

SOUSLIN'S HYPOTHESIS AND CONVERGENCE IN CATEGORY

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ABSTRACT. A sequence of functions $f_n : X \rightarrow \mathbb{R}$ from a Baire space X to the reals \mathbb{R} is said to converge in category iff every subsequence has a subsequence which converges on all but a meager set. We show that if there exists a Souslin tree, then there exists a nonatomic Baire space X such that every sequence which converges in category converges everywhere on a comeager set. This answers a question of Wagner and Wilczynski who proved the converse.

Suppose that $S \subseteq P(X)$ is a σ -field of subsets of X and $I \subseteq S$ is a σ -ideal. If I has the countable chain condition (ccc), i.e., every family of disjoint sets in $S \setminus I$ is countable, then S/I is a complete boolean algebra. A boolean algebra is atomic iff there is an atom beneath every nonzero element.

A function $f : X \rightarrow \mathbb{R}$ is S -measurable iff $f^{-1}(U) \in S$ for every open set U . A sequence of S -measurable functions $f_n : X \rightarrow \mathbb{R}$ converges I -a.e. to a function f iff there exists $A \in I$ such that $f_n(x) \rightarrow f(x)$ for all $x \in (X \setminus A)$. If (X, S, μ) is a finite measure space, then a sequence of measurable functions $f_n : X \rightarrow \mathbb{R}$ converges in measure to a function f iff for any $\epsilon > 0$ there exists N such that for any $n > N$:

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \epsilon.$$

In this case if I is the ideal of measure zero sets, then f_n converges to f in measure iff every subsequence $\{f_n : n \in A\}$ (where $A \subseteq \mathbb{N}$) has a subsequence $B \subseteq A$ such that $\{f_n : n \in B\}$ converges I -a.e. This allows us to define convergence in measure without mentioning the measure, only the ideal I . So in the abstract setting define the following: f_n converges to f with respect to I iff every subsequence $\{f_n : n \in A\}$ has a subsequence $B \subseteq A$ such that $\{f_n : n \in B\}$ converges I -a.e. (where A and B range over infinite sets of natural numbers.) For more background on this subject in case I is the ideal of meager sets, see Poreda, Wagner-Bojakoska, and Wilczyński [PWW] and Ciesielski, Larson, and Ostaszewski [CLO].

Marczewski [M] showed that if (X, S, μ) is an atomic measure and I the μ -null sets, then ' I -a.e. convergence' is the same as 'convergence with respect to I '.

Gribanov [G] proved the converse, if (X, S, μ) is a finite measure space and I the μ -null sets, then if ' I -a.e. convergence' is the same as 'convergence with respect to I ' then μ is an atomic measure.

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Souslin's Hypothesis (SH) is the statement that there are no Souslin lines. It is known to be independent (see Solovay and Tennenbaum [ST]). It was the inspiration for Martin's Axiom.

Theorem 1 (Wagner and Wilczyński [WW]). *Assume SH. Then for any σ -field S and ccc σ -ideal $I \subseteq S$ the following are equivalent:*

- '*I*-a.e. convergence' is the same as 'convergence with respect to *I*' for *S*-measurable sequences of real-valued functions, and
- the complete boolean algebra S/I is atomic.

At the real analysis meeting in Łódź, Poland in July 1994, Wilczyński asked whether or not SH is needed for the Theorem above. We show here that the conclusion of Theorem 1 implies Souslin's Hypothesis.

Theorem 2. *Suppose SH is false (so there exists a Souslin tree). Then there exists a regular topological space X such that*

1. X has no isolated points,
2. X is ccc (every family of disjoint open sets is countable),
3. every nonempty open subset of X is nonmeager, and
4. if I is the σ -ideal of meager subsets of X , then '*I*-a.e. convergence' is the same as 'convergence with respect to *I*' for any sequence of Baire measurable real-valued functions.

Hence if S is the σ -ideal of sets with the property of Baire and I the σ -ideal of meager sets, then S/I is ccc and nonatomic, but the two types of convergence are the same.

Proof. Define $(T, <)$ to be an ω_1 -tree iff it is a partial order and for each $s \in T$ the set $\{t \in T : t < s\}$ is well-ordered by $<$ with some countable order type, $\alpha < \omega_1$. We let

$$T_\alpha = \{s \in T : \{t \in T : t < s\} \text{ has order type } \alpha\}.$$

Also

$$T_{<\alpha} = \bigcup \{T_\beta : \beta < \alpha\}.$$

Define $C \subseteq T$ is a chain iff for every $s, t \in C$ either $s \leq t$ or $t \leq s$.

Define $A \subseteq T$ is an antichain iff for any $s, t \in A$ if $s \leq t$, then $s = t$, i.e. distinct elements are \leq -incomparable.

Define T is a Souslin tree iff T is an ω_1 -tree in which every chain and antichain is countable. (Note that since T_α is an antichain, it must be countable.)

SH is equivalent to saying there is no Souslin tree. Every Souslin tree contains a normal Souslin tree, i.e., a Souslin tree T such that for every $\alpha < \beta < \omega_1$ and $s \in T_\alpha$ there exists a $t \in T_\beta$ with $s < t$. (Just throw out nodes of T which do not have extensions arbitrarily high in the tree.) For more on Souslin trees see Todorčević [T].

Now we are ready to define our space X . Let the elements of X be maximal chains of T . For each $s \in T$ let

$$C_s = \{b \in X : s \in b\}$$

and let

$$\{C_s : s \in T\}$$

be an open basis for the topology on X . Note that $C_s \cap C_t$ is either empty or equal to either C_s or C_t depending on whether s and t are incomparable, or $t \leq s$ or $s \leq t$, respectively. Each C_s is clopen since its complement is the union of C_t for t which are incomparable to s . X has no isolated points, since given any $s \in T$ there must be incomparable extensions of s (because T is normal) and therefore at least two maximal chains containing s , so C_s is not a singleton. Clearly X has the countable chain condition. \square

Lemma 3. *Open subsets of X are nonmeager. In fact, the intersection of countably many open dense sets contains an open dense set.*

Proof. The proof is quite standard and can be found in the reference books: Kunen [K] or Jech [J]. For the convenience of the reader we include it.

Suppose $(U_n : n \in \omega)$ is a sequence of open dense subsets of X . Let $A_n \subseteq T$ be an antichain which is maximal with respect to the property that $C_s \subseteq U_n$ for each $s \in A_n$. Since U_n is open dense in X , A_n will be a maximal antichain in T .

Let

$$V_n = \bigcup \{C_s : s \in A_n\}.$$

Then $V_n \subseteq U_n$ and V_n is open dense. (It is dense, because given any C_t there exist $s \in A_n$ and $r \in T$ with $t \leq r$ and $s \leq r$, hence $C_r \subseteq V_n \cap C_t$.)

Choose $\alpha < \omega_1$ so that for each $n \in \omega$ the (necessarily countable) antichain $A_n \subseteq T_{<\alpha}$. Let

$$U = \bigcup \{C_s : s \in T_\alpha\}.$$

Note that since T is normal, U is an open dense set. Also

$$U \subseteq \bigcap_{n < \omega} V_n \subseteq \bigcap_{n < \omega} U_n.$$

$U \subseteq V_n$ because for any $b \in U$ if $b \in C_s$ for some $s \in T_\alpha$ there must be $t \in A_n$ comparable to it, since A_n is a maximal antichain, and since $A_n \subseteq T_{<\alpha}$, it must be that $t < s$ and so $b \in C_t \subseteq V_n$. \square

Lemma 4. *Suppose $f : X \rightarrow \mathbb{R}$ is a real-valued Baire function. Then there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ the function f is constant on C_s .*

Proof. Let \mathcal{B} be a countable open basis for \mathbb{R} . For each $B \in \mathcal{B}$ the set $f^{-1}(B)$ has the property of Baire (open modulo meager). So there exists an open U_B such that

$$U_B \Delta f^{-1}(B) \text{ is meager.}$$

By the proof of Lemma 3 we may assume that

$$U_B = \bigcup \{C_s : s \in A_B\}.$$

By the proof of Lemma 3 there exists an $\alpha < \omega_1$ such that

- each $A_B \subseteq T_{<\alpha}$ and
- if U is the open dense set $\bigcup \{C_s : s \in T_\alpha\}$, then U is disjoint from $U_B \Delta f^{-1}(B)$ for each $B \in \mathcal{B}$.

But now, f is constant on each C_s for $s \in T_\alpha$. Otherwise, suppose that $f(b) \neq f(c)$ for some $b, c \in C_s$ for some $s \in T_\alpha$. Then suppose that $f(b) \in B$ and $f(c) \notin B$ for some $B \in \mathcal{B}$. Because $b \in (f^{-1}(B) \cap U)$ and U is disjoint from $U_B \Delta f^{-1}(B)$, it must be that $b \in U_B$. Hence there exists $t \in T_{<\alpha}$ such that $C_t \subseteq U_B$ and $b \in C_t$. Since $t < s$, it must be that $c \in C_t$ and so $c \in f^{-1}(B)$, which contradicts $f(c) \notin B$. \square

Steprańs [S] shows that every continuous function on a Souslin tree takes on only countably many values.

Lemma 5. *Suppose $\{f_n : X \rightarrow \mathbb{R} : n \in \omega\}$ is a countable set of real-valued Baire functions. Then there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ and $n < \omega$ the function f_n is constant on C_s .*

Proof. Apply Lemma 4 countably many times and take the supremum of the α_n . \square

Finally, we prove the theorem. The idea of the proof is to use the argument of the atomic case, where the ‘atoms’ are supplied by Lemma 5. Since ‘ I -a.e. convergence’ always implies ‘convergence with respect to I ’, it is enough to see the converse. So let $f_n : X \rightarrow \mathbb{R}$ be Baire functions which converge to $f : X \rightarrow \mathbb{R}$ with respect to I , i.e. every subsequence has a subsequence which converges on a comeager set to f . By Lemma 5 there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ and $n < \omega$ the function f_n is constant on C_s . Since every subsequence has a convergent subsequence, it must be that for each fixed $s \in T_\alpha$ the constant values of f_n on C_s converge to a constant value. It follows that the sequence $f_n(x)$ converges to $f(x)$ on the dense open set $\{C_s : s \in T_\alpha\}$.

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