

## STABLE SPLITTINGS OF $BO(2n)$ AND $BU(2n)$

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ABSTRACT. Using the Snaith-Mitchell-Priddy splittings of  $BO(2n)$  and  $BU(2n)$ , we can give new stable splittings of  $BO(2n)$  and  $BU(2n)$  respectively.

In [5] and [6] Snaith, Mitchell, and Priddy showed that the natural filtrations on  $BO(n)$ ,  $BU(n)$ , and  $BSP(n)$  stably split, respectively, and in [2] Henn and Mui showed the corresponding splitting for  $BSO(2n+1)$  and  $BSU(2n+1)$ . The purpose of this note is to give new stable splittings of  $BO(2n)$  and  $BU(2n)$ , respectively.

Let  $g_{2n}: O(2n) \rightarrow SO(2n+1)$  be defined by  $g_{2n}(\alpha) = \det(\alpha) \oplus \alpha$ . Then we have

$$Bg_{2n}: BO(2n) \rightarrow BSO(2n+1).$$

Let  $Y_{2n}$  be the stable fibre of  $Bg_{2n}$ , that is,

$$Y_{2n} \xrightarrow{\delta_{2n}} BO(2n) \xrightarrow{Bg_{2n}} BSO(2n+1)$$

is a stable fibration.

**Theorem A.** *Localized at the prime 2, the stable fibre  $Y_{2n}$  of  $Bg_{2n}$  is a stable summand of the stable fibre  $Y_{2n+2}$  of  $Bg_{2n+2}$ .*

**Corollary B.** *Localized at the prime 2, for each  $n$  there are 2-local spectra  $F_{2i}$ ,  $1 \leq i \leq n$ , such that*

$$F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n} \rightarrow BO(2n) \xrightarrow{Bg_{2n}} BSO(2n+1)$$

is a stable filtration and

$$BO(2n) \cong BSO(2n+1) \vee F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n}.$$

*Remark 1.* In [3], Mitchell and Priddy proved that the homotopy type of  $F_2$  is  $\sum^{-2} \overline{SP^4} S^0 \vee P_1^\infty$  (see [3]).

*Remark 2.* We have analogous theorems for  $BU(2n)$  and  $BSU(2n+1)$ .

From now on we will suppress details for  $BU(2n)$ , which can be obtained easily from those for  $BO(2n)$ , and all spaces or spectra are implicitly localized at prime 2.

**Lemma 1.** *Let  $Y_{2n} \xrightarrow{\delta_{2n}} BO(2n) \xrightarrow{Bg_{2n}} BSO(2n+1)$  be a stable fibration. Then  $BO(2n) \cong BSO(2n+1) \vee Y_{2n}$ .*

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*Proof.* Since the fibre of

$$Bg_{2n}: \text{BO}(2n) \rightarrow \text{BSO}(2n + 1)$$

is  $\text{SO}(2n + 1)/O(2n) \cong RP^{2n}$ , the Becker-Gottlieb transfer [1] associated to this fibration is a stable map

$$q_{2n}: \text{BSO}(2n + 1) \rightarrow \text{BO}(2n)$$

such that  $(q_{2n})^* \circ (Bg_{2n})^*$  is multiplication by the Euler characteristic  $\chi(RP^{2n}) = 1 \pmod{2}$ . Thus  $q_{2n}$  provides a stable splitting of  $Bg_{2n}$ , and

$$\text{BO}(2n) \cong \text{BSO}(2n + 1) \vee Y_{2n}.$$

This completes the proof.

*Proof of Corollary B.* This follows immediately from Theorem A and Lemma 1 by induction.

**Lemma 2.** *There is a stable map  $\lambda_{2n}: \text{BO}(2n + 2) \rightarrow \text{BO}(2n)$  such that the composite map*

$$\text{BO}(2n) \xrightarrow{Bi_{2n}} \text{BO}(2n + 2) \xrightarrow{\lambda_{2n}} \text{BO}(2n)$$

*is homotopic to the identity map, where the first map is the natural inclusion map.*

*Proof.* This lemma follows immediately from [5] and [6].

Now we can prove Theorem A.

*Proof of Theorem A.* Let  $g_{2n}$  be the map  $g_{2n}(\alpha) = (\det \alpha) \oplus \alpha$ . Then the diagram of groups

$$\begin{array}{ccc} O(2n) & \xrightarrow{g_{2n}} & \text{SO}(2n + 1) \\ i_{2n} \downarrow & & \downarrow j_{2n+1} \\ O(2n + 2) & \xrightarrow{g_{2n+2}} & \text{SO}(2n + 3) \end{array}$$

is strictly commutative, where  $i_{2n}$  and  $j_{2n+1}$  are the inclusions

$$(\det(\alpha) \oplus \alpha) \oplus 1 = \det(\alpha \oplus 1) \oplus (\alpha \oplus 1).$$

This yields a strictly commutative diagram of classifying spaces, and if  $Y_{2n}$  denotes the stable fibre of  $Bg_{2n}$ , then we have a commutative diagram of stable maps:

$$\begin{array}{ccccc} Y_{2n} & \xrightarrow{\delta_{2n}} & \text{BO}(2n) & \xrightarrow{Bg_{2n}} & \text{BSO}(2n + 1) \\ \downarrow k & & \downarrow Bi_{2n} & & \downarrow Bj_{2n+1} \\ Y_{2n+2} & \xrightarrow{\delta_{2n+2}} & \text{BO}(2n + 2) & \xrightarrow{Bg_{2n+2}} & \text{BSO}(2n + 3) \end{array}$$

Now by Lemma 1,  $Bg_{2n}$  has a right inverse (in the stable category), that is, the fibration sequence splits, that is,  $\delta_{2n}$  has a left inverse, say  $\eta_{2n}$ . By Lemma 2,  $Bi_{2n}$  has a left inverse, say  $\lambda_{2n}$ . It follows that  $k$  has a left inverse

$$(\eta_{2n} \lambda_{2n} \delta_{2n+2})k = \eta_{2n} \lambda_{2n} Bi_{2n} \delta_{2n} = \eta_{2n} \delta_{2n} = 1.$$

This completes the proof of Theorem A.

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