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DEGREE OF IRRATIONALITY OF A PRODUCT OF TWO ELLIPTIC CURVES

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ABSTRACT. We consider the degree of irrationality $d_r(S)$ of some algebraic surface S. Firstly we give an estimate of $d_r(S)$ for a surface S with a structure of a fiber space. Secondly we prove the existence of a nonsingular curve of genus 3 on $E \times E$ for a certain elliptic curve E with complex multiplications. As a corollary, we obtain that $d_r(E \times E) = 3$.

1. INTRODUCTION

Let V be an n-dimensional algebraic variety defined over a field k, and let $k(V)$ be the rational function field of V. The degree of irrationality of V is defined to be the least number m such that $m = [k(V)\colon k(x_1, \ldots, x_n)]$, where $x_1, \ldots, x_n$ are algebraically independent elements of $k(V)$ (cf. [6], [9]). By definition this number is a birational invariant and we denote it by $d_r(V)$. In other words it is the minimal degree of a dominant rational map from V to the projective n-space. In the case when $n = 1, d_r(V)$ coincides with the gonality of a curve and has been studied mainly for plane curves (see, e.g., [3]).

In what follows we assume that $k = \mathbf{C}$ and we work in the category of algebraic varieties over $\mathbf{C}$. When $n = 2$ and $d_r(V) = 2$, some results are obtained in [8]. For an abelian variety A, it is proved that $d_r(A) \geq n + 1$ in [1]. Clearly we have that $d_r(A) = 2$ if $n = 1$. It seems to be important to determine the value $d_r(A)$ when $n = 2$, but only a few results have been obtained; for example, if A is a double covering of a Jacobian variety of a curve, then $d_r(A) = 3$ (see [7]). In this paper we will give an estimate of d_r for a surface with a structure of a fiber space and prove the existence of a nonsingular curve of genus 3 on $E \times E$, where E is a certain elliptic curve with complex multiplications. As a corollary, we obtain that $d_r(E \times E) = 3$.

2. STATEMENT OF RESULTS

First we present an estimate of d_r for a surface with a structure of a fiber space.

Proposition 1 (cf. [7])**.** *Let S and C be a nonsingular projective surface and curve, respectively. Suppose that there is a surjective morphism $f\colon S \to C$, whose general*

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fiber F is irreducible. Let $\mathrm{g}(F)$ denote the genus of F. Then we have the following assertions:

(1) *If $\mathrm{g}(F) = 0$, then $d_r(S) = d_r(C)$.*
(2) *If $\mathrm{g}(F) = 1$ and f has a section, then $d_r(S) \leq 2d_r(C)$.*
(3) *If $\mathrm{g}(F) \geq 2$ and $d_r(F) = 2$, then $d_r(S) \leq 2d_r(C)$.*
(4) *If $\mathrm{g}(F) = 3, d_r(F) \neq 2$ and f has a section, then $d_r(S) \leq 3d_r(C)$.*

If we drop the assumption that f has a section in (2), then the conclusion does not hold true. For example let S be a hyperelliptic surface; then it has a structure of an elliptic fiber space with multiple singular fibers. We can shown that $d_r(S) \geq 3$ for some S. The case (4) has been proved by a different method in [7]. Using this proposition, we get

Theorem 2. *If there is a nonsingular curve of genus 3 on an abelian surface A, then $d_r(A) = 3$.*

Note that if there is a nonsingular curve C of genus 2 on A, then A is a Jacobian variety $J(C)$ of C. If an abelian surface B is a double covering of $J(C)$, then $d_r(B) = 3$ by this theorem.

In the sequel we use the following notation. Let m be a positive square free integer. Put $\omega = \sqrt{-m}$ [resp. $\frac{1+\sqrt{-m}}{2}$] if $-m \equiv 2$ or $3 \pmod 4$ [resp. $-m \equiv 1 \pmod 4$]. Let $K = \mathbf{Q}(\sqrt{-m})$ be an imaginary quadratic field. For each $\xi \in K \backslash \mathbf{Q}$, let $a\xi^2 + b\xi + c = 0$ be the equation for ξ with $a, b, c \in \mathbf{Z}, a > 0$ and $(a, b, c) = 1$. Let L be the lattice generated by $\{1, \xi\}$ and let E be the elliptic curve $\mathbf{C}/L$.

Theorem 3. *Under the situation above, suppose that at least one of a, b, c is an even number. Then there exist two elliptic curves E_1 and E_2 on $A = E \times E$ satisfying $(E_1, E_2) = 2$, where (E_1, E_2) denotes the intersection number of E_1 and E_2. Especially there exists a nonsingular curve of genus 3 on A, hence $d_r(A) = 3$.*

Note 4. Of course there are many elliptic curves E satisfying the condition in Theorem 3. In fact, if $-m \equiv 2$ or $3 \pmod 4$, then b is even, because $a\xi$ becomes an integer. Hence every ξ enjoys the condition. For the remaining case, letting k and l $(\neq 0)$ be rational integers, we have the following.

(i) If $-m \equiv 1 \pmod 8$, then $\xi = k + l\omega$ and $\frac{1}{2} + l\omega$ are the suitable ones.
(ii) If $-m \equiv 5 \pmod 8$, then $\xi = k + 2l\omega$ and $\frac{1}{2} + l\omega$ are the suitable ones.

However we notice the following assertion.

Proposition 5. *Suppose that $\xi = \omega$ and $m = 3, 11, 19, 43, 67$ or 163. Then there exist no elliptic curves E_1 and E_2 satisfying $(E_1, E_2) = 2$ on $E \times E$.*

Remark 6. When $m = 3$ and $\xi = \omega$, we consider the quotient of $E \times E$ by the automorphism $(z_1, z_2) \mapsto (\omega z_1, \omega z_2)$. Then the quotient space turns out to be a rational surface and hence $d_r(E \times E) = 3$ (cf. [7]).

Example 7. As an application of Theorem 2, we take an example from [2, (1.8)]. Let D be a nonsingular curve of genus 3 which admits an elliptic involution $\pi \colon D \to E$. Let $e_0 \in D$ be a branchpoint for π and embed $D \to J = \mathrm{Pic}^0(D)$ via $p \mapsto \mathcal{O}_D(p - e_0)$. By using $x_0 = \pi e_0$, we identity $E \to \mathrm{Pic}^0(E)$ via $x \mapsto \mathcal{O}_E(x - x_0)$. Then we have a natural injection $\pi^* \colon E \to J$. We put $A = J/\pi^* E$; then the map $D \hookrightarrow J \to A$ turns into an embedding. Hence we have $d_r(A) = 3$.

3. Proof

First we prove Proposition 1. Let us treat the case (1). Since S is birationally equivalent to $C \times \mathbf{P}^1$, we have that $d_r(S) = d_r(C \times \mathbf{P}^1)$. Then we get $d_r(C \times \mathbf{P}^1) = d_r(C)$ (cf. [9]). Proofs of (2), (3) and (4) are done simultaneously. Let $\mathcal{K}_S$ denote the canonical divisor on S and let Γ be the section in (2) and (4). Let $\mathcal{F}$ be the sheaf on S equal to $\mathcal{O}_S(2\Gamma)$, $\mathcal{O}_S(\mathcal{K}_S + F)$ and $\mathcal{O}_S(\mathcal{K}_S - \Gamma)$, corresponding to the cases (2), (3) and (4) respectively. Since $f_*\mathcal{F}$ is a coherent sheaf on C, we have a projective fiber space $\mathbf{P}(f_*\mathcal{F}) \to C$ associated with $f_*\mathcal{F}$ and a rational map $g\colon S - \to \mathbf{P}(f_*\mathcal{F})$. Let X be the image of g. Then we see that X is a ruled surface over C. In the case (2) or (3), the degree of g is 2, hence we conclude that $d_r(S) \le 2d_r(C)$ by (1). On the other hand, in the case (4), since

$$\dim H^0(F, \mathcal{O}(\mathcal{K}_F - F \cap \Gamma)) = h^0(F, \mathcal{O}(\mathcal{K}_F - F \cap \Gamma)) = 2$$

for a general fiber F and it is not hyperelliptic, the rational map g has degree 3. Hence, similarly we infer that $d_r(S) \le 3d_r(C)$.

Next we prove Theorem 2. Let C be the nonsingular curve of genus 3. Since $C^2 = (C, C) = 4$, we see that C is ample and $h^0(A, \mathcal{O}(C)) = 2$ from the Riemann-Roch theorem. The rational map defined by the complete linear system $|C|$ has 4 base points. By blowing-up these points, we get a morphism $f\colon \widetilde{A} \to \mathbf{P}^1$. Clearly f has 4 sections. As we mentioned in the Introduction, we have that $d_r(A) \ge 3$, hence it is sufficient to show that a general fiber is not hyperelliptic. Suppose that except for finitely many fibers every fiber is hyperelliptic. Then we have $d_r(A) = 2$ by (3), which is a contradiction. Hence a general fiber must be non-hyperelliptic, because in the moduli space of curves of genus 3, hyperelliptic ones consist of an analytic subspace. Thus by (4) we obtain $d_r(A) = 3$.

Before the proof of Theorem 3 we provide two lemmas. The next one may be well-known.

Lemma 8. *Let E be an elliptic curve on an abelian surface A. Then $h^0(A, \mathcal{O}(E)) = 1$.*

Proof. From the exact sequence of sheaves

$$0 \to \mathcal{O}_A(-E) \to \mathcal{O}_A \to \mathcal{O}_E \to 0,$$

we get the long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^1(A, \mathcal{O}(-E)) \to H^1(A, \mathcal{O}_A) \xrightarrow{r} H^1(E, \mathcal{O}_E) \\ \to H^2(A, \mathcal{O}(-E)) \to H^2(A, \mathcal{O}_A) \to 0. \end{aligned}$$

From this sequence and by the Serre duality theorem, we infer that $h^0(A, \mathcal{O}(E)) = \dim H^1(A, \mathcal{O}(-E)) = h^1(A, \mathcal{O}(-E))$. On the other hand, referring to [4, p. 571], we see that $\dim \ker r$ is the number of linearly independent holomorphic 1-forms on A which vanish on E. Whence we have that $h^1(A, \mathcal{O}(-E)) \le 1$, which proves the assertion. □

Lemma 9. *If there are two elliptic curves E_1 and E_2 satisfying $(E_1, E_2) = 2$ on an abelian surface A, then there is a nonsingular curve of genus 3 on A.*

Proof. Putting $D = E_1 + E_2$, we see that D is an ample divisor and hence $h^0(A, \mathcal{O}(D)) = 2$. By the above lemma the pencil $|D|$ has no fixed component. Hence by Bertini's theorem its general member is an irreducible nonsingular curve of genus 3 (cf. [2, (1.4)]). □

Now we proceed to the proof of Theorem 3. Let $\varphi_{\alpha,\beta}\colon E \to E\times E$ be a morphism defined by $\varphi_{\alpha,\beta}(z) = (\alpha z, \beta z)$, where α and $\beta \in \mathrm{End}(E)$. Note that $\mathrm{End}(E)$ is generated by 1 and $a\xi$ over $\mathbf{Z}$. Put $E_{\alpha,\beta} = \varphi_{\alpha,\beta}(E)$. By taking a suitable $(\alpha,\beta;\gamma,\delta)$, we may obtain that $(E_{\alpha,\beta}, E_{\gamma,\delta}) = 2$. For example $(E_{0,1}, E_{2,\lambda}) = 2$ if we take λ as follows: In case a is even, let $\lambda = a\xi$. On the contrary, in case a is odd, let $\lambda = x + y(a\xi)$, where x and y ($\neq 0$) are given as follows: If b and c are even, then let x be even and y be odd. If b or c is odd, then let x and y be odd. By simple calculations we see that the number of the elements of the set $\{(2z, \lambda z) \in E_{2,\lambda} | 2z = 0 \text{ in } E\}$ is 2. Since $E_{0,1}$ and $E_{2,\lambda}$ meet transversally, we have that $(E_{0,1}, E_{2,\lambda}) = 2$. Using Lemma 9, we finish the proof of Theorem 3.

Next we prove Proposition 5. Since $\mathrm{End}(E_i)$ becomes a maximal order of K in this case, we make use of the results of [5]. Suppose that such curves E_i $(i = 1, 2)$ exist. Then E_i is a translation of E_{α_i,β_i} for some $\alpha_i, \beta_i \in \mathrm{End}(E)$ (cf. [5, Lemma 1]). Hence

$$(E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}) = (E_1, E_2) = 2.$$

Moreover, by [5, Corollary 1 on p. 6], we have that

$$(E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}) = \frac{N(\alpha_1\beta_2 - \alpha_2\beta_1)}{N(\alpha_1,\beta_1)N(\alpha_2,\beta_2)},$$

where N denotes the norm. Clearly we also have that

$$(E_{\overline{\alpha}_1\alpha_1,\overline{\alpha}_1\beta_1}, E_{\overline{\alpha}_2\alpha_2,\overline{\alpha}_2\beta_2}) = 2.$$

We can write $\overline{\alpha}_i\alpha_i = c_i a_i$, $\overline{\alpha}_i\beta_i = c_i b_i + c_i\omega$, where $a_i, b_i, c_i \in \mathbf{Z}$ $(i = 1, 2)$ and we may assume that $(c_i a_i, c_i b_i + c_i\omega)$ form a canonical basis. Then we infer from the above that $\gamma\overline{\gamma} = 2a_1a_2$, where $\gamma = (a_1b_2 - b_1a_2) + (a_1 - a_2)\omega$. Since 2 is a prime number in K and the class number of K is 1, we see that a_1 and a_2 are even numbers. Putting $a_i = 2a_i'$, we obtain that $\gamma'\overline{\gamma}' = 2a_1'a_2'$, where $\gamma' = (a_1'b_2 - b_1a_2') + (a_1' - a_2')\omega$. We can repeat the same argument finitely many times, which gives rise to a contradiction.

Finally we mention a problem concerning d_r.

Problem 10. Find the value $d_r(A)$ for each abelian surface A. Especially we ask whether the following assertions hold true:

(1) Is there an abelian surface A satisfying $d_r(A) \geq 4$? For example, is it true that $d_r(E_1 \times E_2) = 4$ if E_1 and E_2 are not isogenous?

(2) If two abelian surfaces A_1 and A_2 are isogenous, then is it true that $d_r(A_1) = d_r(A_2)$?

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