

ON CONTRAVARIANT FINITENESS OF SUBCATEGORIES OF MODULES OF PROJECTIVE DIMENSION $\leq I$

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ABSTRACT. Let Λ be an artin algebra. This paper presents a sufficient condition for the subcategory $\mathcal{P}^i(\Lambda)$ of $\text{mod } \Lambda$ to be contravariantly finite in $\text{mod } \Lambda$, where $\mathcal{P}^i(\Lambda)$ is the subcategory of $\text{mod } \Lambda$ consisting of Λ -modules of projective dimension less than or equal to i . As an application of this condition it is shown that $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ for each i when Λ is stably equivalent to a hereditary algebra.

INTRODUCTION AND PRELIMINARIES

Throughout this paper, all algebras are artin algebras, all modules are finitely generated left modules, and all subcategories are full subcategories. For an artin algebra Λ , we denote by $\text{mod } \Lambda$ the category of all finitely generated Λ -modules.

The notions of contravariantly and covariantly finite subcategories of $\text{mod } \Lambda$ were first introduced and studied by Auslander and Smalø in connection with the study of the existence of almost split sequences in a subcategory of $\text{mod } \Lambda$ (see [2] and [3]). We now recall these notions from [2].

Definition. A full subcategory \mathcal{A} of \mathcal{B} is said to be

- (i) contravariantly finite in \mathcal{B} if for each object X in \mathcal{B} , the representable functor $\text{hom}_{\mathcal{B}}(_, X)$ restricted to \mathcal{A} is finitely generated as a functor on \mathcal{A} ,
- (ii) covariantly finite in \mathcal{B} if for each object Y in \mathcal{B} , the representable functor $\text{hom}_{\mathcal{B}}(Y, _)$ restricted to \mathcal{A} is finitely generated, and
- (iii) functorially finite if \mathcal{A} is both contravariantly and covariantly finite in \mathcal{B} .

For an artin algebra Λ , an interesting class of subcategories of $\text{mod } \Lambda$ is the subcategory $\mathcal{P}^i(\Lambda)$ which consists of all Λ -modules of projective dimension $\leq i$ for $i \geq 0$, as well as the subcategory $\mathcal{P}^\infty(\Lambda)$ consisting of all Λ -modules of finite projective dimension. The contravariant and covariant finiteness of $\mathcal{P}^i(\Lambda)$ and $\mathcal{P}^\infty(\Lambda)$ is studied by many authors; see, for example, [1], [5], and [6].

The aim of this paper is to present a condition which is sufficient for the subcategory $\mathcal{P}^i(\Lambda)$ to be contravariantly finite in $\text{mod } \Lambda$. As an application of this condition we show in section 2 that $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ for each i when Λ is stably equivalent to a hereditary algebra. The main idea of the

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proofs is from [1]. Note that in [1] Auslander and Reiten gave a sufficient condition for $\mathcal{P}^\infty(\Lambda)$ to be contravariantly finite in $\text{mod } \Lambda$ and proved that $\mathcal{P}^\infty(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ when Λ is stably equivalent to a hereditary algebra.

The terminology used throughout this article is taken from [1].

1. A SUFFICIENT CONDITION FOR $\mathcal{P}^i(\Lambda)$ TO BE CONTRAVARIANTLY FINITE

The proofs in this section are analogous to those in sect.4 in [1], but for the completeness of the article, we give here the proofs. Before presenting the sufficient condition, we first give the following more general result.

Proposition 1.1. *Suppose that \underline{a} is an ideal in Λ with $\text{pd}_\Lambda \Lambda / \underline{a} \leq i$ such that if M is a Λ -module with $\text{pd}_\Lambda M \leq i$, then $M / \underline{a}M$ is a projective Λ / \underline{a} -module. Let C be a Λ / \underline{a} -module. Then we have the following.*

- (a) *A map $B \rightarrow C$ in $\text{mod } \Lambda / \underline{a}$ is a right $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of C if and only if it is a right $\mathcal{P}^i(\Lambda)$ -approximation of C .*
- (b) *If $A \rightarrow C$ is a right $\mathcal{P}^i(\Lambda)$ -approximation of C , then $A / \underline{a}A \rightarrow C$ is a right $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of C .*

Proof. (a) Suppose $f : B \rightarrow C$ is a right $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of C . Since B is a projective Λ / \underline{a} -module and $\text{pd}_\Lambda \Lambda / \underline{a} \leq i$, as a Λ -module B is in $\mathcal{P}^i(\Lambda)$. Let $g : X \rightarrow C$ be a morphism in $\text{mod } \Lambda$ with X in $\mathcal{P}^i(\Lambda)$. Then g is the composition of the canonical projection $\pi : X \rightarrow X / \underline{a}X$ and the induced map $g_1 : X / \underline{a}X \rightarrow C$. Since $X / \underline{a}X$ is a projective Λ / \underline{a} -module, the morphism g_1 can be lifted to B , so g can be lifted to B , that is, $f : B \rightarrow C$ is a right $\mathcal{P}^i(\Lambda)$ -approximation of C .

Conversely, suppose that a morphism $f : B \rightarrow C$ in $\text{mod } \Lambda / \underline{a}$ is a right $\mathcal{P}^i(\Lambda)$ -approximation of C . Since $\text{pd}_\Lambda B \leq i$, as a Λ / \underline{a} -module $B = B / \underline{a}B$ is projective. Hence $f : B \rightarrow C$ is a right $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation of C .

- (b) This is trivial.

Corollary 1.2. *Let \underline{a} be an ideal in Λ satisfying the hypothesis of Proposition 1.1. If $\underline{a}S = 0$ for each simple Λ -module S with $\text{pd}_\Lambda S > i$, then $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$.*

Proof. First note that $\mathcal{P}^i(\Lambda)$ is a resolving subcategory (i.e. $\mathcal{P}^i(\Lambda)$ satisfying the following three conditions: (a) closed under extension, (b) closed under kernels of surjections, and (c) contains all projective Λ -modules). Then by [1, Proposition 3.7], $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ if and only if each simple Λ -module has a right $\mathcal{P}^i(\Lambda)$ -approximation. Suppose S is a simple Λ -module. If $\text{pd}_\Lambda S \leq i$, we are done. Suppose now $\text{pd}_\Lambda S > i$; then S is a Λ / \underline{a} -module. Hence there is a right $\mathcal{P}^0(\Lambda / \underline{a})$ -approximation $A \rightarrow S$ of S since $\mathcal{P}^0(\Lambda / \underline{a})$ is contravariantly finite in $\text{mod } \Lambda / \underline{a}$. By (a) of Proposition 1.1, $A \rightarrow S$ is also a right $\mathcal{P}^i(\Lambda)$ -approximation of S . Therefore, $\mathcal{P}^i(\Lambda)$ is contravariantly finite.

For an artin algebra Λ , we denote by $\Omega(\text{mod } \Lambda)$ the subcategory consisting of the syzygy modules $\Omega(C)$ of all C in $\text{mod } \Lambda$. Further, we denote by $\tau_{\mathcal{A}}(M)$ the trace of a category \mathcal{A} in the module M , that is, the submodule of M generated by the images of all maps $A \rightarrow M$ with A in \mathcal{A} . Finally, for each $i \geq 0$, we denote by \underline{a}_i the trace of $\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)$ in \underline{r} , i.e. $\underline{a}_i = \tau_{\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)}(\underline{r})$, where \underline{r} denotes the radical of Λ . It is obvious that $\underline{a}_i \subset \underline{r}$ is an ideal in Λ .

Proposition 1.3. *If $\text{pd}_{\wedge \underline{a}_i} \leq i$, then $\mathcal{P}^{i+1}(\wedge)$ is contravariantly finite in $\text{mod } \wedge$.*

Proof. We first observe that $\underline{a}_i P = \tau_{\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)}(\underline{r}P)$ if P is a projective module, where $\underline{r}P$ is the radical of P .

Let M be a \wedge -module with projective dimension $\leq i + 1$. We then consider the following exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

with f a projective cover of M . Since $\text{pd}_{\wedge} \Omega(M) \leq i$, one gets that $\Omega(M)$ is in $\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)$. Therefore, it holds that $\Omega(M) \subset \tau_{\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)}(\underline{r}P) = \underline{a}_i P$ and that $P/\underline{a}_i P \cong M/\underline{a}_i M$, that is, $M/\underline{a}_i M$ is a projective \wedge/\underline{a}_i -module. The condition $\text{pd}_{\wedge} \underline{a}_i \leq i$ implies that $\text{pd}_{\wedge} \wedge/\underline{a}_i \leq i + 1$. Because each simple \wedge -module is annihilated by \underline{a}_i , one has by Corollary 1.2 that $\mathcal{P}^{i+1}(\wedge)$ is contravariantly finite.

Corollary 1.4. *Suppose $\text{pd}_{\wedge} \underline{a}_i \leq i$. Then the \wedge -modules of projective dimension $\leq i + 1$ are the summands of modules M which have filtrations $M = M_0 \supset M_1 \supset \dots \supset M_n = 0$ such that each subquotient M_i/M_{i+1} is an indecomposable projective \wedge/\underline{a}_i -module.*

Proof. Note that the minimal right $\mathcal{P}^0(\wedge/\underline{a})$ -approximation of a \wedge/\underline{a} -module is its projective cover. It then follows that the \wedge/\underline{a} -projective covers of the simple \wedge -modules are just the minimal right $\mathcal{P}^i(\wedge)$ -approximations of the simple \wedge -modules. The corollary then follows directly from [1, Proposition 3.7].

Combining the above results with the results in [1] and [3], we get the following corollary.

Corollary 1.5. *Suppose \underline{a}_0 is projective. Then $\mathcal{P}^1(\wedge)$ has almost split sequences.*

Proof. From [1] one knows that $\mathcal{P}^1(\wedge)$ is covariantly finite in $\text{mod } \wedge$ for all artin algebras. By Proposition 1.3, one has that $\mathcal{P}^1(\wedge)$ is also contravariantly finite in $\text{mod } \wedge$, so $\mathcal{P}^1(\wedge)$ is functorially finite. Then from [3, Theorem 2.4] it follows that $\mathcal{P}^1(\wedge)$ has almost split sequences.

As seen the example in [5], one knows that $\mathcal{P}^i(\wedge)$ is not always contravariantly finite. In the following we construct an algebra with finite global dimension $n \geq 2$ which satisfies that $\mathcal{P}^i(\wedge)$ are not contravariantly finite for $1 \leq i \leq n - 1$.

Example. Let k be an algebraically closed field. For each $n \geq 2$, let \wedge_n be given as the path algebra of the following quiver

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_{n-2}} n \xrightarrow{\gamma_{n-1}} n+1$$

modulo the ideal generated by the paths $\gamma_1 \alpha$ and $\gamma_{i+1} \gamma_i$, $1 \leq i \leq n - 2$. It is easy to calculate that \wedge_n has global dimension n and that the simple module S_1 corresponding to the vertex 1 has projective dimension n . On one hand, the module S_1 is a preinjective module (see [2]) and each indecomposable preinjective module has projective dimension n . On the other hand, there is a family of indecomposable \wedge -modules $\{Y_\lambda \mid 0 \neq \lambda \in k\}$ in $\mathcal{P}^1(\wedge)$ of dimension 2 with support containing vertices 1 and 2 only (i.e. the Y_λ can be considered as modules over double arrows; see [4, sect. 8]). Moreover, it holds that $\text{hom}_{\wedge}(Y_\lambda, Y_\mu) = 0$ for $\lambda \neq \mu$ and there is a

nonzero morphism from each Y_λ to S_1 . Suppose $f : X \rightarrow S_1$ is a right $\mathcal{P}^1(\Lambda)$ -approximation. One can easily show that each Y_λ should appear as a summand of X , but this is impossible. Hence S_1 has no right $\mathcal{P}^1(\Lambda)$ -approximation, that is, $\mathcal{P}^1(\Lambda)$ is not contravariantly finite in $\text{mod } \Lambda$. Similarly, one can prove that $\mathcal{P}^i(\Lambda)$ is not contravariantly finite in $\text{mod } \Lambda$ for $2 \leq i \leq n - 1$.

An easy observation shows that $\text{pd}_\Lambda \underline{a}_i = n - 1$ for $0 \leq i \leq n - 2$.

2. AN APPLICATION

As an application of the sufficient condition given in section 1, in this section we prove that $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ when Λ is stably equivalent to a hereditary algebra. The proof is based on the following two lemmas.

Lemma 2.1 [1, Proposition 4.12]. *For an artin algebra, the following are equivalent.*

- (a) Λ satisfies the conditions:
 - (i) A simple module S is a torsion module (i.e. $S^* = \text{hom}_\Lambda(S, \Lambda) = 0$) if it is a composition factor of $\underline{r}P/\text{soc } P$ for some indecomposable projective module P .
 - (ii) Every indecomposable torsionless module (i.e. a submodule of a free Λ -module) is simple or projective.
- (b) Λ is stably equivalent to a hereditary algebra.

Lemma 2.2. *Suppose that Λ is stably equivalent to a hereditary algebra. Then it holds that $\text{pd}_\Lambda \underline{a}_i \leq i$ for each $i \geq 0$, where $\underline{a}_i = \tau_{\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)}(\underline{r})$.*

Proof. By Lemma 2.1 one gets that

$$\underline{a}_i = P \oplus S_1 \oplus \cdots \oplus S_t,$$

where P is projective and the S_j are simple modules.

Suppose now that $\text{pd}_\Lambda \underline{a}_i > i$. Then there is a simple module $S := S_j$ satisfying that $\text{pd}_\Lambda S > i$. By the construction of \underline{a}_i , there is an epimorphism $f : L \rightarrow S$ with L indecomposable in $\Omega(\text{mod } \Lambda) \cap \mathcal{P}^i(\Lambda)$. We claim that $f(\text{soc}(L)) = 0$. Otherwise, f would be a splittable epimorphism, so $\text{pd}_\Lambda S \leq i$, but this is impossible. Thus S is a composition factor of $L/\text{soc } L$. Since L is in $\Omega(\text{mod } \Lambda)$, it follows that $L \subset \underline{r}Q$ for some projective module Q . Then $L/\text{soc } L \subset \underline{r}Q/\text{soc } Q$; that is, S is a composition factor of $\underline{r}Q/\text{soc } Q$. This implies by Lemma 2.1 (ii) that S is a torsion module. This contradicts that $S \subset \underline{a}_i$. Therefore, it holds that $\text{pd}_\Lambda \underline{a}_i \leq i$.

Proposition 2.3. *Suppose that Λ is stably equivalent to a hereditary algebra. Then $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ for each $i \geq 0$.*

Proof. By Lemma 2.2 one knows that $\text{pd}_\Lambda \underline{a}_i \leq i$ for each i . Then by applying Proposition 1.3 one gets that $\mathcal{P}^{i+1}(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$. The proposition then follows from the fact that the subcategory $\mathcal{P}^0(\Lambda)$ is always contravariantly finite in $\text{mod } \Lambda$.

Remark. In [1] it is proved that $\mathcal{P}^\infty(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ when Λ is stably equivalent to a hereditary algebra. Combining this result with Proposition 2.3, one can see that if Λ is stably equivalent to a hereditary algebra, then $\mathcal{P}^i(\Lambda)$ is contravariantly finite in $\text{mod } \Lambda$ for all $i \in \mathbb{N} \cup \{\infty\}$.

REFERENCES

1. M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. in Math. **86** (1991), 111–152. MR **92e**:16009
2. M. Auslander and S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra **66** (1980), 61–122. MR **83a**:16039
3. ———, *Almost split sequences in subcategories*, J. Algebra **69** (1981), 426–454. MR **82j**:16048a
4. P. Gabriel, B. Keller, and A. V. Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia of Math. Sci., vol. 73, Springer, Berlin, 1992. MR **94h**:16001b
5. K. Igusa, S. O. Smalø, and G. Todorov, *Finite projectivity and contravariant finiteness*, Proc. Amer. Math. Soc. **109** (1990), 937–941. MR **91b**:16010
6. S. O. Smalø, *Functorial finite subcategories over triangular matrix rings*, Proc. Amer. Math. Soc. **111** (1991), 651–656. MR **91f**:16016

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