

## SINGLE VALUEDNESS FROM WEAKLY COERCIVE HAMILTONIANS

RICHARD T. NEWCOMB II AND JIANZHONG SU

(Communicated by Barbara Lee Keyfitz)

ABSTRACT. We define a class of weakly coercive Hamiltonians and then demonstrate the single valuedness of the associated Hamilton-Jacobi operators (in the viscosity sense).

### 1. INTRODUCTION

The question of the single valuedness of the Hamilton-Jacobi operator of first order has been addressed by various authors and, in general, has been answered in the affirmative based on some special behavior of the associated Hamiltonian at gradients with large norm. The appropriate notion of solution used is that of the theory of viscosity solutions, and hence is a type of weak solution. In particular, single valuedness refers to the property that two Hamilton-Jacobi equations which share the same viscosity solution and the same Hamiltonian operator are identical equations.

We introduce a new class of Hamiltonians which we term “weakly coercive” and show that single valuedness of the associated Hamilton-Jacobi operator holds. The class of weakly coercive Hamiltonians contains the class of “strongly coercive” Hamiltonians introduced in [4] and is distinct from other known classes of Hamiltonians for which single valuedness has been shown. Moreover, the condition for a Hamiltonian to be weakly coercive can be satisfied by restricting the behavior of the Hamiltonian on sets of arbitrarily small measure in the gradient space; however, this is not the case with other known classes for which single valuedness holds.

More precisely, if  $\Omega$  is an open subset of  $\mathbb{R}^N$  we say that the Hamiltonian  $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is *weakly coercive* if  $H$  is continuous and if for each  $F \subset \subset \Omega \times \mathbb{R}$  there exists a sequence  $\{\alpha_j\}$  of positive real numbers tending to infinity, so that, for each  $j \in \mathbb{N}$ , the set

$$\{p \mid H(x, r, p) \geq \alpha_j \quad \forall (x, r) \in F\}$$

contains a subset  $\Gamma_j$  which is the boundary of a compact convex set with the origin in its interior. Our main result can now be stated.

---

Received by the editors November 9, 1993.

1991 *Mathematics Subject Classification*. Primary 35F20, 35B65, 26B25.

*Key words and phrases*. Viscosity solution, coerciveness, single valuedness, Hamilton-Jacobi equations.

**Main Theorem.** *Let  $u \in C(\Omega)$  be a viscosity solution of both*

$$(1.1) \quad H(x, u, Du) = f(x) \quad \text{and} \quad H(x, u, Du) = g(x),$$

where  $f, g \in C(\Omega)$ . If  $H$  is weakly coercive, then it follows that  $f = g$  on  $\Omega$ .

Previously, L. C. Evans has shown that single valuedness holds when  $H$  is uniformly continuous. If  $H$  is strongly coercive, that is, if  $H(p) \rightarrow \infty$  as  $|p| \rightarrow \infty$ , M. G. Crandall and P. L. Lions ([4]) have shown that single valuedness holds. In [5], H. Frankowska has shown that sectorial coerciveness suffices. Sectorial coerciveness means that if  $|p| \rightarrow \infty$  for  $p$  remaining in a certain open sector of  $\mathbb{R}^N$ , then  $H(p) \rightarrow \infty$ . In the latter two cases the single valuedness of the Hamilton-Jacobi operator follows from the regularity of solutions  $u$  of (1.1); either that  $u$  is locally Lipschitz or that  $u$  is directionally Lipschitz. In our case we shall show that weakly coercive Hamiltonians have only locally Lipschitz solutions. Note also that there are weakly coercive Hamiltonians which are not uniformly continuous and do not possess sectorial coerciveness; for example, one may take  $H(p) \equiv |p|^2 \sin(|p|)$ .

## 2. PRELIMINARIES

For the sake of completeness, some preliminary definitions and results are included here. We define viscosity superdifferential and subdifferential sets and give several lemmas concerning convex bodies and their support functions. With the exception of (2.2d) below, these results are all classical; in fact, they are due to Minkowski.

For  $u \in C(\Omega)$  and  $x_0 \in \Omega$ , we define the viscosity *superdifferential* set  $D^+u(x_0)$  (respectively *subdifferential*  $D^-u(x_0)$ ) as follows:

$$D^\pm u(x_0) \equiv \{p \in \mathbb{R}^N \mid u(x) \leq (\text{resp. } \geq) u(x_0) + p \cdot (x - x_0) + o(|x - x_0|)\}.$$

We refer the reader to the early paper [2] and the more recent “user’s guide” [3] for further definitions and results in the theory of viscosity solutions.

We recall that a *convex body*  $C$  is a closed convex subset of  $\mathbb{R}^N$  such that  $\text{int } C \neq \emptyset$ . We will also assume throughout the paper that each convex body considered here has the origin in its interior. The *support function* of  $C$  is defined by

$$(2.1) \quad h(x) \equiv \sup_{p \in C} \{p \cdot x\}.$$

**Lemma 2.1.** *Let  $C$  be a compact convex body in  $\mathbb{R}^N$ . Then the support function  $h$  of  $C$  enjoys the following properties.*

$$(2.2a) \quad h(x) \geq \text{dist}(\partial C, 0)|x|,$$

$$(2.2b) \quad h \text{ is convex on } \mathbb{R}^N,$$

$$(2.2c) \quad h(\varepsilon x) = \varepsilon h(x) \quad \forall \varepsilon > 0 \quad \forall x \in \mathbb{R}^N,$$

$$(2.2d) \quad D^-h(0) = C.$$

*Proof.* The proofs of (2.2a)-(2.2c) are elementary. Let  $p \in C$ . From (2.1) it is seen that  $h(x) - (p \cdot x) \geq 0$  for all  $x$  in  $\mathbb{R}^N$ . Since  $h(0) = 0$ , we have  $p \in D^-h(0)$ . Now

suppose  $q \notin C$ . As  $C$  is compact and convex there is a hyperplane  $\pi$  which strictly separates the point  $q$  and the set  $C$ . Fix  $z \in \pi$  and choose a unit normal  $n$  to  $\pi$  so that  $(q - z) \cdot n > 0$  and  $(p - z) \cdot n < 0$  for all  $p \in C$ . As the set  $C$  is compact, there exists a  $\delta > 0$  satisfying

$$(2.3) \quad \max_{p \in C} \{(p - z) \cdot n\} = -\delta.$$

Choose  $p^* \in C$  so that  $h(n) = p^* \cdot n$ . In view of (2.3) we observe that  $(q - p^*) \cdot n > \delta$  and in view of (2.2c) that

$$(2.4) \quad q \cdot \varepsilon n > h(\varepsilon n) + \varepsilon \delta \quad \forall \varepsilon > 0.$$

As  $\delta$  is independent of  $\varepsilon$  in (2.4), we must have  $q \notin D^-h(0)$ . □

In case the set  $C$  is strictly convex as well as compact then for each  $x$  in  $\mathbb{R}^N - \{0\}$  there is a unique point  $\nu(x)$  on  $\partial C$  so that the relation  $h(x) = \nu(x) \cdot x$  holds. The map  $\nu$  plays a role in the following classical result (see §16 of [1]).

**Lemma 2.2.** *If  $C$  is a compact strictly convex body in  $\mathbb{R}^N$ , then the support function  $h$  of  $C$  belongs to  $C^1(\mathbb{R}^N - \{0\})$  and in addition*

$$(2.5) \quad Dh(x) = \nu(x).$$

One last classical result, on the approximation of convex bodies, will be required.

**Lemma 2.3.** *The set of compact strictly convex bodies is dense in the set of compact convex bodies with the Hausdorff metric.*

*Proof.* In fact Minkowski in [6] has shown that real analytic compact strictly convex bodies are dense. □

### 3. PROOF OF THE MAIN THEOREM

First we show the following regularity result for subsolutions.

**Theorem 3.1.** *Let  $\Omega$  be an open set in  $\mathbb{R}^N$  and suppose  $f \in C(\Omega)$ . Let  $u \in C(\Omega)$  be a viscosity subsolution of  $H(x, u, Du) = f$  in  $\Omega$ . If  $H$  is weakly coercive, then  $u$  is locally Lipschitz on  $\Omega$ .*

*Proof.* Assume to the contrary that there is a closed ball  $K$  contained in  $\Omega$  so that  $u$  is not Lipschitz on  $K$ . Let  $\delta = \text{dist}(K, \partial\Omega)$ ,  $\beta = \max_{\tilde{K}} f$ , and  $\gamma = \max_{\tilde{K}} |u|$ , where

$$\tilde{K} = \{x \in \Omega \mid \text{dist}(x, K) \leq \frac{\delta}{4}\}.$$

With  $F = \tilde{K} \times [-\gamma, \gamma]$ , choose sequences  $\{\alpha_j\}$  and  $\{\Gamma_j\}$  according to the definition of the weak coerciveness of  $H$ . Put  $a_j = \min\{|p| : p \in \Gamma_j\}$ . The continuity of  $H$  and  $u$  forces  $\lim_{j \rightarrow \infty} a_j = +\infty$ . Pick  $j_0$  so that

$$(3.1) \quad \alpha_{j_0} \geq \beta + 2 \quad \text{and} \quad a_{j_0} \geq 8\delta^{-1}\gamma.$$

Now the norms of the elements of  $D^-u(x)$  cannot be bounded above uniformly for all  $x$  in  $K$ , since any such uniform bound would be a Lipschitz constant for  $u$  on  $K$ .

Hence, there exists an  $x_0 \in K$  and a  $p_0 \in D^-u(x_0)$  so that  $p_0 \notin C_0$ , where  $C_0$  is the closed convex hull of  $\Gamma_{j_0}$ . Let  $h_0$  be the support function of  $C_0$ . Grant for the moment that  $C_0$  is strictly convex and put

$$(3.2) \quad \phi(x) \equiv u(x_0) + h_0(x - x_0).$$

Note that by using (2.2a), (2.2b), and (3.1) we have

$$(3.3) \quad h_0(x - x_0) \geq \frac{\delta}{4} a_{j_0} \geq 2\gamma \quad \forall x \in \partial B_{\frac{\delta}{4}}(x_0).$$

If it were true that  $\phi \geq u$  on  $B_{\frac{\delta}{4}}(x_0)$ , then (2.2d) would imply that

$$D^-u(x_0) \subset D^-h_0(0) = C_0,$$

which contradicts the choice of  $p_0$  made above. Therefore the difference  $u - \phi$  has a global positive maximum relative to  $\tilde{B}_{\frac{\delta}{4}}(x_0)$ , which according to (3.3) must occur at some  $y_0 \in B_{\frac{\delta}{4}}(x_0) - \{x_0\}$ . Lemma 2.2 implies that  $\phi$  is smooth at  $y_0$ ; consequently, in view of (2.5), (3.2), and a well-known characterization of viscosity superdifferentials ([2]), we obtain

$$D\phi(y_0) = \nu(y_0 - x_0) \in \Gamma_{j_0} \cap D^+u(y_0).$$

On the one hand, we have from our choice of  $\beta$

$$H(y_0, u(y_0), D\phi(y_0)) \geq \alpha_{j_0} \geq \beta + 1 > f(y_0);$$

on the other hand, by the definition of viscosity subsolution

$$H(y_0, u(y_0), D\phi(y_0)) \leq f(y_0).$$

This contradiction establishes the theorem in case  $C_0$  is strictly convex. If  $C_0$  is convex, by Lemma 2.3 and the continuity of  $H$ , there exists a strictly convex body  $\tilde{C}$  so that  $H(x, r, p) \geq \alpha_{j_0} - 1$  holds for  $(x, r, p) \in \tilde{K} \times [-\gamma, \gamma] \times \partial\tilde{C}$ . Reasoning as above, we reach the same conclusion.  $\square$

*Proof of the Main Theorem.* From Theorem 3.1 it follows that  $u$  is locally Lipschitz on  $\Omega$ . Since  $u$  is differentiable almost everywhere in  $\Omega$  by Rademacher's theorem, and  $f = g$  at all points of differentiability of  $u$ , the continuity of  $f$  and  $g$  allows us to conclude.  $\square$

#### ACKNOWLEDGEMENT

The first author would like to thank Professor M. G. Crandall for calling this problem to his attention and the Research Enhancement Program of the University of Texas at Arlington for its support.

## REFERENCES

1. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Chelsea, New York, 1948. MR **51**:8954
2. M. G. Crandall, L. C. Evans, and P. L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **282** (1984), 487–502. MR **86a**:35031
3. M. G. Crandall, H. Ishii, and P. L. Lions, *User's Guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N. S.) **27** (1) (1992), 1–67. MR **92j**:35050
4. M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42. MR **85g**:35029
5. H. Frankowska, *On the Single Valuedness of Hamilton-Jacobi Operators*, Nonlinear Anal. **10** (1986), 1477–1483. MR **88g**:35050
6. H. Minkowski, *Volumen und Oberfläche*, Math. Ann. **57** (1903), 447–495.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT ARLINGTON, ARLINGTON, TEXAS 76019

*E-mail address:* `jzsu@utam.uta.edu`

*Current address:* R. T. Newcomb: Cistercian Preparatory School, One Cistercian Road, Irving, Texas 75039