similar to those described in the context of p-groups. Our arguments are general enough to hold for

Received by the editors October 1, 1993 and, in revised form, December 12, 1993.
1991 *Mathematics Subject Classification.* Primary 20K15, 20K30; Secondary 20J05.
Key words and phrases. Homomorphism group, Martin's Axiom, p-group, torsion-free group.

modules over integral domains. We want to remind the reader that the concepts of quasi-isomorphism and quasi-equality carry over from torsion-free abelian groups to torsion-free modules over an integral domain R. Theorem 3.2 shows that two torsion-free R-modules A and C whose ranks do not exceed $m < \omega$ are determined up to quasi-isomorphism by the modules $\mathrm{Hom}_R(A, G)$ where $G \in \mathcal{X} = \{G | G$ is torsion-free and $r_0(G) \leq m\}$. Example 3.5 shows that the set $\mathcal{X}$ does not determine A and C up to isomorphism in general.

2. A negative answer to Fuchs' Problem 34

The first result of this section shows that there is a negative answer to [4, Problem 34]. Since the groups A and C which are to be constructed will be p-groups, it is enough to restrict our attention to the case that $\mathcal{X}$ is a set of abelian p-groups. In the following, we write $f_\sigma(A)$ for the σth-Ulm-Kaplansky-Invariant $\dim_{\mathbb{Z}/p\mathbb{Z}} p^\sigma A[p]/p^{\sigma+1} A[p]$ of an abelian p-group A.

Theorem 2.1. *For any set $\mathcal{X}$ of reduced abelian p-groups, there exist two nonisomorphic, totally projective p-groups A_1 and A_2 such that the groups* $\mathrm{Hom}(A_1, G)$ *and* $\mathrm{Hom}(A_2, G)$ *are isomorphic for all $G \in \mathcal{X}$.*

Proof. Let $\tau = \sup\{\mathrm{length}(G) | G \in \mathcal{X}\}$, and choose a cardinal $\kappa > \tau + \omega$. For an ordinal $\sigma < \tau$, define $g_1(\sigma) = g_2(\sigma) = \kappa$. For $n < \omega$, set $g_1(\tau + n) = \aleph_0$ if n is even, and $g_1(\tau + n) = 0$ if n is odd. Similarly, $g_2(\tau + n) = 0$ for even n and $\aleph_0$ for odd n. Clearly, $\sup\{\sigma + 1 | g_i(\sigma) \neq 0\} = \tau + \omega$. Moreover, if σ is an ordinal with $\sigma + \omega < \tau + \omega$, then $\sigma < \tau$ and $\Sigma_{n<\omega} g_i(\sigma + n) \geq g_i(\sigma) = \kappa$ while $\Sigma_{\rho \geq \sigma+\omega} g_i(\rho) \leq \kappa$. This shows that g_1 and g_2 are $\tau + \omega$-admissible functions. By [4, Theorem 83.6], there are non-isomorphic totally projective p-groups A_1 and A_2 of length $\tau + \omega$ such that $f_\sigma(A_i) = g_i(\sigma)$ for all $\sigma < \tau + \omega$.

Observe that A_1 and A_2 have p-basic subgroups B_1' and B_2' isomorphic to $\bigoplus_{n<\omega} \bigoplus_\kappa \mathbb{Z}/p^n\mathbb{Z}$. Since $\bigoplus_{n<\omega} \mathbb{Z}/p^n\mathbb{Z}$ contains a proper p-basic subgroup, we can find a p-basic subgroup B_i of B_i' with $B_i'/B_i \cong \bigoplus_\kappa \mathbb{Z}(p^\infty)$. Since $|A_i| = \Sigma_{\sigma<\tau+\omega} g_i(\sigma) = \kappa$, we have $A_i/B_i \cong \bigoplus_\kappa \mathbb{Z}(p^\infty)$. If D is a divisible p-group, then

$$\begin{aligned} \mathrm{Hom}(A_i, D) &\cong \mathrm{Hom}(B_i, D) \oplus \mathrm{Hom}(A_i/B_i, D) \\ &\cong \mathrm{Hom}(B_i, D) \oplus \left[\prod_\kappa \mathrm{Hom}(\mathbb{Z}(p^\infty), D)\right]. \end{aligned}$$

Therefore, $\mathrm{Hom}(A_1, D) \cong \mathrm{Hom}(A_2, D)$.

Let $G \in \mathcal{X}$. Write $G = D \oplus E$ such that D is divisible and E is reduced. By what has been shown so far, it is enough to establish $\mathrm{Hom}(A_1, E) \cong \mathrm{Hom}(A_2, E)$. The choice of τ guarantees that E is a reduced p-group of length at most τ. Therefore, $\mathrm{Hom}(A_i, E) \cong \mathrm{Hom}(A_i/p^\tau A_i, E)$. Since $A_1/p^\tau A_1 \cong A_2/p^\tau A_2$, we indeed have $\mathrm{Hom}(A_1, E) \cong \mathrm{Hom}(A_2, E)$. □

We now investigate properties shared by p-groups A and C for which $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all G in a family $\mathcal{X}$ of p-groups. Our first result shows that the finite Ulm-Kaplansky invariants can be recovered in this way:

Theorem 2.2. *Consider the following conditions for a class $\mathcal{X}$ of p-groups*:

(a) *If abelian p-groups A and C satisfy* $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all $G \in \mathcal{X}$, then $f_n(A) = f_n(C)$ for all $n < \omega$.*

(b) *For every* $0 < n < \omega$, *there is a group* $G \in \mathcal{X}$ *with* $0 < |p^n G[p]| < \infty$ *and* $f_{n-1}(G) = 0$.

Then, (a) *always implies* (b)*, and the converse holds if one assumes the weak GCH, i.e.* $\kappa < \lambda$ *implies* $2^\kappa < 2^\lambda$ *for all cardinals* κ *and* λ.

Proof. Let G be any abelian p-group with $p^n G[p]$ finite; and choose a p-basic subgroup $B = \bigoplus_{n=1}^\infty B_n$ of G where $B_n \cong \bigoplus_{I_n} \mathbb{Z}(p^n)$ for some index-set I_n. Then, $G = B_1 \oplus \cdots \oplus B_n \oplus D$ for some subgroup D of G. Since D has no cyclic summands of order less than p^{n+1} and $p^n D[p]$ is finite, $D = T \oplus [\bigoplus_m \mathbb{Z}(p^\infty)]$ where T is a finite group and $m < \omega$.

(a) $\Rightarrow$ (b): Suppose that (b) fails for the positive integer n. We consider the groups $A_i = \bigoplus_\omega \mathbb{Z}(p^i)$ for $i = 1, \dots, n-1$ and $i = n+1$. Set $A = A_1 \oplus \cdots \oplus A_{n-1} \oplus \mathbb{Z}(p^n) \oplus A_{n+1}$ and $C = A \oplus \mathbb{Z}(p^n)$. Since $f_{n-1}(A) = 1$ and $f_{n-1}(C) = 2$, there is a group $G \in \mathcal{X}$ with $\mathrm{Hom}(A,G) \not\cong \mathrm{Hom}(C,G)$. If $p^n G[p] = 0$, then $p^n G = 0$, so that any map from A_{n+1} into G contains $p^n A = p^n A_{n+1}$ in its kernel. Consequently, $\mathrm{Hom}(A,G) \cong \mathrm{Hom}(A' \oplus (A_{n+1}/p^n A_{n+1}), G)$ and $\mathrm{Hom}(C,G) \cong \mathrm{Hom}(C' \oplus (A_{n+1}/p^n A_{n+1}), G)$ where $A' = A_1 \oplus \cdots \oplus A_{n-1} \oplus \mathbb{Z}(p^n)$ and $C' = A' \oplus \mathbb{Z}(p^n)$. However, $A' \oplus (A_{n+1}/p^n A_{n+1}) \cong C' \oplus (A_{n+1}/p^n A_{n+1})$, so that $G[p^n] = 0$ is impossible. Suppose that $p^n G[p]$ is infinite, and choose a p-basic subgroup $B = \bigoplus_{n=1}^\infty B_n$ of G where each B_n is a direct sum of cyclics of order p^n. We write $G = B_1 \oplus \cdots \oplus B_n \oplus E$ for some subgroup E of G. Observe $\mathrm{Hom}(A_i, G) \cong \Pi_\omega \mathrm{Hom}(\mathbb{Z}(p^i), G) \cong \Pi_\omega G[p^i]$ for $i = 1, \dots, n-1$, $n+1$ and $\mathrm{Hom}(\mathbb{Z}(p^n), G) \cong G[p^n] = B_1 \oplus \cdots \oplus B_n \oplus E[p^n]$. We can write $E[p^n] = L_1 \oplus \cdots \oplus L_n$ where each L_i is a direct sum of cyclic groups of order p^i. By Szele's Theorem, $B_i \oplus L_i$ is a direct summand of $G[p^i]$, and hence $\Pi_\omega G[p^i] \oplus B_i \oplus L_i \cong \Pi_\omega G[p^i]$. This shows $\mathrm{Hom}(A,G) \cong \Pi_\omega [\bigoplus_{i=1}^{n-1} G[p^i] \oplus G[p^{n+1}]] \oplus B_n \oplus L_n$ while $\mathrm{Hom}(C,G) \cong \mathrm{Hom}(A,G) \oplus B_n \oplus L_n$. Once we have shown that $B_n \oplus L_n$ is infinite, we obtain $(B_n \oplus L_n)^2 \cong B_n \oplus L_n$ and $\mathrm{Hom}(A,G) \cong \mathrm{Hom}(C,G)$.

If $B_n \oplus E$ has an infinite p-basic subgroup B', then B' has no cyclic direct summands of order less than p^n. In particular, $B'[p^n]$ is an infinite direct sum of groups isomorphic to $\mathbb{Z}(p^n)$. By Szele's Theorem, $B'[p^n]$ is a direct summand of $G[p^n]$, which is only possible if $B_n \oplus L_n$ is infinite. If $B_n \oplus E$ has a finite p-basic subgroup, then $B_n \oplus E \cong T \oplus \bigoplus_I \mathbb{Z}(p^\infty)$ for some index-set I and some finite group T. If I were finite, then $p^n G = p^n E$ would have a finite p-socle, contrary to our assumption. Thus, I is infinite, and $B_n \oplus E$ contains a subgroup D which is an infinite direct sum of cyclic groups of order p^n. As before, D is a direct summand of $G[p^n]$, and $B_n \oplus L_n$ is infinite. Consequently, $p^n G[p]$ is finite.

Since (b) fails for n and $0 < |p^n G[p]| < \infty$, we have $f_{n-1}(G) \neq 0$. Since E has no cyclic summands of order less than p^{n+1}, we obtain $p^{n-1} E[p] = p^n E[p]$. Therefore, $p^{n-1} G[p] / p^n G[p] \cong p^{n-1} B_n[p]$ is non-zero. By the initial remarks to this proof, E is finitely cogenerated; and L_n is finite. We write $G[p^{n+1}] = B_1 \oplus \cdots \oplus B_n \oplus K_1 \oplus \cdots \oplus K_{n+1}$ where $K_i \cong \bigoplus_{J_i} \mathbb{Z}(p^i)$ is a subgroup of E. Since $B_n \neq 0$, we have $[\Pi_\omega B_n] \oplus B_n \oplus L_n \cong \Pi_\omega B_n$, and $[\Pi_\omega G[p^{n+1}]] \oplus B_n \oplus L_n \cong \Pi_\omega G[p^{n+1}]$. This shows $\mathrm{Hom}(A,G) \cong \Pi_\omega [\bigoplus_{i=1}^{n-1} G[p^i] \oplus G[p^{n+1}]] \cong \mathrm{Hom}(C,G)$. The resulting contradiction establishes (b).

(b) $\Rightarrow$ (a): Assume the weak continuum hypotheses. Let A and C be abelian p-groups with $\mathrm{Hom}(A,G) \cong \mathrm{Hom}(C,G)$ for all $G \in \mathcal{X}$. Choose a p-basic subgroup $B = \bigoplus_{n=1}^\infty B_n$ of A with B_n a direct sum of cyclics of order p^n. If $B_n \bigoplus_{\sigma_n} \mathbb{Z}(p^n)$, then $f_{n-1}(A) = \sigma_n$. Choose a group $G \in \mathcal{X}$ with $0 < |p^n G[p]| < \infty$ and $f_{n-1}(G) =$

0. By the initial remarks of this proof, $G = D_0 \oplus [\bigoplus_{i=1}^m D_i] \oplus S$ where $p^{n-1}D_0 = 0$, $D_i \cong \bigoplus_{t_i} \mathbb{Z}(p^{s_i})$ with integers $t_i > 0$ and $s_i > n$ for $i = 1, \dots, m$, and $S \cong \bigoplus_r \mathbb{Z}(p^\infty)$ for some $r < \omega$. Observe that G has no direct summand isomorphic to $\mathbb{Z}(p^n)$ since $f_{n-1}(G) = 0$. We now determine the $(n-1)^{st}$-Ulm-Kaplansky-invariant of the torsion subgroup $t\mathrm{Hom}(A, G)$ of $\mathrm{Hom}(A, G)$:

Write $A = B_1 \oplus \cdots \oplus B_n \oplus E$, and obtain a decomposition

$$\mathrm{Hom}(A, G) \cong \mathrm{Hom}(B_1 \oplus \cdots \oplus B_{n-1}, G) \oplus \mathrm{Hom}(B_n \oplus E, G).$$

Since the first summand in this direct sum is annihilated by p^{n-1}, we have

$$f_{n-1}(t\mathrm{Hom}(A, G)) = f_{n-1}(t\mathrm{Hom}(B_n \oplus E, G))$$

is the sum of $f_{n-1}(\mathrm{Hom}(B_n, \bigoplus_{i=1}^m D_i \oplus S))$ and $f_{n-1}(t\mathrm{Hom}(E, \bigoplus_{i=1}^m D_i \oplus S))$. To show that the second term vanishes, we observe that

$$\mathrm{Hom}(E, D_i) \cong \mathrm{Hom}(E/p^{s_i}E, D_i) \cong \mathrm{Hom}\left(\bigoplus_{j=n+1}^{\infty} B_j/p^{s_i}B_j, D_j\right)$$

is bounded, and hence a direct sum of cyclic groups of order at least p^{n+1}. Therefore, $f_{n-1}(\mathrm{Hom}(E, D_i)) = 0$. If $S \neq 0$, then

$$\mathrm{Hom}(E, S) \cong \bigoplus_r [\Pi_{j \geq n+1}(\Pi_{L_j}\mathbb{Z}(p^j)) \oplus \Pi_L J_p]$$

for index-sets L and $L_{n+1}, \dots$. Therefore, $\bigoplus_r [\bigoplus_{j \geq n+1}(\Pi_{L_j}\mathbb{Z}(p^j))]$ is a p-basic subgroup of $t\mathrm{Hom}(E, S)$. Consequently, $f_{n-1}(t\mathrm{Hom}(E, S)) = 0$. This shows

$$\begin{aligned} f_{n-1}(t\mathrm{Hom}(A, G)) &= f_{n-1}\left(\mathrm{Hom}\left(B_n, \bigoplus_{i=1}^m D_i \oplus S\right)\right) \\ &= \begin{cases} \sigma_n(t_1 + \cdots + t_m + r) & \text{if } \sigma_n \text{ is finite,} \\ 2^{\sigma_n} & \text{otherwise.} \end{cases} \end{aligned}$$

If $f_{n-1}(C) = \tau_n$, then a similar computation shows that σ_n is infinite if and only if τ_n is. For finite values of σ_n, we have $\sigma_n = \tau_n$ since $t_1 + \cdots + t_n + r \neq 0$. If σ_n is infinite, then $2^{\sigma_n} = 2^{\tau_n}$ yields $\sigma_n = \tau_n$ because we assume the weak GCH. □

The weak GCH in the proof of implication (b) ⇒ (a) in the last result can be removed if $\mathcal{X}$ contains all bounded p-groups. We want to remind the reader that a homomorphism $\alpha : G \to H$ between p-groups is small if, for every $m < \omega$, there is $n < \omega$ such that $\alpha(p^n A[p^m]) = 0$. The collection of all small homomorphisms between G and H forms a subgroup of $\mathrm{Hom}(G, H)$ which is denoted by $\mathrm{Hom}_S(G, H)$.

Theorem 2.3. *Let $\mathcal{X}$ be a class of p-groups which contains all bounded p-groups. The following conditions are equivalent for abelian p-groups A and C:*

(a) $f_n(A) = f_n(C)$ *for all* $n < \omega$.
(b) $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all torsion-complete p-groups G.*
(c) $\mathrm{Hom}_S(A, G) \cong \mathrm{Hom}_S(C, G)$ *for all* $G \in \mathcal{X}$.
(d) $t\mathrm{Hom}(A, G) \cong t\mathrm{Hom}(C, G)$ *for all* $G \in \mathcal{X}$.

Proof. (a) ⇒ (b): If B is a p-basic subgroup of A, then $B = \bigoplus_{n=1}^\infty B_n$ with $B_n \cong \bigoplus_{f_{n-1}(A)} \mathbb{Z}(p^n)$. Since torsion-complete groups are injective with respect to pure-exact sequences of abelian p-groups, we obtain an exact sequence $\mathrm{Hom}(A/B, G) \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G) \to 0$. If G is torsion-complete, then G is reduced so that

$\mathrm{Hom}(A/B, G) = 0$. Therefore, $\mathrm{Hom}(A, G)$ is completely determined by B. Since A and C have isomorphic p-basic subgroups, we have $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$.

(a) $\Rightarrow$ (c): Let G be in $\mathcal{X}$. As before, let B be a p-basic subgroup of A. We consider the induced exact sequence $0 \to \mathrm{Hom}(A/B, G) \to \mathrm{Hom}(A, G) \xrightarrow{\alpha} \mathrm{Hom}(B, G)$ in which α is the restriction map. Clearly, $\alpha(\mathrm{Hom}_S(A, G)) \subseteq \mathrm{Hom}_S(B, G)$. By [4], for every small homomorphism $\sigma : B \to G$, there is a small homomorphism $\tau : A \to G$ with $\tau|B = \sigma$. Hence, $\alpha(\mathrm{Hom}_S(A, G)) = \mathrm{Hom}_S(B, G)$. Let $\sigma : A \to G$ be a small homomorphism with $\sigma(B) = 0$. For any $m < \omega$, there is an $n < \omega$ with $\sigma(p^n A[p^m]) = 0$. Let $a \in A[p^m]$. Since $p^n A + B = A$, we can write $a = p^n a' + b$ for some $a' \in A$ and $b \in B$. We have $p^{n+m} a' + p^m b = 0$. Therefore, there is $b' \in B$ with $p^{n+m} a' = p^{n+m} b'$. Then, $p^n a' - p^n b' \in p^n A[p^m]$. Thus, $\sigma(a) = \sigma(p^n a' - p^n b') + \sigma(b + p^n b') = 0$. This shows that $\alpha : \mathrm{Hom}_S(A, G) \to \mathrm{Hom}_S(B, G)$ is an isomorphism, which establishes $\mathrm{Hom}_S(A, G) \cong \mathrm{Hom}_S(C, G)$.

(c) $\Rightarrow$ (d): Since every torsion element of $\mathrm{Hom}(A, G)$ is a small homomorphism, the given isomorphism $\mathrm{Hom}_S(A, G) \cong \mathrm{Hom}_S(C, G)$ induces the desired isomorphism between $t\mathrm{Hom}(A, G)$ and $t\mathrm{Hom}(C, G)$.

(b) or (d) $\Rightarrow$ (a): Suppose that one of the two conditions holds, and let n be given. For a chosen cardinal κ, let $G = \bigoplus_\kappa \mathbb{Z}(p^{n+1})$, an element of $\mathcal{X}$. Then, $\mathrm{Hom}(A, G)$ is bounded, and hence $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$. In a way similar to that used in the proof of implication (b) $\Rightarrow$ (a) of Theorem 2.2, we obtain

$$f_n(\mathrm{Hom}(A, G)) = \begin{cases} f_n(A)\kappa & \text{if } f_n(A) \text{ is finite,} \\ \kappa^{f_n(A)} & \text{otherwise.} \end{cases}$$

If $f_n(A)$ is finite, we choose κ to be finite, from which $f_n(A) = f_n(C)$ follows. We now consider the case that $f_n(A)$ and $f_n(C)$ are infinite. Without loss of generality, we may assume $f_n(A) \leq f_n(C)$. Let κ be a strong limit cardinal with $\mathrm{cf}(\kappa) = (f_n(A))^+$ and $f_n(C) < \kappa$. For every $\lambda < \kappa$ we have $\lambda^{f_n(C)} < \kappa$ since κ is a strong limit cardinal. Therefore, $\kappa^{f_n(C)} = \kappa^{\mathrm{cf}(\kappa)} > \kappa$ by König's Theorem. On the other hand, if $f_n(A) < f_n(C)$, then $\kappa^{f_n(A)} = \kappa$. This shows $\mathrm{Hom}(A, G) \not\cong \mathrm{Hom}(C, G)$, a contradiction. Thus, $f_n(A) = f_n(C)$. □

As a consequence of the arguments used in the proof of the last implication of Theorem 2.3, we obtain

Corollary 2.4. *Let A and C be either both direct sums of cyclics or both torsion-complete. The following statements are equivalent*:

(a) $A \cong C$.
(b) $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all p-groups G which are direct sums of cyclics.*
(c) $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all p-groups G.*

However, without some immediate restrictions on the structure of A as in Corollary 2.4, it is impossible to show that A is determined up to isomorphism by the structure of the groups $\{\mathrm{Hom}(A, G) | G \text{ is a } p\text{-group}\}$.

Example 2.5. *There exist abelian p-groups A and C such that $f_n(A) = f_n(C)$ for all $n < \omega$, but $\mathrm{Hom}(A, G) \not\cong \mathrm{Hom}(C, G)$ for some p-group G.*

Proof. Let $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)$ and C be the torsion-complete group with p-basic subgroup A. Then, $\mathrm{Hom}(A, \mathbb{Z}(p^\infty)) \cong \prod_{n=1}^{\infty} \mathbb{Z}(p^n)$ has cardinality $2^{\aleph_0}$. On the other hand, $\mathrm{Hom}(C, \mathbb{Z}(p^\infty))$ contains $\mathrm{Hom}(C/A, \mathbb{Z}(p^\infty))$ as a subgroup. Since

$C/A \cong \bigoplus_{2^{\aleph_0}} \mathbb{Z}(p^\infty)$, we have $|\mathrm{Hom}(C, \mathbb{Z}(p^\infty))| \geq |\Pi_{2^{\aleph_0}} J_p| = 2^{2^{\aleph_0}}$. Therefore, $\mathrm{Hom}(A, \mathbb{Z}(p^\infty)) \not\cong \mathrm{Hom}(C, \mathbb{Z}(p^\infty))$, while $f_n(A) = f_n(C) = 1$ for all $n < \omega$. □

Although Theorem 2.3 shows that it is possible to remove the set-theoretic assumptions from Theorem 2.2 if $\mathcal{X}$ is a class which contains all bounded p-groups, this is not possible if $\mathcal{X}$ is a set as is required in [4, Problem 34]:

Corollary 2.6. (a) *The following conditions are equivalent if both A and C are countable p-groups or the weak GCH holds*:

(i) $f_n(A) = f_n(C)$ *for all* $n < \omega$.
(ii) $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all cyclic p-groups G.*
(iii) $\mathrm{char}(A) \cong \mathrm{char}(C)$.

(b) *Let $\aleph_\nu$ be an uncountable cardinal number, and assume that Martin's Axiom and $2^{\aleph_0} > \aleph_\nu$ hold in addition to ZFC. There exist abelian p-groups A and C with $|A|$, $|C| \leq \aleph_\nu$ which satisfy $f_0(A) \neq f_0(C)$ and $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all abelian groups G with $|G| \leq \aleph_\nu$.*

Proof. (a) It remains to consider the case that A and C are countable: The arguments of the proof of Theorem 2.3 can be used to establish $f_n(A) = f_n(C)$ if $f_n(A)$ or $f_n(C)$ is finite. Therefore, $f_n(A)$ and $f_n(C)$ are either both finite or both infinite. They have to coincide in the latter case since $f_n(A)$ and $f_n(C)$ are countable.

(b) Let $A = \bigoplus_{\aleph_0} \mathbb{Z}(p)$ and $C = \bigoplus_{\aleph_1} \mathbb{Z}(p)$. If $G[p] = 0$, then $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G) = 0$. If $|G| \leq \aleph_\nu$ and $G[p] \neq 0$, then $\mathrm{Hom}(A, G) \cong \Pi_{\aleph_0} G[p]$ has cardinality $|G[p]|^{\aleph_0} = 2^{\aleph_0}$ since $2^{\aleph_0} > \aleph_\nu$. On the other hand, $\mathrm{Hom}(C, G)$ has cardinality $|G[p]|^{\aleph_1} = 2^{\aleph_1}$ for these G by the same argument. By the Martin-Solovay Theorem [5, Theorem 52], $2^{\aleph_1} = 2^{\aleph_0}$, and $\mathrm{Hom}(A, G)$ and $\mathrm{Hom}(C, G)$ are isomorphic since both are $\mathbb{Z}(p)$-vector-spaces. □

In particular, the abelian groups A and C given in the proof of part (b) of the last corollary have the following property: For a set $\mathcal{X}$ of abelian groups, choose an uncountable cardinal number $\aleph_\nu$ with $\aleph_\nu > \sup\{|G| \,|G \in \mathcal{X}\}$. By part (b) of the last corollary, we can find a model of ZFC in which $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all $G \in \mathcal{X}$.

As a consequence of the last result, we obtain that the statement "Two p-groups A and C which are either torsion-complete or a direct sum of cyclics are isomorphic if and only if $\mathrm{char}(A) \cong \mathrm{char}(C)$" is undecidable in ZFC.

3. Torsion-free groups

In this section, we restrict our attention to the case in which we compare the groups $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, B)$ for B chosen from a family of torsion-free abelian groups. The first result is categorical in nature:

Theorem 3.1. *Let R be an integral domain. Two R-modules A and C are isomorphic if and only if the modules $\mathrm{Hom}_R(A, B)$ and $\mathrm{Hom}_R(C, B)$ are naturally isomorphic for all R-modules B.*

Proof. If $\phi : A \to C$ is an isomorphism, then the induced isomorphisms

$$\mathrm{Hom}(\phi, B) : \mathrm{Hom}_R(C, B) \to \mathrm{Hom}_R(A, B)$$

are natural.

Conversely, consider a collection $\{\Phi_B : \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(C,B) | B$ is an R-module$\}$ of natural isomorphisms. Let $\alpha = \Phi_C^{-1}(1_C)$ and $\beta = \Phi_A(1_A)$. By the naturality of the maps Φ, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(A,\ A) & \xrightarrow{\mathrm{Hom}(A,\ \alpha)} & \mathrm{Hom}_R(A,\ A) \\ \Phi_A \big\downarrow \wr & & \Phi_C \big\downarrow \wr \\ \mathrm{Hom}_R(C,\ A) & \xrightarrow{\mathrm{Hom}(C,\ \alpha)} & \mathrm{Hom}_R(C,\ C) \end{array}$$

commutes. Thus, $[\Phi_C \mathrm{Hom}(A,\alpha)](1_A) = [\mathrm{Hom}(C,\alpha)\Phi_A(1_A)$, from which we obtain]

$$1_C = \Phi_C(\alpha) = \Phi_C \mathrm{Hom}(A,\alpha)(1_A) = \mathrm{Hom}(C,\alpha)\Phi_A(1_A) = \alpha\beta.$$

Since the maps Φk_B^{-1} also form a natural collection of isomorphisms, we obtain $\beta\alpha = 1$ in the same way. $\square$

Evidently, the last theorem carries over literally to the quasi-category of torsion-free modules over an integral domain R. However, we can improve it in this case by not only removing the naturality requirement on the isomorphism between $\mathrm{Hom}_R(A,B)$ and $\mathrm{Hom}_R(C,A)$, but also by restricting the ranks of the modules B which need to be considered. We want to remind the reader that the rank of a torsion-free module M over an integral domain R is the dimension of the vector-space $Q(R) \otimes_R M$ where $Q(R)$ is the field of quotients of R.

Theorem 3.2. *Let R be an integral domain, and $m < \omega$. Two torsion-free R modules A and C of rank at most m are quasi-isomorphic if and only if* $\mathrm{Hom}_R(A,B)$ *and* $\mathrm{Hom}_R(C,B)$ *are quasi-isomorphic for all torsion-free R-modules B whose rank is at most m.*

Proof. Let A and M be torsion-free R-modules whose ranks are at most m. It is enough to verify that A and C are quasi-isomorphic when $\mathrm{Hom}_R(A,B)$ and $\mathrm{Hom}_R(C,B)$ are quasi-isomorphic for each torsion-free module B of rank not exceeding m. The A-*socle of* M, which is denoted by $S_A(M)$, is defined to be $\mathrm{Hom}_R(A,M)A$. Since $\mathrm{Hom}_R(A,C) = \mathrm{Hom}_R(A,S_A(C))$, we have that $\mathrm{Hom}_R(C,S_A(C))$ and $\mathrm{Hom}_R(C,C)$ are quasi-isomorphic R-modules. Observing that the module $\mathrm{Hom}_R(C,C)$ has finite rank, we obtain that $\mathrm{Hom}_R(C,S_A(C))$ contains $d\mathrm{Hom}_R(C,C)$ for some non-zero $d \in R$. Therefore, $d1_C \in \mathrm{Hom}_R(C,S_A(C))$. This shows $dC \subseteq S_A(C)$. Likewise, $d'A \subseteq S_C(A)$ for some non-zero $d' \in R$. Without loss of generality, we may assume $A = A_1 \oplus \cdots \oplus A_n$ and $C = C_1 \oplus \cdots \oplus C_m$ where each A_j and C_i is a strongly indecomposable R-module since $Q(R)E_R(A)$ and $Q(R)E_R(C)$ are Artinian rings. Let N be the nilradical of the endomorphism ring, $E_R(C)$, of C. We know that N contains $\mathrm{Hom}_R(C_i,C_j)$ whenever C_i is not quasi-isomorphic to C_j. Also, for quasi-isomorphic pairs C_i and C_j, the ideal N contains all those elements of $\mathrm{Hom}_R(C_i,C_j)$ which are not monomorphisms.

Since N is nilpotent, C/NC has positive torsion-free rank. If this were false, then C/N^kC would be a torsion module for all $k < \omega$ which is not possible. Since $S_C(A) \doteq A$ and $S_A(C) \doteq C \nsubseteq NC$, there is a map $\phi : S_C(A) \to C$ such that $\mathrm{Im}\,\phi \nsubseteq NC$. Since $S_C(A) = S_C(A_1) \oplus \cdots \oplus S_C(A_n)$, we may assume without loss of generality that $\phi(S_C(A_1)) \nsubseteq NC$. Moreover, $S_C(A_1) = \Sigma S_{C_j}(A_1)$ yields $\phi(S_{C_j}(A_1)) \nsubseteq NC$ for some j. Choose elements $c_1, \ldots, c_r \in C_j$ and $\gamma_1, \ldots, \gamma_r \in$

$\mathrm{Hom}_R(C_j, A_1)$ with $\phi(\Sigma_{i=1}^r \gamma_i(c_i)) \notin NC$. Evidently, not all the maps $\phi\gamma_i$ can be in N, say $\phi\gamma_1 \notin N$.

Let $\pi_i : C \to C_i$ denote the natural projection. Observe that $\pi_i\phi\gamma_1$ is an endomorphism of C. However, not all the elements of the form $\pi_i\phi\gamma_1$ are in N, say $\pi_1\phi\gamma_1 \notin N$. Since $\mathrm{Hom}_R(C_1, C_j) \subseteq N$ if C_1 and C_j are not quasi-isomorphic, we obtain $C_1 \sim C_j$ and the fact that $\pi_1\phi\gamma_1$ is a monomorphism. In particular, γ_1 has to be a monomorphism.

Since $C_j \sim C_1$, there is a monomorphism $\delta : C_1 \to A_1$ such that $\pi_1\phi\delta \in E(C_1)$ is a monomorphism. Therefore, $\pi_1\phi\delta$ is invertible in $Q(R)E_R(C_1)$ since C_1 is strongly indecomposable. This shows that δ quasi-splits. Since A_1 is strongly indecomposable, this is only possible if $A_1 \sim C_1$.

For a torsion-free module B of finite rank, we have

$$\begin{aligned}\mathrm{Hom}_R(A, B) &\sim \mathrm{Hom}_R(A_1, B) \oplus \mathrm{Hom}_R(A_2 \oplus \cdots \oplus A_n, B)\\ &\sim \mathrm{Hom}_R(C_1, B) \oplus \mathrm{Hom}_R(C_2 \oplus \cdots \oplus C_n, B) = \mathrm{Hom}_R(C, B).\end{aligned}$$

By the Krull-Schmidt Theorem in the category of quasi-homomorphisms,

$$\mathrm{Hom}_R(A_2 \oplus \cdots \oplus A_n, B) \sim \mathrm{Hom}_R(C_2 \oplus \cdots \oplus C_m, B).$$

An induction argument shows $A_2 \oplus \cdots \oplus A_n \sim C_2 \oplus \cdots \oplus C_m$. □

Corollary 3.3. *Let R be an integral domain. Two torsion-free R-modules A and C of finite rank are quasi-isomorphic if and only if* $\mathrm{Hom}_R(A, M) \sim \mathrm{Hom}_R(C, M)$ *for all torsion-free R-modules* (*of finite rank*).

If A is a J-group, i.e. a torsion-free group of finite rank such that all groups $C \sim A$ are isomorphic to A, then we obtain

Corollary 3.4. *Let A be a J-group, and C be a torsion-free group of finite rank. Then, $C \cong A$ if and only if* $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(C, B)$ *for all torsion-free modules B of finite rank.*

However, considering the stronger requirement $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(C, B)$ for all groups B with $r_0(B) \leq m$ in Theorem 3.2 does not ensure that torsion-free abelian groups A and C or rank at most m are isomorphic as the following example demonstrates. We want to remind the reader that every strongly indecomposable torsion-free group of rank 2 has a commutative endomorphism ring. It was shown in [3] that such a group is a J-group if and only it is a Murley group, i.e. $r_p(A) \leq 1$ for all primes p.

Example 3.5. *There are torsion-free abelian groups A and C of rank* 2 *which are quasi-isomorphic, but not nearly isomorphic and satisfy* $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(C, B)$ *for all torsion-free groups B with $r_0(B) \leq 2$.*

Proof. We may use the construction in [1] to find a homogeneous torsion-free group A of rank 2 whose type is the type of $\mathbb{Z}_{pq}$, the localization of $\mathbb{Z}$ at the distinct primes p and q, and has the following properties:

(a) $E(A) = \mathbb{Z}_{pq}$.
(b) Every rank 1 image of A is isomorphic to $\mathbb{Z}_p$.

Having done this, we observe that A is not a Murley group as mentioned above since $r_p(A) = 2$. In particular, A is not a J-group, and we can find a torsion-free group C of rank 2 which is quasi-isomorphic, but not isomorphic to A. Since A is semi-local, C is not nearly isomorphic to A either. Let C be quasi-isomorphic to A, and consider a torsion-free abelian group B of rank at most 2. In the first instance, consider the case when rank $B = 1$. If B is not isomorphic to $\mathbb{Q}$ or $\mathbb{Z}_p$, then we have $\mathrm{Hom}(A, B) = \mathrm{Hom}(C, B) = 0$. On the other hand, $\mathrm{Hom}(A, \mathbb{Z}_p)$ is a $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ since every rank 1 quotient of A is isomorphic to $\mathbb{Z}_p$. This shows $\mathrm{Hom}(A, \mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \cong \mathrm{Hom}(C, \mathbb{Z}_p)$. Clearly, $\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(C, \mathbb{Q})$.

Now let B have rank 2. Without loss of generality, we may assume that B is indecomposable. If $\mathrm{Hom}(A, B) \neq 0$, then $S_A(B)$ has rank 1 or 2. In the former case, we can apply the arguments of the previous paragraph to show $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(C, B)$ since $S_C(B)$ is quasi-equal to $S_A(B)$. Therefore, it remains to consider the case that $S_A(B)$ has rank 2. Then, A embeds as a subgroup into B. Because $r_{p'}(A) \leq 1$ for all primes $p' \neq p$, we have that C is isomorphic to a subgroup C' of A whose index is a power of p. If B is p-divisible, we have $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(C, B)$. We, therefore, assume that B is not p-divisible.

In the case that $q^\omega B \neq 0$, we observe that B contains a pure subgroup U isomorphic to $\mathbb{Z}_p$. Since every rank-1 quotient of B contains one of A, the group B/U is a $\mathbb{Z}_p$-module too. This implies that B itself is a $\mathbb{Z}_p$-module. The sequence $0 \to \mathbb{Z}_{pq} \to A \overset{\alpha}{\to} \mathbb{Z}_p \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_p, B) \overset{\alpha_*}{\to} \mathrm{Hom}(A, B) \overset{\gamma}{\to} \mathrm{Hom}(\mathbb{Z}_{pq}, B) \to \mathrm{Ext}(\mathbb{Z}_p, B)$$

for which $\mathrm{Ext}(\mathbb{Z}_p, B) \cong \mathrm{Ext}_{\mathbb{Z}_p}(\mathbb{Z}_p, B) = 0$. Moreover, the groups $\mathrm{Hom}(\mathbb{Z}_p, B)$ and $\mathrm{Hom}(\mathbb{Z}_{pq}, B)$ are isomorphic to B.

Observe, the fact that B is indecomposable guarantees that it is strongly indecomposable in this case. We want to remind the reader that a strongly indecomposable p-local group of rank 2 is a Murley group. In particular, $E(B)$ is a principal ideal domain. Choose an epimorphism $\beta : A \to \mathbb{Z}_p$ independent of α. The map β induces an embedding $\beta_* : \mathrm{Hom}(\mathbb{Z}_p, B) \to \mathrm{Hom}(A, B)$. If $g \in \mathrm{Im}\,\alpha_* \cap \mathrm{Im}\,\beta_*$, then $g = \gamma_1\alpha = \gamma_2\beta$ for some maps $\gamma_i : \mathbb{Z}_p \to B$. For non-zero elements $a \in \ker\alpha$ and $b \in \ker\beta$, we have $A = \langle a, b\rangle_*$. Therefore, $g(a) = 0 = g(b)$, and $g = 0$. Define $\theta : \mathrm{Hom}(\mathbb{Z}_{pq}, B) \to \mathrm{Im}\,\beta_*$ to be any isomorphism. Since $\gamma\theta$ is a non-zero element of $E(\mathrm{Hom}(\mathbb{Z}_{pq}, B)) \cong E(B)$ and $E(B)$ is a principal ideal domain, this implies that $\gamma\theta$ is invertible in $QE(\mathrm{Hom}(\mathbb{Z}_{pq}, B))$. Therefore, the sequence $0 \to \mathrm{Hom}(\mathbb{Z}_p, B) \overset{\alpha_*}{\to} \mathrm{Hom}(A, B) \overset{\gamma}{\to} \mathrm{Hom}(\mathbb{Z}_{pq}, B) \to 0$ quasi-splits. Since $\mathrm{Ext}(B, B)$ is torsion-free, the last sequence splits, i.e. $\mathrm{Hom}(A, B) \cong B \oplus B$. Now $\mathrm{Hom}(C, B) \sim B \oplus B$, so there is a quasi-splitting sequence $0 \to B_1 \to \mathrm{Hom}(C, B) \to B_2 \to 0$ with $B_i \sim B$. But B is a J-group, and so $B_i \cong B$. As before, $\mathrm{Hom}(C, B) \cong B \oplus B$.

It remains to consider the case $q^\omega B = 0$. We shall show that $\mathrm{Hom}(A, B)$ has rank 1. Since $\mathrm{Hom}(C, B)$ is quasi-isomorphic to $\mathrm{Hom}(A, B)$, these two groups have to be isomorphic as well. Set F to be a subgroup of A for which $A/F \cong \mathbb{Z}(q^\infty)$. We then observe that F is a free $\mathbb{Z}_{pq}$-module, and that $A/F \subseteq B/F \subseteq \mathbb{Q}A/F \cong \mathbb{Z}^2(p^\infty) \oplus \mathbb{Z}^2(q^\infty)$. In the case under consideration, B/A does not contain a subgroup isomorphic to $\mathbb{Z}(q^\infty)$. If B/A is finite, then $A \sim B$ and $\mathrm{Hom}(A, B) \cong \mathbb{Z}_{pq}$. If B/A contains $\mathbb{Z}^2(p^\infty)$, then $p^\omega B = B$, a case which has been settled earlier. Thus, we may assume that $B/A \cong \mathbb{Z}(p^\infty) \oplus T$ where T is finite. Let $\phi \in \mathrm{Hom}(A, B)$. Since neither A nor $\phi(A)$ map onto $\mathbb{Z}(p^\infty)$, the subgroup $(\phi(A) + A)/A$ of B/A must be finite. For some non-zero integer n, we have $n\phi(A) \subseteq A$. In particular, $n\phi \in E(A)$, and $\mathrm{Hom}(A, B)$ has rank 1. $\square$

We conclude this paper with a result which shows that the consideration of groups of the form $\mathrm{Hom}(A,T)$ with T torsion does not yield any further information beyond what has been already obtained.

Proposition 3.6. *Let p be a prime. The following conditions are equivalent for torsion-free abelian groups A and C of finite rank*:

(a) $r_p(A)=r_p(C)$*; and* $r_0(A)=r_p(A)$ *if and only if* $r_0(C)=r_p(C)$.
(b) $\mathrm{Hom}(A,G)\cong \mathrm{Hom}(C,G)$ *for all p-groups G.*

Proof. (a) $\Rightarrow$ (b): Let G be a p-group, and F a p-basic subgroup of A. There is an induced exact sequence $0\to F_p\to A_p\to H\to 0$ in which H is a torsion-free divisible group whose rank is $m=r_0(A/F)=r_0(A)-r_p(A)$. This sequence induces the exact sequence $0\to \mathrm{Hom}(\bigoplus_m \mathbb{Q},G)\to \mathrm{Hom}(A_p,G)\to \mathrm{Hom}(F_p,G)\to \mathrm{Ext}(\bigoplus_m \mathbb{Q},G)$. Since $\mathrm{Hom}(\bigoplus_m \mathbb{Q},G)$ and $\mathrm{Ext}(\bigoplus_m \mathbb{Q},G)$ are torsion-free divisible, the short exact sequence $0\to \mathrm{Hom}(\bigoplus_m \mathbb{Q},G)\to \mathrm{Hom}(A_p,G)\to \mathrm{Hom}(F_p,G)\to 0$ splits. Thus, $\mathrm{Hom}(A_p,G)\cong \mathrm{Hom}(\bigoplus_m \mathbb{Q},G)\oplus \bigoplus_{r_p(A)} G$.

Suppose that $m\neq 0$. Then, $r_0(\mathrm{Hom}(\bigoplus_m \mathbb{Q},G))=mr_0(\mathrm{Hom}(\mathbb{Q},G))$. If G is reduced, then $\mathrm{Hom}(\mathbb{Q},G)=0$. Otherwise, $\mathrm{Hom}(\mathbb{Q},G)$ contains a direct summand isomorphic to $\mathrm{Hom}(\mathbb{Q},\mathbb{Z}(p^\infty))$ which has torsion-free rank $2^{\aleph_0}$. Consequently,

$$\mathrm{Hom}(A_p,G)\cong \begin{cases} \bigoplus_{r_p(A)} G & \text{if } r_0(A)=r_p(A),\\ \mathrm{Hom}(\mathbb{Q},G)\oplus \bigoplus_{r_p(A)} G & \text{otherwise.}\end{cases}$$

This shows $\mathrm{Hom}(A_p,G)\cong \mathrm{Hom}(C_p,G)$. On the other hand, there is an exact sequence $0\to A\to A_p\to H'\to 0$ in which H' is a divisible torsion group with $H'[p]=0$. This induces the exact sequence $0=\mathrm{Hom}(H',G)\to \mathrm{Hom}(A_p,G)\to \mathrm{Hom}(A,G)\to \mathrm{Ext}(H',G)=0$.

(b) $\Rightarrow$ (a): Observe $\mathrm{Hom}(A,\mathbb{Z}(p))\cong A/pA\cong \bigoplus_{r_p(A)}\mathbb{Z}(p)$. Therefore, $r_p(A)=r_p(C)$. Assume $r_p(A)<r_0(A)$, but $r_p(C)=r_0(C)$. Using the arguments from the proof of the previous implication, we obtain that $\mathrm{Hom}(\mathbb{Q},\mathbb{Z}(p^\infty))$ is a direct summand of $\mathrm{Hom}(A,\mathbb{Z}(p^\infty))$ while $\mathrm{Hom}(C,\mathbb{Z}(p^\infty))$ is a torsion group. This proves the proposition. □

References

1. R. A. Beaumont and R. S. Pierce, *Torsion-free groups of rank two*, Memoirs of the AMS, Vol. 38 (1961). MR **24:**A162
2. R. A. Beaumont and R. J. Wisner, *Rings with additive group which is torsion-free of rank two*, Acta. Sci. Math. (Szeged) **20** (1959), 105–116. MR **21:**5651
3. T. Faticoni and H. P. Goeters, *On torsion-free* Ext, Comm. in Algebra **16(9)** (1988), 1853–1876. MR **90d:**20098
4. L. Fuchs, *Infinite Abelian Groups*, Vol. I/II, Academic Press, New York, London (1970/73). MR **41:**333; MR **50:**2362
5. T. Jech, *Set Theory*, Academic Press, New York, London (1978). MR **80a:**03062

Department of Mathematics, Auburn University, Auburn, Alabama 36849
E-mail address: albreuf@mail.auburn.edu
E-mail address: goetehp@mail.auburn.edu