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A MODEL FOR INVERTIBLE COMPOSITION OPERATORS ON H^2

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Abstract. A model is obtained for invertible hyperbolic and parabolic composition operators on H^2. This model shows that the adjoints of these composition operators are similar to block Toeplitz matrices constructed with weighted bilateral shifts and rank one operators.

In their paper, Nordgren, Rosenthal and Wintrobe [3, Thm. 5.7] present a model for the composition operator C_φ, where φ is a parabolic or hyperbolic disk automorphism. A brief summary of their model, which will be considered in greater detail and contrasted with the model from this paper later on, follows. A subspace $\mathcal{K}$ is defined, which is invariant under C_φ. Then H^2 is decomposed into a nonorthogonal direct sum of two invariant subspaces M and N constructed from $\mathcal{K}$. Finally, $C_\varphi|_M$ and $C_\varphi|_N$ are each shown to be unitarily equivalent to inflations of $C_\varphi|_\mathcal{K}$.

This paper presents a model for C_φ^* (and hence for C_φ also), where φ is a parabolic or a hyperbolic disk automorphism. It is noted that when φ is an elliptic disk automorphism, C_φ is similar to a diagonal operator and is therefore less interesting. The model presented in this paper is similar to but simpler and more detailed than the model which they obtained. H^2 is decomposed into an orthogonal direct sum $H^2 = \sum_{i=0}^\infty \oplus B^i\mathcal{L}_0$ ($\mathcal{L}_0$ will be defined shortly). The compression of C_φ^* to $B^i\mathcal{L}_0$ is similar to a weighted bilateral shift. The restriction of C_φ^* to $B^i\mathcal{L}_0$ is the sum of this compression and a rank-one operator. A formula for the rank-one operator is given and its range is a subset of $B^{i-1}\mathcal{L}_0$. In conclusion, it is shown that C_φ^* is similar to a block Toeplitz matrix.

For any disk automorphism φ, define the iterates $\varphi^{(n)}$ as

$$\varphi^{(0)} = z$$

and for $n = 1, 2, 3, \ldots$

$$\varphi^{(n)} = \varphi \circ \varphi^{(n-1)} \quad \text{and} \quad \varphi^{(-n)} = \varphi^{-1} \circ \varphi^{(-n+1)}.$$

Let $z_n = \varphi^{(n)}(0)$.

If φ is a parabolic or a hyperbolic disk automorphism, Nordgren, Rosenthal and Wintrobe [3, p. 333] show that $\{z_n\}$ is an interpolating Blaschke sequence. Furthermore, if φ has fixed points in $\{-1, 1\}$, with Denjoy-Wolff point 1, and $B(z) =$

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$z\prod_{\substack{-\infty\\ n\neq 0}}^{\infty}\frac{\bar{z}_n}{|z_n|}\frac{z_n-z}{1-\bar{z}_n z} = z\prod_{\substack{-\infty\\ n\neq 0}}^{\infty}\frac{\bar{z}_n}{|z_n|}\varphi^{(n)}(z)$ (the Blaschke product with zeros z_n and $B'(0) > 0$), they show that

$$B\circ\varphi = \tau B \tag{1}$$

where $\tau = -1$ in the hyperbolic case and $\tau = 1$ in the parabolic case.

Let $k_n = \sqrt{1-|z_n|^2}/(1-\bar{z}_n z)$. Note that $k_n = K_{z_n}/\|K_{z_n}\|$, where K_{z_n} is the reproducing kernel for evaluation at the point z_n, that is, $\langle f, K_{z_n}\rangle = f(z_n)$ for f in H^2. Now define $\mathcal{L}_0$ as

$$\mathcal{L}_0 = \text{closed span}\{k_n\}_{-\infty}^{\infty}.$$

Since $\{z_n\}$ is an interpolating sequence, a result of Shapiro and Shields [4] implies that $\{k_n\}$ is similar to an orthonormal set (see also Cowen [2, p. 23]). Noting that $C_\varphi^* k_n = \sqrt{1-|z_n|^2}k_{n+1}/\sqrt{1-|z_{n+1}|^2}$, it follows that $C_\varphi^*|_{\mathcal{L}_0}$ is similar to the weighted bilateral shift with weight sequence $\{\sqrt{1-|z_n|^2}/\sqrt{1-|z_{n+1}|^2}\}$. Also note that $B^i\mathcal{L}_0 \perp B^j\mathcal{L}_0$ for $i \neq j \quad i,j \geq 0$. For i a positive integer, make the definition

$$\mathcal{L}_i = \mathcal{L}_0 \oplus B\mathcal{L}_0 \oplus \cdots \oplus B^i\mathcal{L}_0.$$

This leads to the following Lemma.

Lemma 1. *If φ is a parabolic or a hyperbolic disk automorphism with fixed points in $\{-1,1\}$ and Denjoy-Wolff point 1, and $\mathcal{L}_i$ is defined as above, then $\mathcal{L}_i = (B^{i+1}H^2)^\perp$ and $\mathcal{L}_i$ is invariant under C_φ^*.*

Proof. Suppose f is in H^2 and $j \leq i$. Then

$$\langle B^{i+1}f, B^j k_n\rangle = \langle B^{i+1-j}f, k_n\rangle = \sqrt{1-|z_n|^2}B^{i+1-j}(z_n)f(z_n) = 0$$

since z_n is a zero of B. This shows that $\mathcal{L}_i \subset (B^{i+1}H^2)^\perp$. The opposite inclusion will now be shown for $\mathcal{L}_0$ and then the other cases will be obtained by induction.

Suppose f is in ${\mathcal{L}_0}^\perp$. Then

$$\langle f, k_n\rangle = \sqrt{1-|z_n|}f(z_n) = 0$$

for all n. This means that all the zeros of B are also zeros of f so $f = Bg$ for some g in H^2 and hence ${\mathcal{L}_0}^\perp \subset BH^2$. Thus,

$$(BH^2)^\perp \subset ({\mathcal{L}_0}^\perp)^\perp = \mathcal{L}_0.$$

Suppose $g \perp \mathcal{L}_i$. Then $g \perp \mathcal{L}_{i-1}$ and by induction $g = B^i f$ for some f in H^2. Now,

$$\langle f, k_n\rangle = \langle B^i f, B^i k_n\rangle = \langle g, B^i k_n\rangle = 0$$

since $g \perp B^i\mathcal{L}_0$. Thus, $f = Bh$ for some h in H^2, so $g = B^{i+1}h$ and it follows that ${\mathcal{L}_i}^\perp \subset B^{i+1}H^2$. This implies $(B^{i+1}H^2)^\perp \subset \mathcal{L}_i$ and we have $\mathcal{L}_i = (B^{i+1}H^2)^\perp$.

Now, to show the invariance, let f be in H^2, let $j \leq i$, and compute:

$$\langle C_\varphi^* B^j k_n, B^{i+1}f\rangle = \langle B^j k_n, C_\varphi(B^{i+1}f)\rangle.$$

Now, by equation (1), we have,

$$\begin{aligned} &= \langle B^j k_n, \tau^{i+1}B^{i+1}f\circ\varphi\rangle \\ &= \tau^{i+1}\langle k_n, B^{i-j+1}f\circ\varphi\rangle = 0 \end{aligned}$$

since $0 \leq j \leq i$.

Thus, $\mathcal{L}_i = (B^{i+1}H^2)^{\perp}$ is invariant under C_φ^*. □

The next lemma is important because it shows that the compression of C_φ^* to $B^i\mathcal{L}_0$ is similar to a bilateral weighted shift, since we already know that $C_\varphi^*|_{\mathcal{L}_0}$ is.

Lemma 2. *If φ is a hyperbolic or a parabolic disk automorphism with fixed points in $\{-1,1\}$ and Denjoy-Wolff point 1, then the compression of C_φ^* to $B^i\mathcal{L}_0, i \geq 1$, is unitarily equivalent to $\tau^i C_\varphi^*|_{\mathcal{L}_0}$, where $\tau = 1$ in the parabolic case and $\tau = -1$ in the hyperbolic case.*

Proof. The compression of C_φ^* to $B^i\mathcal{L}_0$ is $PC_\varphi^*|_{B^i\mathcal{L}_0}$, where P is projection onto $B^i\mathcal{L}_0$. Let $U_i = T_{B^i}|_{\mathcal{L}_0}$ for $i \geq 1$, where T_{B^i} is the Toeplitz operator with symbol B^i. In this case it is multiplication by B^i since B^i is in H^∞. Note that T_{B^i} is an isometry of $\mathcal{L}_0$ onto $B^i\mathcal{L}_0$. Thus, U_i is unitary and it is easily seen that ${U_i}^* = {T_{B^i}}^*|_{B^i\mathcal{L}_0}$.

Let f be in $B^i\mathcal{L}_0$, so $f = B^i g$ for some g in $\mathcal{L}_0$. Let h be in $\mathcal{L}_i$, so $h = h_1 + h_2$ where h_1 is in $\mathcal{L}_{i-1}$ and h_2 is in $B^i\mathcal{L}_0$. Then

$$\begin{aligned}
\langle \tau^i U_i C_\varphi^* {U_i}^* f, h\rangle &= \tau^i \langle U_i C_\varphi^* g, h_2\rangle \\
&= \tau^i \langle C_\varphi^* g, {U_i}^* h_2\rangle \\
&= \tau^i \langle g, C_\varphi {U_i}^* h_2\rangle \\
&= \tau^i \langle B^i g, B^i C_\varphi {U_i}^* h_2\rangle \\
&= \tau^i \langle B^i g, \tau^i C_\varphi B^i {U_i}^* h_2\rangle
\end{aligned}$$

as a consequence of equation (1)

$$\begin{aligned}
&= \langle f, C_\varphi h_2\rangle \\
&= \langle C_\varphi^* f, h_2\rangle \\
&= \langle P C_\varphi^* f, h\rangle.
\end{aligned}$$

So $U_i(\tau^i C_\varphi^*){U_i}^* = PC_\varphi^*|_{B^i\mathcal{L}_0}$ □

Theorem 3. *Let φ be a hyperbolic disk automorphism with fixed points $\{-1,1\}$ and Denjoy-Wolff point 1. Then, for $i \geq 1$, $C_\varphi^*|_{B^i\mathcal{L}_0} = W_i + R_i$, where W_i is the compression of C_φ^* to $B^i\mathcal{L}_0$ and R_i is the rank one operator of $B^i\mathcal{L}_0$ into $B^{i-1}\mathcal{L}_0$ given by*

$$R_i(B^i f) = \frac{f(0)\varphi(0)(-1)^{i-1}B^i}{z - \varphi(0)}$$

for all f in $\mathcal{L}_0$.

Before proceeding with the proof, a few comments about the statements of this theorem and the next theorem are in order. First of all, the case $i = 0$ was excluded because it has been shown that $\mathcal{L}_0$ is invariant under C_φ^*. Now, if we say that $W_0 = C_\varphi^*|_{\mathcal{L}_0}$, then for $i \geq 0$ we can conclude by Lemma 2 that W_i is similar to a weighted bilateral shift. Lastly, in the formula for R_i we can divide by $z - \varphi(0)$ since $\varphi(0)$ is a zero of B^i of multiplicity i.

Proof. If φ is a hyperbolic disk automorphism with fixed points 1 and -1, and the Denjoy-Wolff point is 1, then φ can be written as $\varphi(z) = (z+r)(1+rz)^{-1}$, where r is real and nonzero. Convenient ways to write φ and its inverse are

$$\varphi(z) = \tfrac{z+\varphi(0)}{1+\varphi(0)z}, \qquad \varphi^{-1}(z) = \tfrac{z-\varphi(0)}{1-\varphi(0)z}.$$

A fact that is evident now and that will be used throughout the rest of the proof is that z_n is real for all n. One of the conclusions of Lemma 1 is that $\text{Range}(C_\varphi^*|_{B^i\mathcal{L}_0}) \subset \mathcal{L}_i$. Note that in particular, $\text{Range}(C_\varphi^*|_{B\mathcal{L}_0}) \subset B\mathcal{L}_0 \oplus \mathcal{L}_0$. It will now be shown that $\text{Range}(C_\varphi^*|_{B^i\mathcal{L}_0}) \subset B^i\mathcal{L}_0 \oplus B^{i-1}\mathcal{L}_0$ for $i \geq 2$.

Let f be in $\mathcal{L}_0$ and $i \geq j \geq 2$. Then

$$\begin{aligned}\langle C_\varphi^* B^i f, B^{i-j}k_n\rangle &= \langle B^i f, C_\varphi B^{i-j}k_n\rangle \\ &= \langle B^i f, (-1)^{i-j} B^{i-j} C_\varphi k_n\rangle\end{aligned}$$

from equation (1)

$$\begin{aligned}
&= (-1)^{i-j}\sqrt{1-z_n{}^2}\langle B^j f, \frac{1}{1-z_n\varphi(z)}\rangle \\
&= (-1)^{i-j}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\langle B^j f, \frac{1+\varphi(0)z}{1-\varphi^{-1}(z_n)z}\rangle \\
&= (-1)^{i-j}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\langle B^j f, \frac{1}{1-z_{n-1}z} + \frac{\varphi(0)z}{1-z_{n-1}z}\rangle \\
&= (-1)^{i-j}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\left(\langle B^j f, K_{z_{n-1}}\rangle + \varphi(0)\langle\left(\frac{B^j}{z}\right)f, K_{z_{n-1}}\rangle\right) \\
&= (-1)^{i-j}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\left((B(z_{n-1}))^j f(z_{n-1}) + \varphi(0)\frac{(B(z_{n-1}))^j}{z_{n-1}}f(z_{n-1})\right) \\
&= 0
\end{aligned}$$

since z_{n-1} is a zero of both B^j and B^j/z for $j \geq 2$.

Thus, $\text{Range}(C_\varphi^*|_{B^i\mathcal{L}_0}) \subset B^i\mathcal{L}_0 \oplus B^{i-1}\mathcal{L}_0$. Now, using the same technique as above, for f in $\mathcal{L}_0$,

$$\begin{aligned}
&\langle C_\varphi^* B^i f, B^{i-1}k_n\rangle \\
&\quad = (-1)^{i-1}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\left(B(z_{n-1})f(z_{n-1}) + \varphi(0)\left(\frac{B}{z}\right)(z_{n-1})f(z_{n-1})\right) \\
&\quad = (-1)^{i-1}\frac{\sqrt{1-z_n{}^2}}{1-z_n\varphi(0)}\varphi(0)\left(\frac{B}{z}\right)(z_{n-1})f(z_{n-1}) \\
&\quad = \begin{cases} (-1)^{i-1}\frac{f(0)\varphi(0)}{\sqrt{1-(\varphi(0))^2}}\left(\frac{B}{z}\right)(0) & \text{if } n=1, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \frac{B^i}{z-\varphi(0)}, B^{i-1}k_n\rangle &= \langle \frac{B}{z-\varphi(0)}, k_n\rangle \\ &= \langle \frac{-B(\varphi^{-1}(z))}{z-\varphi(0)}, k_n\rangle\end{aligned}$$

from equation (1)

$$
\begin{aligned}
&= -\langle \left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty} \frac{z_m - \varphi^{-1}(z)}{1 - z_m \varphi^{-1}(z)} \right) \left(\frac{-\varphi^{-1}(z)}{z - \varphi(0)} \right), k_n \rangle \\
&= \langle \left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty} \frac{z_m - \varphi^{-1}(z)}{1 - z_m \varphi^{-1}(z)} \right) \left(\frac{1}{1 - \varphi(0) z} \right), \sqrt{1 - {z_n}^2} K_{z_n} \rangle \\
&= \begin{cases} \left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty} z_m \right) \frac{1}{\sqrt{1-(\varphi(0))^2}} & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\frac{B}{z} \right)(0) \left(\frac{1}{\sqrt{1-(\varphi(0))^2}} \right) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

So

$$
\begin{aligned}
\langle R_i(B^i f), B^{i-1} k_n \rangle &= \begin{cases} (-1)^{i-1} \frac{f(0)\varphi(0)}{\sqrt{1-(\varphi(0))^2}} \left(\frac{B}{z} \right)(0) & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases} \\
&= \langle C_\varphi^* B^i f, B^{i-1} k_n \rangle.
\end{aligned}
$$

Now, the conclusion will follow if $R_i(B^i f)$ is in $B^{i-1}\mathcal{L}_0$. For this, it suffices to show that $B/(z - \varphi(0))$ is in $\mathcal{L}_0 = (BH^2)^{\perp}$. Let h be in H^2, so

$$\langle \frac{B}{z - \varphi(0)}, Bh \rangle$$

$$
\begin{aligned}
&= \langle \left(z \prod_{\substack{-\infty \\ m\neq 0,1}}^{\infty} \frac{z_m - z}{1 - z_m z} \right) \left(\frac{\varphi(0) - z}{1 - \varphi(0) z} \right) \left(\frac{1}{z - \varphi(0)} \right), \left(z \prod_{\substack{-\infty \\ m\neq 0}}^{\infty} \frac{z_m - z}{1 - z_m z} \right) h \rangle \\
&= \langle \frac{-1}{1 - \varphi(0) z}, \frac{\varphi(0) - z}{1 - \varphi(0) z} h \rangle \\
&= \langle -K_{\varphi(0)}, \frac{\varphi(0) - z}{1 - \varphi(0) z} h \rangle \\
&= 0. \quad \square
\end{aligned}
$$

The next theorem gives a similar result for the parabolic case. The basic structure for the proof is the same as that for the hyperbolic case. However, many of the details are different because the general formula for a parabolic disk automorphism is a little more complicated than that of a hyperbolic disk automorphism.

Theorem 4. *Let φ be a parabolic disk automorphism with Denjoy-Wolff point 1. Then $C_\varphi^*|_{B^i\mathcal{L}_0} = W_i + R_i$, where W_i is the compression of C_φ^* to $B^i\mathcal{L}_0$ and R_i is* *the rank one operator of* $B^i\mathcal{L}_0$ *into* $B^{i-1}\mathcal{L}_0$ *given by*

$$R_i(B^i f) = \frac{-\overline{\varphi(0)}f(0)(1-\varphi(0))^2 B^i}{(1-|\varphi(0)|^2)(z-\varphi(0))}$$

for all f *in* $\mathcal{L}_0$.

Proof. The general form for a parabolic disk automorphism with fixed point 1 is

$$\varphi(z) = \frac{(s-2i)z-s}{sz-s-2i},$$

where s is nonzero and real. It is immediately evident that $z_1 = \varphi(0) = s/(s+2i)$. In fact, more generally, we have $z_n = ns/(ns+2i)$ (see, for example, [3, p. 332]). From this, it follows that

$$\bar{z}_n = z_{-n}. \tag{2}$$

We also have

$$\varphi(z) = \frac{(2\varphi(0)-1)z-\varphi(0)}{\varphi(0)z-1}. \tag{3}$$

It follows that the inverse of φ is given by

$$\varphi^{-1}(z) = \frac{z-\varphi(0)}{\varphi(0)z-2\varphi(0)+1}. \tag{4}$$

As in the hyperbolic case, it is now shown that $\mathrm{Range}(C_\varphi^*|_{B^i\mathcal{L}_0})$ is a subset of $B^i\mathcal{L}_0 \oplus B^{i-1}\mathcal{L}_0$.

Let f be in $\mathcal{L}_0$ and let $i \geq j \geq 2$. Then

$$\langle C_\varphi^* B^i f, B^{i-j}k_n\rangle = \langle B^i f, C_\varphi B^{i-j}k_n\rangle.$$

Now, equation (1) gives

$$\begin{aligned} &= \langle B^i f, B^{i-j}C_\varphi k_n\rangle \\ &= \sqrt{1-|z_n|^2}\langle B^j f, \frac{1}{1-\bar{z}_n\varphi(z)}\rangle \end{aligned}$$

and it follows from equation (3) that

$$\begin{aligned} &= \sqrt{1-|z_n|^2}\langle B^j f, \frac{\varphi(0)z-1}{\varphi(0)z-1-\bar{z}_n((2\varphi(0)-1)z-\varphi(0))}\rangle \\ &= \sqrt{1-|z_n|^2}\langle B^j f, \frac{\varphi(0)z-1}{\varphi(0)\bar{z}_n-1-(2\varphi(0)\bar{z}_n-\bar{z}_n-\varphi(0))z}\rangle \\ &= \frac{\sqrt{1-|z_n|^2}}{\overline{\varphi(0)}z_n-1}\langle B^j f, \frac{\varphi(0)z-1}{1-\left(\frac{(2\varphi(0)-1)\bar{z}_n-\varphi(0)}{\varphi(0)\bar{z}_n-1}\right)z}\rangle \\ &= \frac{\sqrt{1-|z_n|^2}}{\overline{\varphi(0)}z_n-1}\langle B^j f, \frac{\varphi(0)z-1}{1-\varphi(\bar{z}_n)z}\rangle \\ &= \frac{\sqrt{1-|z_n|^2}}{\overline{\varphi(0)}z_n-1}\langle B^j f, \varphi(0)zK_{z_{n-1}}-K_{z_{n-1}}\rangle \end{aligned}$$

is a consequence of equation (2)

$$= \frac{\sqrt{1-|z_n|^2}}{\overline{\varphi(0)}z_n - 1}\left(\frac{(B(z_{n-1}))^j}{z_{n-1}} f(z_{n-1})\overline{\varphi(0)} - (B(z_{n-1}))^j f(z_{n-1})\right) = 0.$$

Now, it is evident that

$$\begin{aligned}\langle C_\varphi^* B^i f, B^{i-1}k_n\rangle &= \frac{\sqrt{1-|z_n|^2}}{\overline{\varphi(0)}z_n - 1}\overline{\varphi(0)}\langle\left(\frac{B}{z}\right)f, \frac{1}{1-\bar{z}_{n-1}z}\rangle \\ &= \begin{cases} \frac{-\overline{\varphi(0)}f(0)}{\sqrt{1-|\varphi(0)|^2}}\left(\frac{B}{z}\right)(0) & \text{if } n=1, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\frac{B^i}{z-\varphi(0)}, B^{i-1}k_n\rangle &= \langle\frac{B}{z-\varphi(0)}, k_n\rangle \\ &= \langle\frac{B(\varphi^{-1}(z))}{z-\varphi(0)}, k_n\rangle\end{aligned}$$

follows from equation (1) and

$$\begin{aligned}&= \langle\left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty}\frac{\bar{z}_m}{|z_m|}\frac{z_m-\varphi^{-1}(z)}{1-\bar{z}_m\varphi^{-1}(z)}\right)\left(\frac{\varphi^{-1}(z)}{z-\varphi(0)}\right), k_n\rangle \\ &= \langle\left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty}\frac{\bar{z}_m}{|z_m|}\frac{z_m-\varphi^{-1}(z)}{1-\bar{z}_m\varphi^{-1}(z)}\right)\left(\frac{1}{\varphi(0)z-2\varphi(0)+1}\right), \sqrt{1-|z_n|^2}K_{z_n}\rangle\end{aligned}$$

is obtained by equation (4)

$$\begin{aligned}&= \begin{cases}\left(\prod_{\substack{-\infty \\ m\neq 0}}^{\infty}\frac{\bar{z}_m z_m}{|z_m|}\right)\frac{\sqrt{1-|z_1|^2}}{(1-\varphi(0))^2} & \text{if } n=1, \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases}\left(\frac{B}{z}\right)(0)\frac{\sqrt{1-|\varphi(0)|^2}}{(1-\varphi(0))^2} & \text{if } n=1, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

So

$$\begin{aligned}\langle R_i(B^i f), B^{i-1}k_n\rangle &= \begin{cases}\frac{-\overline{\varphi(0)}f(0)}{\sqrt{1-|\varphi(0)|^2}}\left(\frac{B}{z}\right)(0) & \text{if } n=1, \\ 0 & \text{otherwise}\end{cases} \\ &= \langle C_\varphi^* B^i f, B^{i-1}k_n\rangle.\end{aligned}$$

Again, the proof will be complete after it is shown that $R_i(B^i f)$ is in $B^{i-1}\mathcal{L}_0$, which will follow if $B/(z-\varphi(0))$ is in $\mathcal{L}_0 = (BH^2)^\perp$. Let h be in H^2 and compute

$$\begin{aligned}
&\langle \frac{B}{z-\varphi(0)}, Bh\rangle \\
&= \langle \left(z \prod_{\substack{-\infty \\ m\neq 0,1}}^{\infty} \frac{z_m - z}{1-\bar{z}_m z}\right)\left(\frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}\right)\left(\frac{1}{z-\varphi(0)}\right), \left(z \prod_{\substack{-\infty \\ m\neq 0}}^{\infty} \frac{z_m - z}{1-\bar{z}_m z}\right) h\rangle \\
&= \langle \frac{-1}{1-\overline{\varphi(0)}z}, \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z} h\rangle \\
&= \langle -K_{\varphi(0)}, \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z} h\rangle \\
&= 0. \quad \square
\end{aligned}$$

The next theorem is important because, while the study of composition operators is relatively new, a lot of effort has been put into the study of block Toeplitz matrices.

Theorem 5. *If φ is a hyperbolic or a parabolic disk automorphism, then C_φ^* is similar to a block Toeplitz matrix of the form*

$$\begin{bmatrix}
W & R & 0 & 0 & 0 & \cdots \\
0 & \tau W & \tau R & 0 & 0 & \\
0 & 0 & W & R & 0 & \\
0 & 0 & 0 & \tau W & \tau R & \\
\vdots & & & & \ddots & \ddots \\
& & & & &
\end{bmatrix}$$

where W is the weighted bilateral shift on l^2 with weight sequence

$$\frac{\sqrt{1-|z_n|^2}}{\sqrt{1-|z_{n+1}|^2}},$$

R is a rank-one operator, and $\tau = -1$ in the hyperbolic case and $\tau = 1$ in the parabolic case.

Note that if $\tau = -1$, then we have a block Toeplitz matrix with the blocks down the main diagonal and the diagonal above the main diagonal, respectively, being

$$\begin{bmatrix} W & R \\ 0 & -W \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ -R & 0 \end{bmatrix}.$$

Proof. There exists a disk automorphism ω that moves the fixed points of φ to the special ones. Then ${C_\omega}^{-1} C_\varphi C_\omega = C_{\omega\varphi\omega^{-1}}$. Thus, we may assume that φ satisfies the hypotheses of one of the two previous theorems, which say that

$$C_\varphi^* = \begin{bmatrix} W_0 & R_1 & 0 & 0 & 0 & \cdots \\ 0 & W_1 & R_2 & 0 & 0 & \\ 0 & 0 & W_2 & R_3 & 0 & \\ 0 & 0 & 0 & W_3 & R_4 & \\ \vdots & & & & \ddots & \ddots \\ & & & & & \end{bmatrix}$$

on $\mathcal{L}_0 \oplus B\mathcal{L}_0 \oplus B^2\mathcal{L}_0 \oplus \cdots$. Now, if U_i is defined as in Lemma 2, then by Lemma 2, ${U_i}^* W_i U_i = \tau^i C_\varphi^*|_{\mathcal{L}_0} = \tau^i W_0$, and a calculation shows that ${U_{i-1}}^* R_i U_i = \tau^{i-1} R_1 U_1$. So if U is defined to be the unitary operator represented by the matrix on $\mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \cdots$ with diagonal entries $I, U_1, U_2, U_3, \ldots$, then

$$U^* C_\varphi^* U = \begin{bmatrix} W_0 & R_1U_1 & 0 & 0 & 0 & \cdots \\ 0 & \tau W_0 & \tau R_1 U_1 & 0 & 0 & \\ 0 & 0 & W_0 & R_1U_1 & 0 & \\ 0 & 0 & 0 & \tau W_0 & \tau R_1 U_1 & \\ \vdots & & & & \ddots & \ddots \\ & & & & & \end{bmatrix}$$

on $\mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \cdots$. Now the conclusion follows since W_0 is similar to the bilateral weighted shift W with the proper weights. □

Let's contrast this with what Nordgren, Rosenthal, and Wintrobe obtained (see [3, Thm. 5.7]). First, let $\mathcal{K}_0 = \operatorname{span}\{\varphi^{(n)} : n \in Z\}$. They show that $\mathcal{K}_0 = (zBH^2)^\perp = C \oplus \mathcal{L}$, C being the complex numbers, and that the compression of C_φ to $\mathcal{L}$ is similar to the weighted bilateral shift with weight sequence $\{\sqrt{1-|z_{-n}|^2} / \sqrt{1-|z_{-n-1}|^2}\}$. By reversing the order of the basis, it can be seen that the operator W obtained in the previous theorem is unitarily equivalent to the adjoint of this weighted bilateral shift. Now, let $\mathcal{K} = \mathcal{K}_0 + B\mathcal{K}_0$. They show that $\mathcal{K}$ is equal to $(zB^2H^2)^\perp$. Furthermore, $B^i\mathcal{K}$ is invariant under C_φ for $i = 0, 1, 2, \ldots$ and $B^{2n}\mathcal{K} \perp B^{2m}\mathcal{K}$ whenever $|n-m| \geq 2$. Define the two subspaces

$$M = \sum_{i=0}^{\infty} \oplus B^{4i}\mathcal{K}, \qquad N = \sum_{i=0}^{\infty} \oplus B^{4i+2}\mathcal{K}.$$

It is shown that $H^2 = M + N$ and $C_\varphi|_M$ and $C_\varphi|_N$ are each unitarily equivalent to inflations of $C_\varphi|_\mathcal{K}$, that is, to the block diagonal matrix

$$\begin{bmatrix} C_\varphi|_\mathcal{K} & 0 & 0 & \cdots \\ 0 & C_\varphi|_\mathcal{K} & 0 & \\ 0 & 0 & C_\varphi|_\mathcal{K} & \\ \vdots & & & \ddots \end{bmatrix}.$$

It follows from this that C_φ is similar to a diagonal block Toeplitz matrix which, in some ways, is easier to deal with than the model of this paper. However, the operator on the diagonal, $C_\varphi|_\mathcal{K}$, is not as detailed as the operators W and R above. Also, this doesn't give a model for the adjoint, C_φ^*, very easily.

References

[1] C.C. Cowen, Composition Operators on Hilbert Spaces of Analytic Functions: A Status Report, *Proceedings of Symposia in Pure Mathematics* **51**(1990), 131–145. MR **91m:**47043

[2] C.C. Cowen, The commutant of an analytic Toeplitz operator, *Trans. Amer. Math. Soc.* **239**(1978), 1–31. MR **58:**2420

[3] E.A. Nordgren, P. Rosenthal, and F.S. Wintrobe, Invertible Composition Operators on H^p, *J. Functional Analysis* **73**(1987), 324–344. MR **89c:**47044

[4] H.S. Shapiro and A.L. Shields, On some interpolation problems for analytic functions, *American J. Math.* **83**(1961), 513–532. MR **24:**A3280

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