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A K-FUNCTIONAL AND THE RATE OF CONVERGENCE OF SOME LINEAR POLYNOMIAL OPERATORS

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ABSTRACT. We show that the K-functional

$$K(f,n^{-2})_p = \inf_{g\in C^2[-1,1]}\big(\|f-g\|_p + n^{-2}\|P(D)g\|_p\big),$$

where $P(D)=\frac{d}{dx}(1-x^2)\frac{d}{dx}$, is equivalent to the rate of convergence of a certain linear polynomial operator. This operator stems from a Riesz-type summability process of expansion by Legendre polynomials. We use the operator above to obtain a linear polynomial approximation operator with a rate comparable to that of the best polynomial approximation.

1. Introduction

For polynomial approximation in $L_p[-1,1]$, $1\le p\le\infty$, it was shown [6, Chapter 7] that

(1.1)
$$E_n(f)_p \equiv \inf_{P\in\Pi_n}\|f-P\|_p \le \inf_{g\in C^r[-1,1]}\big(\|f-g\|_p + n^{-r}\|\varphi^r g^{(r)}\|_p\big) \equiv K_{r,\varphi}(f,n^{-r})_p$$

where $\varphi=\sqrt{1-x^2}$, Π_n is the class of polynomials of degree n and $\|\ \|_p$ is the $L_p[-1,1]$ norm. In fact, a polynomial $P\in\Pi_n$ was constructed to satisfy

$$\|f-P\|_p \le K_{r,\varphi}(f,n^{-r})_p.$$

The result (1.1) is best possible in the sense that a weak converse inequality for $E_n(f)_p$ exists. As a result of the considerations in this paper we obtain

$$E_n(f)_p \le \inf_{g\in C^{2r}[-1,1]}\big(\|f-g\|_p + n^{-2r}\|P(D)^r g\|_p\big) \equiv K_{2r}(f,n^{-2r})_p \tag{1.2}$$

(where $P(D)=\frac{d}{dx}(1-x^2)\frac{d}{dx}$), and this inequality is best possible in the same sense, that is, a weak converse inequality with $E_n(f)_p$ is valid. The K-functionals $K_{2r,\varphi}(f,n^{-2r})_p$ and $K_{2r}(f,n^{-2r})_p$ are not equivalent though related for $1<p<\infty$ (see for explicit results [3, (2.5)] in the case $r=1$). For $r=1$ and $f(x)=x$,

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$K_{2,\varphi}(f,n^{-2})_p = 0$ and $K_2(f,n^{-2})_p \sim n^{-2}$. Hence, (1.2) cannot be derived from (1.1) or vice versa.

For Bernstein polynomials

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.3}$$

it was shown by Totik [9] (generalizing [7, section 8] and settling a conjecture there) that

$$\|f - B_n f\|_{C[0,1]} \sim K_{2,\varphi}(f,n^{-1})_\infty \tag{1.4}$$

where $L_\infty[0,1]$ takes the place of $L_\infty[-1,1]$ and $\varphi = \sqrt{x(1-x)}$ takes the place of $\varphi = \sqrt{1-x^2}$ in the definition of the K-functional in (1.1). Hence,

$$E_n(f)_{C[0,1]} \le C\|f - B_{n^2} f\|_{C[0,1]}. \tag{1.5}$$

For $p \ne \infty$ and for a higher degree of smoothness, the Bernstein polynomials are no longer applicable. In this case one can define the Durrmeyer-Bernstein polynomials

$$M_n(f,x) = \frac{1}{n+1} \sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(y)\, f(y)\, dy. \tag{1.6}$$

As a result of [4] we have for $1 \le p \le \infty$

$$K_{2r}(f,n^{-r})_p \sim \|(M_n - I)^r f\|_p \tag{1.7}$$

where in the definition of K_{2r} we use $L_p[0,1]$ instead of $L_p[-1,1]$ and $P(D) = \frac{d}{dx}x(1-x)\frac{d}{dx}$ instead of $P(D) = \frac{d}{dx}(1-x^2)\frac{d}{dx}$. The result of this paper will imply for $1 \le p \le \infty$ and $r \in N$

$$E_n(f)_{L_p[0,1]} \le C\|(M_{n^2} - I)^r f\|_{L_p[0,1]}. \tag{1.8}$$

To achieve this we define a simple linear operator R_n that will yield the strong converse inequality of type A in the sense of [7], and hence the equivalence

$$\|(R_n - I)^r f\|_{L_p[-1,1]} \sim K_{2r}(f,n^{-2r})_p. \tag{1.9}$$

We note that for polynomial expansion in $[-1,1]$ the Legendre polynomials given by

$$P(D)P_k(x) \equiv \frac{d}{dx}(1-x^2)\,\frac{d}{dx}P_k(x) = -k(k+1)P_k(x) \tag{1.10}$$

and the orthonormality condition

$$\langle P_k, P_\ell \rangle \equiv \int_{-1}^{1} P_k(x)P_\ell(x)\, dx = \delta_{k,\ell} \tag{1.11}$$

appear naturally. Therefore, the operator $P(D)$ is not a contrived expression in the definition of K_{2r} and in (1.2).

In section 2 the operator $R_n f$ will be introduced. In section 3 the main converse inequality is proved. In section 4 applications and generalizations of this result will be discussed. The relation with best polynomial approximation is given in section 5.

2. A summability operator

For the Legendre polynomial P_k given in (1.10) and (1.11) the formal expansion is given by

$$f(x) \sim \sum_{k=0}^{\infty} a_k P_k(x), \qquad a_k = \langle f, P_k \rangle = \int_{-1}^{1} f(y) P_k(y)\, dy. \tag{2.1}$$

Pollard [8] showed that for $\frac{4}{3} < p < 4$

$$\|S_n f\|_p \le C_p \|f\|_p \tag{2.2}$$

where

$$S_n(f, x) = \sum_{k=0}^{n} a_k P_k(x). \tag{2.3}$$

Askey and Hirschman [2] (see Theorem 2a there with $\alpha = 1$ and $\nu = \frac{1}{2}$) proved that for $1 \le p \le \infty$

$$\|\sigma_n f\|_p \le C \|f\|_p \tag{2.4}$$

where $\sigma_n f$ is the Cesàro summability

$$\sigma_{n-1}(f, x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) a_k P_k(x) = \frac{1}{n} \sum_{k=0}^{n-1} S_k(f, x), \quad \sigma_0(f, x) = a_0 P_0. \tag{2.5}$$

The summability method treated here is given by

$$R_n(f, x) = \sum_{k=0}^{n} \left(1 - \frac{k(k+1)}{n(n+1)}\right) a_k P_k(x), \tag{2.6}$$

and (2.6) may be reformulated by

$$R_n(f, x) = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} (k+1) S_k(f, x), \tag{2.7}$$

which places it as a special case of the Riesz summability. Being a linear operator whose range is in Π_n, $R_n f$ and its derivates are linear polynomial operators.

We can now show that, as a result of (2.4), $R_n f$ is also a bounded operator.

Lemma 2.1. *For $1 \le p \le \infty$ and R_n given by (2.6) we have*

$$\|R_n f\|_p \le C \|f\|_p. \tag{2.8}$$

Proof. The following proof was given by V. Totik and is much shorter than the original proof by the author. (The earlier proof was longer and yielded some further results which are not necessary here.)

One can easily verify that

$$\frac{2}{n(n+1)}\sum_{k=0}^{n-1}(n-k)S_k(f,x)=\frac{2}{n(n+1)}\sum_{k=0}^{n-1}(k+1)\sigma_k(f,x). \tag{2.9}$$

Adding (2.9) to (2.7), we have

$$\frac{2}{n(n+1)}(n+1)\sum_{k=0}^{n-1}S_k(f,x)=R_n(f,x)+\frac{2}{n(n+1)}\sum_{k=0}^{n-1}(k+1)\sigma_k(f,x),$$

which implies

$$R_n(f,x)=2\sigma_{n-1}(f,x)-\frac{2}{n(n+1)}\sum_{k=0}^{n-1}(k+1)\sigma_k(f,x).$$

Hence, (2.4) implies (2.8). □

Remark 2.2. For $\frac{4}{3}<p<4$ Pollard's result and (2.7) already imply (2.8).

3. The equivalence result

We can now state and prove our equivalence result.

Theorem 3.1. *For $f\in L_p[-1,1]$, $R_n(f,x)$ given by (2.6) and the K-functional*

$$K(f,t^2)_p\equiv K_2(f,t^2)_p=\inf_{g\in C^2}\big(\|f-g\|_p+t^2\|P(D)g\|_p\big), \tag{3.1}$$

we have

$$\|R_nf-f\|_p\sim K(f,n^{-2})_p \tag{3.2}$$

where $A_n\sim B_n$ if there exists C such that $C^{-1}A_n<B_n<CA_n$.

Proof. As polynomials are dense in L_p or in $C[-1,1]$ when $p=\infty$, (2.8) implies

$$\lim_{n\to\infty}\|R_nf-f\|_p=0. \tag{3.3}$$

This follows from $\lim\limits_{n\to\infty}R_nP_k=\lim\limits_{n\to\infty}(1-\frac{k(k+1)}{n(n+1)})P_k=P_k$ and

$$\|R_nf-f\|_p\le\|R_n(f-P)\|_p+\|f-P\|_p+\|R_nP-P\|_p.$$

To prove the direct result we choose $g\in C^2[-1,1]$ such that

$$\|f-g\|_p+n^{-2}\|P(D)g\|_p\le 2K(f,n^{-2})_p$$

and write

$$\begin{aligned}\|R_nf-f\|_p&\le\|R_n(f-g)-(f-g)\|_p+\|R_ng-g\|_p\\&\le\big(C+1\big)2K(f,n^{-2})_p+\|R_ng-g\|_p.\end{aligned}$$

To estimate the second term we write

$$\begin{aligned}\|R_n g - g\| &\le \|R_n^2 g - R_n g\| + \|R_n^2 g - g\| \\ &\le \|R_n^2 g - R_n g\| + \sum_{m=n}^{\infty} \|R_m^2 g - R_{m+1}^2 g\|,\end{aligned}$$

as (2.8) and (3.3) imply also $\lim_{n\to\infty} \|R_n^2 g - g\| = 0$ for any $g \in C^2[-1,1]$. We now observe that for any $f \in L_1[-1,1]$ (so that the a_k's are defined)

$$n(n+1)R_n(R_n f - f) = P(D)R_n f, \tag{3.4}$$

and that

$$P(D)R_n g = R_n P(D) g \quad \text{for } g \in C^2[-1,1]. \tag{3.5}$$

Therefore,

$$\|R_n^2 g - R_n g\|_p \le \frac{C}{n(n+1)} \|P(D)g\|_p. \tag{3.6}$$

As $R_\ell R_m f = R_m R_\ell f$, we have

$$\|R_m^2 g - R_{m+1}^2 g\|_p \le \|R_m^2 g - R_{m+1} R_m g\|_p + \|R_{m+1}^2 g - R_m R_{m+1} g\|_p.$$

We now note that for $f \in L_1[-1,1]$

$$\begin{aligned} R_m^2 f - R_{m+1} R_m f &= -\frac{2}{m(m+1)(m+2)} P(D) R_m f, \\ R_{m+1}^2 f - R_m R_{m+1} f &= \frac{2}{m(m+1)(m+2)} P(D) R_{m+1} f. \end{aligned} \tag{3.7}$$

Recalling (3.5), we have

$$\begin{aligned}\sum_{m=n}^{\infty} \|R_m^2 g - R_{m+1}^2 g\|_p &\le 4C \|P(D)g\|_p \sum_{m=n}^{\infty} \frac{1}{m^3} \\ &\le \frac{2C}{n^2} \|P(D)g\|_p,\end{aligned}$$

which, together with the above, yields

$$\|R_n f - f\|_p \le (C+1) 2K(f, n^{-2})_p + 3C \cdot 2K(f, n^{-2})_p,$$

and this is the direct estimate of our equivalence.

To prove the converse result, which is a strong converse inequality of type A in the terminology of [7], we note that using (3.1),

$$K(f, n^{-2})_p \le \|f - R_n f\|_p + \frac{1}{n^2} \|P(D) R_n f\|_p.$$

Using (3.4), we now have

$$\|P(D) R_n f\| \le n(n+1) C \|R_n f - f\|_p,$$

which completes the proof. $\square$

Corollary 3.2. *For $f \in L_p[-1,1]$, $1 \le p \le \infty$, we have*

$$(3.8)\quad \|(R_n - I)^r f\|_p \sim \inf_{g \in C^{2r}[-1,1]} \big(\|f-g\|_p + n^{-2r}\|P(D)^r g\|_p\big) \equiv K_{2r}(f, n^{-2r})_p.$$

Proof. We use Theorems 10.2 and 10.3 of [7] (the latter with some obvious modifications) for the direct result and Theorem 10.4 of [7] for the converse result. That is, the necessary inequalities needed for this extension were already proved earlier in this section. □

4. Further results and generalizations

We first deduce as a corollary from Theorem 3.1 and Corollary 3.2 the following "realization" result.

Theorem 4.1. *For $f \in L_p[-1,1]$, $1 \le p \le \infty$, R_n given by (2.6),*

$$(4.1)\quad K_{2r}(f, t^r)_p \equiv \inf_{g \in C^{2r}[-1,1]} (\|f-g\|_p + t^r \|P(D)^r g\|_p)$$

and

$$(4.2)\quad L_{n,r} f = \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} R_n^k f,$$

we have

$$(4.3)\quad \|f - L_{n,r} f\|_p + n^{-2r}\|P(D)^r L_{n,r} f\|_p \sim K_{2r}(f, n^{-2r})_p.$$

Proof. Using Corollary 3.2, all we have to show is

$$(4.4)\quad n^{-2r}\|P(D)^r L_{n,r} f\|_p \le C K_{2r}(f, n^{-2r})_p.$$

The identities (3.4) and (3.5) used repeatedly imply

$$P(D)^r R_n^k f = \big(n(n+1)\big)^r R_n^k (R_n - I)^r f.$$

Hence,

$$\begin{aligned} n^{-2r}\|P(D)^r R_n^k f\|_p &\le C_1 \|(R_n - I)^r f\|_p \\ &\le C_2 K_{2r}(f, n^{-2r})_p, \end{aligned}$$

from which (4.4) follows easily. □

The realization implies the natural relation for hierarchy of measures of smoothness on the K-functionals. Hence, realization, though weaker in a sense than (2.8), is important. The first result actually follows already from Corollary 3.2.

Corollary 4.2. *For $K_{2r}(f, n^{-2r})_p$ given in (4.1)*

$$K_{2r+2}(f, n^{-2(r+1)})_p \le C K_{2r}(f, n^{-2r})_p. \tag{4.5}$$

Proof. In view of Corollary 3.2 we just have to observe that

$$\begin{aligned}\|f - L_{n,r+1}f\|_p &\le \|(I - R_n)(f - L_{n,r}f)\|_p \\ &\le \|f - L_{n,r}f\|_p + \|R_n(f - L_{n,r}f)\|_p,\end{aligned}$$

which, using (2.8), implies (4.5). □

The Marchaud-type inequality also follows.

Theorem 4.3. *For $K_{2r}(f, t)_p$ given in (4.1) we have*

$$K_{2r}(f, t^{2r})_p \le C t^{2r} \left(\int_t^1 \frac{K_{2r+2}(f, u^{2(r+1)})_p}{u^{2r+1}}\, du \right). \tag{4.6}$$

Proof. We write for $n = [1/t]$

$$\begin{aligned}K_{2r}(f, t^{2r})_p &\le \|f - L_{n,r+1}f\|_p + t^{2r}\|P(D)^r L_{n,r+1}f\|_p \\ &\le C_1 K_{2r+2}(f, n^{-2r-2})_p + t^{2r}\|P(D)^r L_{n,r+1}f\|_p.\end{aligned}$$

We now choose k so that $2^k \le n < 2^{k+1}$ and write

$$L_{n,r+1}f = L_{n,r+1}f - L_{2^k,r+1}f + \sum_{\ell=1}^{k}(L_{2^\ell,r+1}f - L_{2^{\ell-1},r+1}f) + L_{1,r+1}f.$$

Using $L_{1,r+1}f = C$, and hence $P(D)^r L_{1,r+1}f = 0$, and

$$\|L_{2^\ell,r+1}f - L_{2^{\ell-1},r+1}f\|_p \le 2C_1 K_{2r+2}(f, 2^{-2(r+1)(\ell-1)})_p,$$

we have, using the Markov-Bernstein inequality,

$$\begin{aligned}K_{2r}(f, t^{2r})_p \le C_1 K_{2r+2}(f, n^{-2r-2})_p &+ t^{2r}\sum_{\ell=0}^{k-1} C_2 2^{2\ell r} 2C_1 K_{2r+2}(f, 2^{-2(r+1)\ell})_p \\ &+ 2C_2 C_1 t^{2r} n^{2r} K_{2r+2}(f, 2^{-2(r+1)k})_p.\end{aligned}$$

Using monotonicity of $K_{2r+2}(f, u)_p$ and of u^α, we now complete the proof. □

We note that we do not have a free term as usual in Marchaud-type inequalities. While this follows from the proof above, it is related intuitively to the fact that if for some r, $P(D)^r g = 0$ and $g \in C^{2r}(-1, 1)$, then $g(x) = A$.

The near best polynomial approximant P_n which is a polynomial satisfying

$$\|f - P_n\|_p \le C E_n(f)_p, \tag{4.7}$$

where $E_n(f)_p = \inf_{P \in \Pi_n} \|f - P\|_p$, can also serve as a realization of the K-functional $K_{2r}(f, t^{2r})_p\|_p$.

Theorem 4.4. *For $1 \le p \le \infty$ and P_n given by (4.7) for the given p we have*

$$\|f - P_n\|_p + n^{-2r}\|P(D)^r P_n\|_p \sim K_{2r}(f, n^{-2r})_p. \tag{4.8}$$

Proof. Obviously,

$$\|f - P_n\|_p \le C\|f - L_{n,r}f\|_p \le CK_{2r}(f, n^{-2r})_p. \tag{4.9}$$

Furthermore,

$$n^{-2r}\|P(D)^r P_n\|_p \le n^{-2r}\|P(D)^r L_{n,r}f\|_p + n^{-2r}\|P(D)^r(P_n - L_{n,r}f)\|_p.$$

Since (4.9) yields the estimate

$$\|P_n - L_{n,r}f\|_p \le 2CK_{2r}(f, n^{-2r})_p,$$

we complete the proof using the Markov-Bernstein inequality. □

One should note that P_n satisfying (4.7) may also serve as a realization of $K_{r,\varphi}(f, n^{-r})_p$ [6, Chapter 7] and that this does not imply equivalence of $K_{2r}(f, n^{-2r})_p$ and $K_{2r,\varphi}(f, n^{-2r})_p$.

We can also define delayed means using $R_n f$ by

$$V_n f = 2\,\frac{2n+1}{3n+1} R_{2n} f - \frac{n+1}{3n+1}\, R_n f \tag{4.10}$$

such that

$$V_n P = P \quad \text{for} \quad P \in \Pi_n. \tag{4.11}$$

Obviously, for $n \ge n_0$

$$\|V_n f\|_p \le 2C\|f\|_p$$

where C is given by Lemma 2.1.

This obviously implies (for $n \ge n_0$)

$$\|V_n f - f\|_p \le (2C+1)E_n(f)_p, \tag{4.12}$$

which makes $V_n f$ a near best polynomial approximant.

In Theorem 4.4 we may replace P_n by $V_n f$ since

$$\|f - V_n f\|_p \le C_1 E_n(f)_p$$

and since for $P_n f$ satisfying (4.4) we have, using the Markov-Bernstein inequality ($V_n f \in \Pi_{2n}$ and $P_n \in \Pi_n \subset \Pi_{2n}$),

$$\begin{aligned}\frac{1}{n^{2r}}\, \|P(D)^r(P_n - V_n f)\|_p &\le C_2\|P_n - V_n f\|_p \\ &\le C_3 E_n(f)_p \le C_4 K_{2r}(f, n^{-2r})_p.\end{aligned}$$

5. Relation with best polynomial approximation

One observes that

$$E_n(f)_p \le \|(R_n - I)^r f\|_p. \tag{5.1}$$

Hence, using (3.8),

$$E_n(f)_p \le C K_{2r}(f, n^{-2r})_p. \tag{5.2}$$

The estimate (5.2) has the following matching weak converse inequality.

Theorem 5.1. *The K-functional $K_{2r}(f, t^{2r})_p$ given in (1.2) is estimated by the rate of best polynomial approximation in the weak-type inequality given by*

$$K_{2r}(f, t^{2r})_p \le C t^{2r} \sum_{0 \le n \le 1/t} (n+1)^{2r-1} E_n(f)_p. \tag{5.3}$$

Proof. The proof is quite routine, choosing ℓ such that $\ell = \max\{k;\ 2^k \le 1/t\}$ and g to be P_{2^k}, the best 2^k degree polynomial approximation to f in $L_p[-1,1]$. We then expand P_{2^ℓ} by

$$P_{2^\ell}(x) = \sum_{k=0}^{\ell} \big(P_{2^{k+1}}(x) - P_{2^k}(x)\big) + P_1(x).$$

The Markov-Bernstein inequality implies

$$\|P(D)^r P_m\|_p \le C m^{2r} \|P_m\|_p \quad \text{for } P_m \in \Pi_m. \tag{5.4}$$

This yields

$$K_{2r}(f, t^{2r})_p \le C t^{2r} \Bigg(\sum_{1 \le 2^k \le \frac{1}{t}} (2^k)^{2r} E_{2^k}(f)_p + E_0(f)_p \Bigg), \tag{5.5}$$

which implies (5.3), using the monotonicity of $E_n(f)_p$, $K_{2r}(f, t^{2r})_p$ and m^{2r}. $\square$

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