

## ON THE EXISTENCE OF SOLUTIONS OF NONLINEAR EQUATIONS

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**ABSTRACT.** Results on the existence of solutions are derived for asymptotically quasilinear, nonlinear operator equations. Applications are given to implicit nonlinear integral equations.

### 1. INTRODUCTION

The purpose of this paper is to study the existence of a solution for the operator equation

$$(1.1) \quad L(x) = N(x) + h,$$

where  $L: H \rightarrow H$  is continuous,  $N: H \rightarrow H$  is continuous compact,  $h \in H$  and  $H$  is a real Hilbert space with the inner product  $(\cdot, \cdot)$  and norm  $|\cdot|$ . We assume that  $L$  is asymptotically linear at infinity and  $N$  is asymptotically quasilinear at infinity. The asymptotic quasilinearity of  $N$  at infinity roughly speaking means the approximability of  $N$  as  $x \rightarrow \infty$  by a family  $\mathcal{M}$  of bounded linear maps. We are motivated by the papers [6], [9] for introducing this notion. Under some conditions on  $\mathcal{M}$  we associate to  $\mathcal{M}$  a generalized Leray–Schauder degree in the sense of the papers [2], [10]. Then we show that (1.1) is solvable for any  $h \in H$ . This is a generalization of the well-known results about asymptotically linear operators (see [4], [7], [8], [15]).

We also study the case when the operators  $L$  and  $N$  leave a wedge in  $H$  invariant. We find sufficient conditions for the existence of a solution of (1.1) in that wedge. This is an extension of similar results of the papers [1], [5], [9], [11].

Applications are given to implicit nonlinear integral equations. We are motivated by the papers [12]–[14] to study implicit integral equations, because boundary value problems of differential equations are studied in [14] when highest-order derivatives are not solvable as well as there are certain stimulations from physics in [12], [13]. Finally, we note that results of this paper can be also applied to problems studied in [2], [10].

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## 2. SOLVABILITY OF (1.1)

The following definitions will be needed in the sequel (see [2, p. 946]).

A mapping  $f: H \rightarrow H$  is:

- *monotone* (denote  $f \in MON$ ), if  $(f(u) - f(v), u - v) \geq 0$  for all  $u, v \in H$ ;
- *pseudomonotone* ( $f \in PM$ ), if for any sequence  $\{u_n\}$  in  $H$  with  $u_n \rightharpoonup u$  (weak convergence) and  $\overline{\lim}(f(u_n), u_n - u) \leq 0$ , it follows that  $f(u_n) \rightharpoonup f(u)$  and  $(f(u_n), u_n) \rightarrow (f(u), u)$ ;
- *of class  $S_+$*  ( $f \in S_+$ ), if for any sequence  $\{u_n\}$  in  $H$  with  $u_n \rightharpoonup u$  and  $\overline{\lim}(f(u_n), u_n - u) \leq 0$ , it follows that  $u_n \rightarrow u$ ;
- *compact* ( $f \in COMP$ ), if it is continuous and for any bounded sequence  $\{u_n\}$  in  $H$  the sequence  $\{f(u_n)\}$  has a convergent subsequence;
- *completely continuous* ( $f \in CC$ ), if for any sequence  $\{u_n\}$  in  $H$  with  $u_n \rightharpoonup u$ , it follows that  $f(u_n) \rightarrow f(u)$ ;
- *bounded*, if it takes any bounded set of  $H$  into a bounded set.

We note that the following relations hold between the above definitions

$$(2.0) \quad \begin{aligned} CC &\subset COMP, & S_+ &\subset PM, & MON &\subset PM; \\ f_1 \in S_+, f_2 \in COMP &\Rightarrow f_1 - f_2 \in S_+; \\ f_1 \in PM, f_2 \in CC &\Rightarrow f_1 - f_2 \in PM; \\ f \in PM &\Rightarrow f + \varepsilon \mathbb{I} \in S_+ \forall \varepsilon > 0, \\ &\text{where } \mathbb{I}: H \rightarrow H \text{ is the identity map.} \end{aligned}$$

In what follows, we shall assume that the mappings are bounded and continuous.

**Definition 2.1.** A map  $N: H \rightarrow H$  is said to be asymptotically quasilinear at infinity if there is a mapping  $N_\infty: H \rightarrow 2^H$  such that

- i)  $N_\infty(u)$  is a closed nonempty subset of  $H$  for any  $u \in H$ ;
- ii)  $\alpha N_\infty(u) = N_\infty(\alpha u) \forall \alpha \geq 0, \forall u \in H$ ;
- iii)  $\overline{\bigcup_{u \in S} N_\infty(u)}$  is a compact subset for  $S = \{u \in H \mid |u| = 1\}$ ;
- iv)  $N_\infty$  is upper-semicontinuous;
- v)  $\forall \varepsilon > 0 \exists R > 0$  such that  $\forall u, |u| \geq R, \exists w_u \in N_\infty(u)$  satisfying

$$\frac{|N(u) - w_u|}{|u|} \leq \varepsilon.$$

$N_\infty$  is said to be the asymptote of  $N$ .

*Remark 2.2.* Let us take

$$N_\infty(u) = \{Ku\} \quad \forall u \in H$$

for a compact linear map  $K: H \rightarrow H$ . Then the assumptions i)–iv) of Definition 2.1 are satisfied trivially. The assumption v) expresses the usual asymptotic linearity at infinity. So a mapping  $L: H \rightarrow H$  is said to be asymptotically linear at infinity if there is a bounded linear map (the so-called asymptote of  $L$ )  $L_\infty: H \rightarrow H$  such that

$$\lim_{|u| \rightarrow \infty} \frac{|L(u) - L_\infty u|}{|u|} = 0.$$

**Theorem 2.3.** *Suppose  $L, N$  are asymptotically linear, quasilinear at infinity with the asymptotes  $L_\infty \in S_+, N_\infty$ , respectively. Assume  $L - N \in PM$  and there is a compact linear map  $A: H \rightarrow H$  such that*

$$(2.1) \quad 0 \notin L_\infty u - \lambda N_\infty(u) - (1 - \lambda)Au \quad \forall u \in S, \forall \lambda \in [0, 1].$$

Then (1.1) has a solution for any  $h \in H$ .

*Proof.* For simplicity, we consider the case  $h = 0$ . First, we show the existence of  $\delta > 0, R_0 > 0$  such that

$$(2.2) \quad \begin{aligned} & |\lambda L(u) - \lambda N(u) + (1 - \lambda)(L_\infty u - Au)| \geq \delta \\ & \forall u, |u| \geq R_0, \quad \forall \lambda \in [0, 1]. \end{aligned}$$

For this purpose, we compute

$$(2.3) \quad \begin{aligned} & \lambda L(u) - \lambda N(u) - (1 - \lambda)Au + (1 - \lambda)L_\infty u \\ & = |u| \left( L_\infty \frac{u}{|u|} - \lambda \frac{w_u}{|u|} - (1 - \lambda)A \frac{u}{|u|} + \lambda \frac{L(u) - L_\infty u}{|u|} + \lambda \frac{w_u - N(u)}{|u|} \right), \end{aligned}$$

where  $w_u$  is from Definition 2.1. Now we show that there is  $c_1 > 0$  such that

$$(2.4) \quad \begin{aligned} & |L_\infty z - \lambda w_z - (1 - \lambda)Az| \geq c_1 \\ & \forall z \in S, \forall w_z \in N_\infty(z), \quad \forall \lambda \in [0, 1]. \end{aligned}$$

If it is not true, then there is a sequence  $\lambda_i \in [0, 1], z_i \in S, w_{z_i} \in N_\infty(z_i)$  such that  $\lambda_i \rightarrow \lambda_0$  and

$$L_\infty z_i - \lambda_i w_{z_i} - (1 - \lambda_i)Az_i \rightarrow 0.$$

We can assume  $z_i \rightharpoonup z_0$  (the weak convergence) and  $w_{z_i} \rightarrow \bar{w}$  (see the assumption iii) of Definition 2.1). So we have both  $Az_i \rightarrow Az_0$ , since  $A$  is compact linear, and

$$L_\infty z_i - \lambda_0 \bar{w} - (1 - \lambda_0)Az_0 \rightarrow 0.$$

Since  $L_\infty \in S_+$ , we can assume  $z_i \rightarrow z_0$ . This gives

$$(2.5) \quad L_\infty z_0 - \lambda_0 \bar{w} - (1 - \lambda_0)Az_0 = 0.$$

Finally, we know  $w_{z_i} \rightarrow \bar{w}, w_{z_i} \in N_\infty(z_i)$  and  $z_i \rightarrow z_0$ . The assumptions i) and iv) of Definition 2.1 imply  $\bar{w} \in N_\infty(z_0)$ . Since (2.5) contradicts (2.1), (2.4) is proved.

We return to (2.3). By the assumption ii) of Definition 2.1 we have

$$\frac{w_u}{|u|} \in N_\infty \left( \frac{u}{|u|} \right)$$

in (2.3). We see now that (2.4) gives (2.2) for  $R_0$  sufficiently large.

According to the assumptions  $L - N \in PM$ ,  $L_\infty \in S_+$ ,  $A \in COMP$  and (2.0), we have that

$$\lambda(L - N) + (1 - \lambda)(L_\infty - A) + \varepsilon\mathbb{I} \in S_+ \quad \forall \lambda \in [0, 1],$$

where  $\mathbb{I}: H \rightarrow H$  is the identity map and  $\varepsilon > 0$ . So the following relations hold:

$$\text{deg}(L - N + \varepsilon\mathbb{I}, B_{R_0}, 0) = \text{deg}(L_\infty - A, B_{R_0}, 0) \neq 0,$$

where  $B_{R_0} = \{u \in H \mid |u| < R_0\}$ ,  $\varepsilon > 0$  is sufficiently small and  $\text{deg}$  is the generalized Leray–Schauder degree in the sense of the papers [2], [10]. This gives the solvability of  $L(u) + \varepsilon u = N(u)$  in  $B_{R_0}$  for any  $\varepsilon > 0$  sufficiently small. By passing to the limit  $\varepsilon \rightarrow 0_+$  and using  $L - N \in PM$ , we obtain the solvability of  $L(u) = N(u)$ . The proof is finished.

*Remark 2.4.* If  $N_\infty(\cdot)$  are, moreover, convex and  $Au \in N_\infty(u) \forall u \in H$ , then (2.1) is simplified to

$$0 \notin L_\infty u - N_\infty(u) \quad \forall u \in S.$$

This case happens in the following way. Let  $\mathcal{M}$  be a nonempty compact convex subset of the set of all compact linear maps from  $H$  into  $H$ . We put

$$N_\infty(u) = \{Ku \mid K \in \mathcal{M}\} \quad \forall u \in H.$$

It is clear that the assumptions i)–iv) of Definition 2.1 hold as well as the convexity of  $N_\infty(\cdot)$ . Now we can take  $A = K_0$  for any fixed  $K_0 \in \mathcal{M}$ . So we can associate the degree  $\text{deg}(L_\infty - K_0, B_1, 0)$  to  $\mathcal{M}$ . Here  $B_1 = \{u \in H \mid |u| < 1\}$ . We note that  $\text{deg}(L_\infty - K_0, B_1, 0) \neq 0$  is constant in  $K_0 \in \mathcal{M}$ . If the remaining assumptions of Theorem 2.3 hold, then (1.1) is solvable for any  $h \in H$ .

Finally, if  $N$  is asymptotically linear at infinity, i.e.  $\mathcal{M} = \{B\}$  for a compact linear map  $B$ , then (2.1) is simplified to

$$0 \neq L_\infty u - Bu \quad \forall u \neq 0.$$

Then Theorem 2.3 is an extension of [7, Theorem XI.1] and the proof of this theorem is related to a proof of that one.

Now we look for special solutions of (1.1). Let  $C \subset H$  be a wedge, i.e.  $C$  is a closed, nonempty, convex subset of  $H$  such that  $\lambda C \subset C \forall \lambda \geq 0$ . We know (see [3, p. 71]) that there is a continuous retraction  $\eta: H \rightarrow C$ , the so-called metric projection, such that  $\eta(\lambda x) = \lambda \eta(x) \forall \lambda \geq 0, \forall x \in H$  and  $|\eta(x)| \leq |x| \forall x \in H$ .

We are interested in the existence of solutions of (1.1) in  $C$ . So in the rest of this section, we solve the equation

$$(2.6) \quad L(x) = N(x), \quad x \in C,$$

where  $N \in COMP$  satisfies  $N(C) \subset C$  and  $L \in MON$  is such that  $(L + \varepsilon\mathbb{I})(C) = C$  for any  $\varepsilon > 0$  sufficiently small. Here  $\mathbb{I}$  is the identity map.

We suppose

(H1)  $L$  is asymptotically linear at infinity with the asymptote  $L_\infty \in S_+$  satisfying  $L_\infty x = 0 \Rightarrow x = 0$  and  $L_\infty(C) = C$ .

(H2)  $N$  is asymptotically quasilinear at infinity with the asymptote  $N_\infty$  satisfying  $N_\infty(u) \subset C \forall u \in C$ .

**Theorem 2.5.** *Assume that (H1), (H2) hold and  $L - N \in PM$ . If for any  $x \in C$ ,  $0 < \lambda \leq 1$ , the condition  $0 \in L_\infty x - \lambda N_\infty(x)$  implies  $x = 0$ , then (2.6) has a solution.*

*Proof.* We solve

$$(2.7) \quad L(x) + \varepsilon x = N(\eta(x))$$

for  $\varepsilon > 0$  small. We know partly that  $L + \varepsilon I$  is strongly monotone (i.e.  $((L(x_1) + \varepsilon x_1) - (L(x_2) + \varepsilon x_2), x_1 - x_2) \geq \varepsilon|x_1 - x_2| \forall x_1, x_2 \in H)$ , so it is invertible (see [3, p. 100]) and partly that  $(L + \varepsilon I)(C) = C$ . Hence any solution of (2.7) belongs to  $C$ .

Now we verify the assumption (2.1) with  $A = 0$  for the map  $L + \varepsilon I - N(\eta)$  with  $\varepsilon > 0$  sufficiently small, i.e. we intend to apply Theorem 2.3 to (2.7). For this purpose, we claim that the asymptote of  $N(\eta)$  is  $N_\infty(\eta)$ . Indeed, the boundedness of  $N$  and the assumptions of Definition 2.1 imply that for any  $\omega > 0$  there is a constant  $c(\omega) > 0$  such that for any  $u \in H$  there is  $w_u \in N_\infty(u)$  satisfying

$$|N(u) - w_u| \leq \omega|u| + c(\omega).$$

Since for any bounded  $B \subset H$ , the compactness of  $\overline{N_\infty(B)}$  follows from the assumptions i)–iv) of Definition 2.1, we can clearly take

$$c(\omega) = \sup \left\{ |N(u)| + |w| \mid u \in H; |u| \leq R; w \in N_\infty(u) \right\},$$

where  $R$  is established by the assumption v) of Definition 2.1 when  $\omega$  is considered instead of  $\varepsilon$ .

Hence we have

$$|N(\eta(u)) - w_{\eta(u)}| \leq \omega|\eta(u)| + c(\omega) \leq \omega|u| + c(\omega).$$

Lastly, the validity of the assumptions i)–iv) of Definition 2.1 for  $N_\infty(\eta(\cdot))$  can be easily verified. The claim is proved.

If (2.1) is not true for this case with  $A = 0$ , then there is a sequence  $\varepsilon_i > 0$ ,  $u_i \in S$ ,  $\lambda_i \in [0, 1]$  such that  $\varepsilon_i \rightarrow 0$ ,  $u_i \rightarrow u_0$ ,  $\lambda_i \rightarrow \lambda_0$  and

$$(2.8) \quad 0 \in L_\infty u_i + \varepsilon_i u_i - \lambda_i N_\infty(\eta(u_i)).$$

So there is  $w_i \in N_\infty(\eta(u_i))$  such that

$$0 = L_\infty u_i + \varepsilon_i u_i - \lambda_i w_i.$$

This equality gives

$$L_\infty u_i - \lambda_0 w_i \rightarrow 0.$$

By the assumption iii) of Definition 2.1, we can assume  $w_i \rightarrow w$ . Hence

$$L_\infty u_i - \lambda_0 w \rightarrow 0.$$

By the condition  $L_\infty \in S_+$  we can assume  $u_i \rightarrow u_0$ . Since  $w_i \in N_\infty(\eta(u_i))$ , we obtain both  $w \in N_\infty(\eta(u_0))$  (see the assumptions i) and iv) of Definition 2.1) and

$L_\infty u_0 = \lambda_0 w$ . The assumptions (H1 – 2) imply the invertibility of  $L_\infty$  and  $w \in C$ . Hence we obtain  $u_0 \in C$ . Finally, we see that the equation (2.8) gives

$$0 \in L_\infty u_0 - \lambda_0 N_\infty(u_0)$$

for some  $u_0 \in S \cap C$ . This contradicts our assumptions. So (2.1) holds for (2.7). Hence Theorem 2.3 is applicable to (2.7) for any  $\varepsilon > 0$  sufficiently small. The proof of Theorem 2.3 gives a constant  $M > 0$  such that (2.7) has a solution  $x_\varepsilon$ ,  $|x_\varepsilon| \leq M$ . We already know  $x_\varepsilon \in C$ . By using both the pseudomonotony of  $L - N$  and the weak closeness of  $C$ , we obtain the desired solution. The proof is finished.

*Remark 2.6.* If  $L \in MON$  holds also in Theorem 2.3, then  $L - N + \varepsilon \mathbb{I} \in S_+$  for any  $\varepsilon > 0$  (see (2.0)). Now, let the assumption  $L - N \in PM$  be dropped in Theorem 2.5 and let this assumption be replaced by  $L \in MON$  in Theorem 2.3. Then the assumptions of both Theorems 2.3 and 2.5, modified in this way, imply only the almost solvability of  $L(u) = N(u)$  in the sense that

$$0 \in \overline{(L - N)(H)}, \quad \text{respectively} \quad 0 \in \overline{(L - N)(C)}.$$

Indeed, by the ends of the proofs of Theorems 2.3 and 2.5, there is a constant  $M > 0$  such that both equations  $L(u) + \varepsilon u = N(u)$  and  $L(u) + \varepsilon u = N(u)$ ,  $u \in C$ , have solutions  $x_\varepsilon$ ,  $|x_\varepsilon| \leq M$ , for any  $\varepsilon > 0$  sufficiently small. So the claim is proved.

*Remark 2.7.* The assumption  $L - N \in PM$  in Theorems 2.3 and 2.5 is satisfied provided that  $L \in PM$  and  $N \in CC$ .

### 3. IMPLICIT INTEGRAL EQUATIONS

In this section, we illustrate the above abstract results for certain integral equations of the type

$$(3.1) \quad p(x, u(x)) = \int_0^1 G(x, t)m(t, u(t)) dt + f(x), \quad x \in [0, 1],$$

where  $p, m \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ ,  $G \in L_2([0, 1] \times [0, 1], \mathbb{R})$  and  $f \in C([0, 1], \mathbb{R})$ . Moreover, we suppose that  $p$  is nondecreasing in  $u \in \mathbb{R}$ . We are motivated by the papers [12]–[14] to study implicit integral equations.

We assume that there are constants  $\alpha > 0$ ,  $\beta_2 \geq \beta_1$  satisfying

$$(3.2) \quad \lim_{|u| \rightarrow \infty} |p(x, u) - \alpha u|/|u| = 0 \quad \text{uniformly in } x \in [0, 1];$$

$$(3.3) \quad \beta_1 \leq \liminf_{|u| \rightarrow \infty} m(x, u)/u \leq \limsup_{|u| \rightarrow \infty} m(x, u)/u \leq \beta_2 \quad \text{uniformly in } x \in [0, 1].$$

**Theorem 3.1.** *Assume that (3.2 – 3) hold and, moreover, suppose that  $p$  is increasing in  $u$ , i.e.*

$$(p(x, u_1) - p(x, u_2))(u_1 - u_2) > 0 \quad \forall x \in [0, 1], u_1 \neq u_2.$$

If there is a constant  $\delta > 0$  such that the equation

$$\alpha u(x) = \int_0^1 G(x, t)\phi(t)u(t) dt$$

has no nonzero solution for any  $\phi \in L_2([0, 1], \mathbb{R})$  satisfying  $\beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta$ , then (3.1) has a solution.

*Proof.* We apply Theorem 2.3 and Remark 2.4 by putting

$$H = L_2([0, 1], \mathbb{R}), \quad L(u) = p(\cdot, u),$$

$$N(u) = \int_0^1 G(\cdot, t)m(t, u(t)) dt, \quad L_\infty u = \alpha u,$$

$$\mathcal{M} = \left\{ u \rightarrow \int_0^1 G(\cdot, t)\phi(t)u(t) dt \quad \left| \quad \phi \in L_2([0, 1], \mathbb{R}); \beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta \right. \right\},$$

$$N_\infty(u) = \{Ku \mid K \in \mathcal{M}\}.$$

It is clear that  $L \in MON$ ,  $N \in COMP$ . We also know that the strict monotony of  $p$  in  $u$  implies  $L \in S_+$  (see [2], [10]). So  $L - N \in PM$  (see (2.0)).

Now we verify the assumptions of Definition 2.1. The assumption i) follows partly from the weak compactness of the set

$$\left\{ \phi u \quad \left| \quad \phi \in L_2([0, 1], \mathbb{R}); \beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta \right. \right\}$$

in  $L_2([0, 1], \mathbb{R})$  for any  $u \in L_2([0, 1], \mathbb{R})$  fixed, and partly from the compactness of the map  $u \rightarrow \int_0^1 G(\cdot, t)u(t) dt$ . So  $N_\infty(\cdot)$  are compact subsets. The assumption ii) is clear. Since the set

$$\left\{ \phi u \quad \left| \quad (\phi, u) \in L_2([0, 1], \mathbb{R}) \times L_2([0, 1], \mathbb{R}); |u|_{L_2} = 1, \beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta \right. \right\}$$

is bounded in  $L_2([0, 1], \mathbb{R})$  and the map  $u \rightarrow \int_0^1 G(\cdot, t)u(t) dt$  is compact, the assumption iii) is proved. The assumption iv) is clear, since  $N_\infty(\cdot)$  are compact subsets and  $\mathcal{M}$  is a set of uniformly bounded linear mappings. To prove v), we have by (3.3) that  $m$  can be decomposed as

$$m(x, u) = m_1(x, u)u + m_2(x, u),$$

where  $m_1, m_2$  are continuous,  $m_2(\cdot, \cdot)$  is bounded on  $[0, 1] \times \mathbb{R}$  and  $\beta_1 - \delta \leq m_1(\cdot, \cdot) \leq \beta_2 + \delta$ . Now it is clear that we can take

$$w_u = \int_0^1 G(x, t)m_1(t, u(t))u(t) dt \quad \forall u \in L_2([0, 1], \mathbb{R})$$

in the assumption v) of Definition 2.1. So  $N_\infty$  is the asymptote of  $N$ . Similarly, by using (3.2) we can show that  $L_\infty$  is the asymptote of  $L$ .

It remains to verify

$$0 \notin L_\infty u - N_\infty(u) \quad \forall u \in S$$

for this case. If it is not true, then there is  $u \in L_2([0, 1], \mathbb{R})$  such that

$$(3.4) \quad \alpha u(x) = \int_0^1 G(x, t)\phi(t)u(t) dt$$

for some  $\phi \in L_2([0, 1], \mathbb{R})$  satisfying  $\beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta$ . But the assumption of this theorem implies  $u = 0$ . So (2.1) is valid. We see that all assumptions of Theorem 2.3 are satisfied. The proof is finished.

**Theorem 3.2.** *Assume that asymptotic behaviors (3.2–3) hold only for  $u \rightarrow +\infty$ . Suppose in addition that*

$$p(\cdot, 0) = 0, \quad f(\cdot) \geq 0, \quad G(\cdot, \cdot) \geq 0, \quad m(\cdot, u) \geq 0 \quad \forall u \in [0, +\infty);$$

*p is increasing in u.*

*If there is a constant  $\delta > 0$  such that the equation*

$$(3.5) \quad \alpha u(x) = \int_0^1 G(x, t)\phi(t)u(t) dt$$

*has no nonzero nonnegative solution for any  $\phi \in L_2([0, 1], \mathbb{R})$  satisfying  $0 \leq \phi(\cdot) \leq \beta_2 + \delta$ , then (3.1) has a nonnegative solution.*

*Proof.* We follow the above proof. Since we are interested in nonnegative solutions of (3.1), we can assume, by modifying  $p, m$  for  $u \rightarrow -\infty$ , the validity of (3.2–3) with  $\beta_1 = 0$ . We take

$$C = \left\{ u \in L_2([0, 1], \mathbb{R}) \mid u \geq 0 \text{ almost everywhere on } [0, 1] \right\}.$$

It is clear that  $N(C) \subset C$ ,  $(L + \varepsilon\mathbb{I})(C) = C$ . Finally, the remaining assumptions of Theorem 2.5 are verified in the same way as above. The difference is only that we now take

$$\mathcal{M} = \left\{ u \rightarrow \int_0^1 G(\cdot, t)\phi(t)u(t) dt \mid \phi \in L_2([0, 1], \mathbb{R}); 0 \leq \phi(\cdot) \leq \beta_2 + \delta \right\}.$$

The proof is finished.

By applying Remark 2.6 we obtain the following

**Theorem 3.3.** *If the strict monotony of  $p$  in  $u$  is dropped in Theorems 3.1 – 2, i.e.  $p$  is only nondecreasing in  $u$ , then (3.1) is almost solvable in these theorems.*

We complete this paper by presenting two examples.

**Example 1.** Consider the equation

$$(E1) \quad (p(x, u))' = m(x, u), \quad u(0) = a,$$

where  $p, m$  possess the properties in (3.1).

**Theorem 3.4.** *If (3.2 – 3) hold and  $p$  is increasing in  $u$ , then (E1) has a solution, i.e. there is  $u \in C([0, 1], \mathbb{R})$  such that  $p(\cdot, u) \in C^1([0, 1], \mathbb{R})$  and  $u$  satisfies (E1).*

*Proof.* We apply Theorem 3.1, since (E1) has the form

$$(3.6) \quad p(x, u) = \int_0^x m(t, u(t)) dt + p(0, a).$$

The linear equation in Theorem 3.1, for this case, has the form

$$\alpha u(x) = \int_0^x \phi(t)u(t) dt,$$

where  $\phi \in L_2([0, 1], \mathbb{R})$  satisfies  $\beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta$ . By using the Gronwall lemma, this equation has only the zero solution. The proof is finished.

**Example 2.** Consider the equation

$$(E2) \quad (p(x, u))'' = m(x, u), \quad u(0) = u(1) = 0,$$

where  $p, m$  possess the properties in (3.1).

**Theorem 3.5.** *Assume that (3.2 – 3) hold with*

$$\beta_1 > -\pi^2(n + 1)^2\alpha, \quad \beta_2 < -\pi^2n^2\alpha$$

*for a nonnegative integer number  $n$ . If  $p$  is increasing in  $u$ , then (E2) has a solution; i.e. there is  $u \in C([0, 1], \mathbb{R})$  such that  $p(\cdot, u) \in C^2([0, 1], \mathbb{R})$  and  $u$  satisfies (E2).*

*Proof.* By observing

$$(p(x, u))'' = (p(x, u) - p(0, 0) - x(p(1, 0) - p(0, 0)))'',$$

we consider that  $p(0, 0) = p(1, 0) = 0$ . We apply Theorem 3.1, since (E2) has the form

$$(3.7) \quad p(x, u) = \int_0^1 G(x, t)m(t, u(t)) dt,$$

where  $G$  is the Green function of  $w'' = z, w(0) = w(1) = 0$ . The linear equation in Theorem 3.1, for this case, has the form

$$\alpha u(x) = \int_0^1 G(x, t)\phi(t)u(t) dt,$$

where  $\phi \in L_2([0, 1], \mathbb{R})$  satisfies  $\beta_1 - \delta \leq \phi(\cdot) \leq \beta_2 + \delta$ . This equation is equivalent to

$$(3.8) \quad \alpha u'' = \phi u, \quad u(0) = u(1) = 0.$$

Since  $\beta_1 > -\pi^2(n + 1)^2\alpha$  and  $\beta_2 < -\pi^2n^2\alpha$ , it is well-known (see [3]) that the equation (3.8) has only the zero solution for a sufficiently small constant  $\delta > 0$ . The proof is finished.

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