

UNIONS OF LOEB NULLSETS

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ABSTRACT. The union of every point-finite, completely measurable family of Loeb nullsets is itself a Loeb nullset, provided the nonstandard model satisfies a simple set-theoretic condition. One application of this result is that every Loeb measurable function into a metric space has a lifting.

1. INTRODUCTION

Recall Lusin's Theorem:

Let f be a measurable function from a Radon probability space (X, \mathcal{B}, P) to a second countable topological space. Then for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that the restriction of f to K is continuous and $P(K) > 1 - \epsilon$.

(By **Radon** we mean that X is topological, \mathcal{B} contains all Borel sets, and P is compact-inner-regular, that is, $P(E) = \sup\{P(K) \mid K \subseteq E, K \text{ compact}\}$ for every $E \in \mathcal{B}$.)

More generally, a function f from a topological probability space (X, \mathcal{B}, P) to a topological space Y is **Lusin measurable** provided for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that the restriction of f to K is continuous and $P(K) > 1 - \epsilon$. Fremlin [F] has proved that if (X, \mathcal{B}, P) is Radon and Y is metric, then f is Lusin measurable.

A corresponding result to Lusin's Theorem in nonstandard measure theory is due originally to Robert Anderson ([SB], Theorem 2.1.4):

Every measurable function from a Loeb probability space to a second countable topological space has a lifting.

The correspondence is made precise in [R2], where it is proved that a function is Lusin measurable if and only if it admits a "two-legged" lifting. In addition, necessary and sufficient conditions for the existence of a lifting are given there; this yields a new sufficient condition for a function to be Lusin measurable.

Prikry and Kupka show in [KuP] that Fremlin's result follows easily once one knows that every point-finite, completely measurable family of Radon nullsets is a Radon nullset (see below). Prikry and Koumoullis [KoP] have extended this latter result to arbitrary compact probability spaces. (See §2 for definitions.) An analogous result about unions of Loeb nullsets, and a corresponding generalization of Anderson's theorem to general metric spaces, would be an immediate consequence of the standard theory, provided Loeb measures were compact.

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In [R1] it is shown that every compact probability space is the image of a Loeb space under a measurable map (Lemma 5.1, below); this suggests that Loeb spaces might indeed be compact. However, in a recent paper J. Aldaz [A] gives an example of a Loeb space which is *not* compact, rendering both the unions-of-nullsets result and the corresponding lifting result nontrivial for Loeb spaces.

This paper gives proofs for both these results. In fact, the unions-of-Loeb-nullsets result is strictly stronger than the corresponding standard result (see Corollary 5.2, below). (Weaker results, requiring the much stronger hypothesis that the nonstandard model is “fully” saturated, appeared in [R3].)

2. PRELIMINARIES

For X a set, write $\mathcal{P}(X)$ for the power set of X , $\mathcal{P}_n(X)$ for $\{E \in \mathcal{P}(X) \mid \text{card}(E) = n\}$, and $\mathcal{P}_{\text{fin}}(X)$ for $\{E \in \mathcal{P}(X) \mid E \text{ finite}\} = \bigcup_{n=0}^{\infty} \mathcal{P}_n(X)$.

Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$. \mathcal{F} is **point-finite** if for every $x \in X$, $\{E \in \mathcal{F} \mid x \in E\}$ is finite. \mathcal{F} has the **finite intersection property** if $E_1 \cap \dots \cap E_n \neq \emptyset$ whenever $\{E_1, \dots, E_n\} \subseteq \mathcal{F}$, $n \in \mathbb{N}$. \mathcal{F} is **compact** provided that for every subfamily $\mathcal{K} \subseteq \mathcal{F}$ with the finite intersection property, $\bigcap \mathcal{K} \neq \emptyset$.

Suppose now that (X, \mathcal{B}, P) is a probability space. A family $\mathcal{F} \subseteq \mathcal{B}$ is **completely measurable** provided $\bigcup \mathcal{K} \in \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{F}$.

The probability space (X, \mathcal{B}, P) is **compact** provided P is inner-regular with respect to a compact family $\mathcal{K} \subseteq \mathcal{B}$.

The reader is assumed to be familiar with nonstandard analysis in general, and Loeb measures in particular, and is referred to [C] or [SB] for background. All nonstandard models will be assumed \aleph_1 -saturated.

Suppose that X is internal and $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$. Call \mathcal{A} **κ -terraced** provided (i) $\kappa = \text{card}(\mathcal{A})$, and (ii) For some nondecreasing sequence $\{\mathcal{A}_i\}_{i < \kappa}$ with each $\mathcal{A}_i \subseteq \mathcal{A}$ compact, $\mathcal{A} = \bigcup_{i < \kappa} \mathcal{A}_i$. For $A \in \mathcal{A}$ let $\text{Level}(A) = \min\{i < \kappa \mid A \in \mathcal{A}_i\}$. (Note this definition depends on choice of terracing; a particular one will always be assumed already chosen.)

Suppose α and κ are cardinals; say that α is **not measurably cofinal in κ** , $\text{NMC}(\alpha, \kappa)$, if either α is not a measurable cardinal (see below) or α is not cofinal in κ . Note that $\text{NMC}(\alpha, \kappa)$ holds for every α precisely when the cofinality of κ is not measurable.

Lemma 2.1. *There exist nonstandard models such that for every internal X and every internal $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$ with $\kappa = \text{card}(\mathcal{A})$ infinite, \mathcal{A} is κ -terraced and $\text{NMC}(\alpha, \kappa)$ for all α .*

Proof. Suppose the nonstandard model satisfies the special model axiom SMA_β (see [R4] or [J]). This means that for any first-order language \mathcal{L} with fewer than β relation, function, and constant symbols, and every \mathcal{L} -model $\mathfrak{A} = (A, \dots)$ with both A and every interpretation of every relation and function symbol internal, \mathfrak{A} is a special model (see [CK] for definitions). It is shown in [R4] that all internal infinite sets have the same external cardinality κ when SMA_β holds, $\beta \geq \aleph_0$.

Suppose that X and $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$ are internal, and that \mathcal{A} is infinite. Let \mathcal{L} be the first-order language with unary predicate symbol $Q(x)$ and binary predicate $E(x, y)$, and let $\mathfrak{A} = (A, \dots)$ be the \mathcal{L} -structure with $A = X \cup \mathcal{A}$, $Q^{\mathfrak{A}} = \mathcal{A}$, and $E^{\mathfrak{A}} = \{(x, y) \in X \times \mathcal{A} \mid x \in y\}$. By SMA_β , there is an elementary chain $\mathfrak{A}_0 \preceq$

$\mathfrak{A}_1 \preceq \dots$ with \mathfrak{A}_i i^+ -saturated and $\mathfrak{A} = \bigcup_{i < \kappa} \mathfrak{A}_i$. Let $\{a_i\}_{i < \kappa}$ be any enumeration of \mathcal{A} , and put $\mathcal{A}_i = Q^{\mathfrak{A}_i} \cap \{a_j\}_{j \leq i}$. Evidently $\mathcal{A} = \bigcup_{i < \kappa} \mathcal{A}_i$. It remains to show that \mathcal{A}_i is compact.

Suppose $\mathcal{K} \subseteq \mathcal{A}_i$ has the finite intersection property. Consider the set $\Gamma(x)$ of $\mathcal{L} \cup \mathcal{A}_i$ -formulas of form $E(x, K)$, $K \in \mathcal{K}$. This set has cardinality $\leq i$, and since \mathcal{K} has the finite intersection property, $\Gamma(x)$ is finitely consistent, so by i^+ -saturation of \mathfrak{A}_i there is an $a \in \mathcal{A}_i$ with $\mathfrak{A}_i \models E(a, K)$ for all $K \in \mathcal{K}$, i.e., $a \in \bigcap \mathcal{K}$, proving compactness.

Thus any nonstandard model where (i) SMA_β holds for some $\beta \geq \aleph_0$, and (ii) κ is not cofinally measurable (for example, κ regular and not measurable) will work; this proves the lemma.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is an internal, *finitely-additive probability space. Denote by $(\Omega, \mathcal{A}_L, \mu_L)$ the (standard) probability space generated from $(\Omega, \mathcal{A}, \mu)$ by the Loeb construction. Recall that a measurable function f from Ω to a topological space Y has a lifting F provided $F : \Omega \rightarrow^* Y$ is internal and $F(\omega) \approx f(\omega)$ μ_L -almost everywhere.

The main results can now be stated; the proofs are deferred to §4.

Theorem 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal *finitely-additive probability space, and let $\mathcal{E} = \{A_i\}_{i < \alpha}$ be a point-finite, completely measurable family of μ_L -nullsets. Suppose that \mathcal{A} is κ -terraced, and that $NMC(\alpha, \kappa)$ holds. Then $\mu_L(\bigcup_{i < \alpha} A_i) = 0$.*

Corollary 2.3. *Suppose $(\Omega, \mathcal{A}, \mu)$ is an internal *finitely-additive probability space, where \mathcal{A} is κ -terraced, and that $NMC(\alpha, \kappa)$ holds for all α . Let f be a measurable function from Ω to a metric space Y . Then for some $\Omega' \subseteq \Omega$ with $\mu_L(\Omega') = 1$, the restriction of f to Ω' has second countable range (and so f has a lifting, by Anderson's lifting theorem).*

3. USEFUL STANDARD LEMMAS

The first lemma is due to Bernstein; see ([KuP], Proposition 3.5) for a proof.

Lemma 3.1. *There is a subset \mathbb{B} of $[0, 1]$ with $\text{card}(\mathbb{B}) = 2^{\aleph_0}$ such that every closed subset of \mathbb{B} is at most countable.*

Let X be an arbitrary set, and $\nu : \mathcal{P}(X) \rightarrow [0, \infty)$. A set $E \in \mathcal{P}(X)$ is an **atom** for ν if $\nu(E) \neq 0$ and $\{\nu(B), \nu(E - B)\} = \{0, \nu(E)\}$ for every $B \subseteq E$. If ν has no atoms it is **atomless**. If α is an infinite cardinal and ν is a finite measure on $(\alpha, \mathcal{P}(\alpha))$ for which (i) α is an atom, (ii) singletons are nullsets, and (iii) $\nu(\bigcup \mathcal{E}) = 0$ whenever \mathcal{E} is a family of nullsets with $\text{card}(\mathcal{E}) < \alpha$, then α is a **measurable cardinal**.

The following is a form of Ramsey's Theorem due to Hajnal and Erdős; see [K] for a proof.

Lemma 3.2. *If α is a measurable cardinal and $\varphi : \mathcal{P}_{\text{fin}}(\alpha) \rightarrow \{0, 1\}$, then there is a subset $\hat{\alpha} \subseteq \alpha$ such that $\text{card}(\hat{\alpha}) = \alpha$ and such that φ is constant on $\mathcal{P}_n(\hat{\alpha})$ for every n .*

The final lemma in this section is ([KoP], Lemma 4).

Lemma 3.3. *If \mathcal{F} is an uncountable family of nonnullsets in a finite measure space, then for some x , $\{A \in \mathcal{F} \mid x \in A\}$ is infinite. (In other words, there is no point-finite uncountable family of nonnullsets in a finite measure space.)*

4. PROOF OF MAIN RESULTS

Proof of Theorem 2.2. Otherwise, let $\mathcal{E} = \{A_i\}_{i < \alpha}$ be a counterexample, that is, a point-finite completely measurable family of Loeb nullsets with $\mu_L(\bigcup_{i < \alpha} A_i) > 0$.

Without loss of generality, α has the least cardinality of any counterexample, and $\mu_L(\bigcup_{i < \alpha} A_i) = 1$. Put $A = \bigcup_{i < \alpha} A_i$.

For $E \subseteq \alpha$ put $\nu(E) = \mu_L(\bigcup_{i \in E} A_i)$. Consider two cases:

Case 1. ν is atomless on $(\alpha, \mathcal{P}(\alpha))$. Inductively define partitions P_λ , $\lambda < \omega_1$, of α as follows. Put $P_0 = \{\alpha\}$. If $\beta < \omega_1$ is a limit ordinal, let P_β consist of all nonempty intersections of the form $\bigcap_{\lambda < \beta} E_\lambda$, where $E_\lambda \in P_\lambda$ for all $\lambda < \beta$. Given

P_λ , construct $P_{\lambda+1}$ by replacing each E in P_λ by itself if $\nu(E) = 0$, otherwise by disjoint sets E_1, E_2 where $E_1 \cup E_2 = E$ and $0 < \nu(E_1) \leq \nu(E_2) \leq \nu(E)$. (We may take $\nu(E_1) \neq 0$ since from the definition of ν , $\nu(E_1 \cup E_2) \leq \nu(E_1) + \nu(E_2)$.)

Let $\alpha = E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$, with $E_\lambda \in P_\lambda$ for all $\lambda < \omega_1$. (Call such a sequence *adapted*.) Suppose (for a contradiction) that $\nu(E_\lambda) \neq 0$ for every λ . Put $B_\lambda = E_\lambda \setminus E_{\lambda+1}$. By the construction, $\nu(B_\lambda) > 0$ for every λ ; but then $\{\bigcup_{i \in B_\lambda} A_i\}_{\lambda < \omega_1}$ is a point-finite uncountable collection of nonnull sets, contradicting

Lemma 3.3.

It follows that every adapted sequence is eventually constant, so there are at most \aleph_0 of them. Let $\{E_\tau\}_{\tau \in T}$ be the set of intersections of adapted sequences; we may take $T \subseteq \mathbb{B}$. For $\tau \in T$ put $A_\tau = \bigcup_{i \in E_\tau} A_i$; note that $\mu_L(A_\tau) = \nu(E_\tau) = 0$, and that $\{E_\tau\}_{\tau \in T}$ forms a partition of α .

For $x \in A$ let $f(x)$ be the smallest (in the sense of the ordering on \mathbb{B} inherited from \mathbb{R}) of the finitely many τ with $x \in A_\tau$. Note that for $0 \leq r \leq 1$, $f^{-1}([0, r]) = \bigcup_{\tau < r} A_\tau$, so $f : A \rightarrow [0, 1]$ is Loeb measurable.

It follows that f has a lifting, and for some internal $K \subseteq A$ with $\mu_L(K) > 0$ and some internal $F : K \rightarrow^* [0, 1]$, $F(x) \approx f(x)$ for all $x \in K$. Since $f(K) = \text{st}(F(K))$ is closed ([SB], Proposition 2.1.1) and $f(K) \subseteq \mathbb{B}$, $f(K)$ is at most countable, and

$$\begin{aligned} 0 &< \mu_L(K) \\ &\leq \mu_L(f^{-1}(f(K))) \\ &= \mu_L\left(\bigcup_{\tau \in f(K)} f^{-1}(\tau)\right) \\ &\leq \mu_L\left(\bigcup_{\tau \in f(K)} A_\tau\right) \quad (\text{since } f(x) = \tau \text{ only if } x \in A_\tau) \\ &= 0 \quad (\text{since } f(K) \text{ countable and } \mu_L(A_\tau) = 0). \end{aligned}$$

This is a contradiction, and proves the theorem in this case.

Case 2. ν has an atom in $(\alpha, \mathcal{P}(\alpha))$. Without loss of generality, α is the atom. By minimality of α , if $E \subseteq \alpha$ and $\text{card}(E) < \alpha$, then $\nu(E) = 0$. It is easy to verify in this case that ν is actually a measure, so α is a measurable cardinal.

For each $\lambda < \alpha$, put $B_\lambda = \bigcup_{i>\lambda} A_i$. Note $\mu_L(B_\lambda) > 0$, so there is an internal $K_\lambda \subseteq B_\lambda$ with $\mu_L(K_\lambda) > 0$. Choose this K_λ so that $\text{Level}(K_\lambda)$ is minimal.

Define $\varphi : \mathcal{P}_{\text{fin}}(\alpha) \rightarrow \{0, 1\}$ by

$$\varphi(E) = \begin{cases} 1 & \text{if } E = \emptyset \text{ or } \bigcap_{\lambda \in E} K_\lambda \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.2 there is a subset $\hat{\alpha} \subseteq \alpha$, $\text{card}(\hat{\alpha}) = \alpha$, such that φ is constant on each $\mathcal{P}_n(\hat{\alpha})$. By Lemma 3.3, $\varphi \equiv 1$ on $\mathcal{P}_n(\hat{\alpha})$ whenever $n > 1$, so $\{K_\lambda\}_{\lambda \in \hat{\alpha}}$ has the finite intersection property.

Let $\mathcal{O} : \hat{\alpha} \rightarrow \kappa$ be the map $\mathcal{O}(\lambda) = \text{Level}(K_\lambda)$. Since B_λ decreases in λ and K_λ is chosen to minimize $\text{Level}(K_\lambda)$ for each λ , \mathcal{O} is nondecreasing on $\hat{\alpha}$.

Define a function $\varphi_1 : \mathcal{P}_{\text{fin}}(\hat{\alpha}) \rightarrow \{0, 1\}$ by

$$\varphi_1(E) = \begin{cases} 1 & \text{if } \mathcal{O}(\lambda_1) = \mathcal{O}(\lambda_2) \ \forall \lambda_1, \lambda_2 \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3.2 again, there is $\hat{\hat{\alpha}} \subseteq \hat{\alpha}$ with $\text{card}(\hat{\hat{\alpha}}) = \alpha$ and φ_1 constant on $\mathcal{P}_n(\hat{\hat{\alpha}})$ for every n .

Suppose (for a contradiction) that $\varphi_1 \equiv 1$ on $\mathcal{P}_2(\hat{\hat{\alpha}})$; then for some $\gamma < \kappa$, $\gamma = \mathcal{O}(\lambda) = \text{Level}(K_\lambda)$ for all $\lambda \in \hat{\hat{\alpha}}$. Since $\{K_\lambda\}_{\lambda \in \hat{\hat{\alpha}}}$ has the finite intersection property and $\{K \in {}^*\mathcal{P}(\Omega) \mid \text{Level}(K) = \gamma\}$ is a compact family, there is an $x \in \bigcap_{\lambda \in \hat{\hat{\alpha}}} K_\lambda$. By point-finiteness of $\{A_i\}_{i<\alpha}$, there is a largest $\beta < \alpha$ with $x \in A_\beta$.

Let $\lambda \in \hat{\hat{\alpha}}$, $\lambda > \beta$; then $x \notin \bigcup_{i \geq \lambda} A_i = B_\lambda$, but $x \in K_\lambda \subseteq B_\lambda$, a contradiction.

Therefore, $\varphi_1 \equiv 0$ on $\mathcal{P}_2(\hat{\hat{\alpha}})$, so \mathcal{O} is strictly increasing on $\hat{\hat{\alpha}}$. The argument used in the last paragraph shows that $\{\mathcal{O}(\lambda)\}_{\lambda \in \hat{\hat{\alpha}}}$ is unbounded in κ , so $\hat{\hat{\alpha}}$, hence α , is cofinal in κ , contradicting the hypothesis on κ . The theorem is proved.

Proof of Corollary 2.3. By paracompactness of Y , for every $n \in \mathbb{Z}^+$ there is a point-finite open cover \mathcal{U}_n of Y by $\frac{1}{n}$ -balls. By Lemma 3.3, we can write $\mathcal{U}_n = \mathcal{U}_n^1 \cup \mathcal{U}_n^2$, where (i) \mathcal{U}_n^1 is countable, (ii) $\mathcal{U}_n^1 \cap \mathcal{U}_n^2 = \emptyset$, and (iii) for every $u \in \mathcal{U}_n^2$, $f^{-1}(u)$ is a μ_L -nullset. Put $\Omega_n = f^{-1}(\bigcup \mathcal{U}_n^1)$; by Theorem 2.2, $f^{-1}(\Omega \setminus \Omega_n) \subseteq \bigcup_{\mu \in \mathcal{U}_n^2} f^{-1}(\mu)$ is

a μ_L -nullset. Put $\Omega' = \bigcap_n \Omega_n$; it is easy to confirm that Ω' satisfies the conditions of the lemma.

5. A STANDARD COROLLARY

Lemma 5.1. *Let (X, \mathcal{B}, P) be a (standard) compact probability space, where the nonstandard model is $\text{card}(\mathcal{B})^+$ -saturated. Then there is a Loeb space $(\Omega, \mathcal{A}_L, \mu_L)$ and a measurable function $\varphi : \Omega \rightarrow X$, so that $P = \mu_L \circ \varphi^{-1}$.*

Proof. See [R1].

Corollary 5.2. *Let (X, \mathcal{B}, P) be a compact probability space, and $\{A_i\}_{i < \alpha}$ a point-finite completely measurable family of P -nullsets. Then $P(\bigcup_{i < \alpha} A_i) = 0$.*

Proof. There exist nonstandard models satisfying SMA_β for β arbitrarily large; it follows from Lemma 2.1 that there is a nonstandard model which is $\text{card}(\mathcal{B})^+$ -saturated, and for some κ every internal family of internal sets is κ -terraced and satisfies $\text{NMC}(\alpha, \kappa)$. Let $(\Omega, \mathcal{A}_L, \mu_L)$ and $\varphi : \Omega \rightarrow X$ be as given in Lemma 5.1. Then $\{\varphi^{-1}(A_i)\}_{i < \alpha}$ is a point-finite completely measurable family of μ_L -nullsets, so $P(\bigcup_{i < \alpha} A_i) = \mu_L(\varphi^{-1}(\bigcup_{i < \alpha} A_i)) = \mu_L(\bigcup_{i < \alpha} \varphi^{-1}(A_i)) = 0$ by Theorem 2.2.

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