

**BANACH SPACES IN WHICH EVERY  
 $p$ -WEAKLY SUMMABLE SEQUENCE  
LIES IN THE RANGE OF A VECTOR MEASURE**

C. PIÑEIRO

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let  $X$  be a Banach space. For  $1 < p < +\infty$  we prove that the identity map  $I_X$  is  $(1, 1, p)$ -summing if and only if the operator  $x^* \in X^* \rightarrow \sum \langle x_n, x^* \rangle e_n \in l_q$  is nuclear for every unconditionally summable sequence  $(x_n)$  in  $X$ , where  $q$  is the conjugate number for  $p$ . Using this result we find a characterization of Banach spaces  $X$  in which every  $p$ -weakly summable sequence lies inside the range of an  $X^{**}$ -valued measure (equivalently, every  $p$ -weakly summable sequence  $(x_n)$  in  $X$ , satisfying that the operator  $(\alpha_n) \in l_q \rightarrow \sum \alpha_n x_n \in X$  is compact, lies in the range of an  $X$ -valued measure) with bounded variation. They are those Banach spaces such that the identity operator  $I_{X^*}$  is  $(1, 1, p)$ -summing.

Let  $X$  be a Banach space. In [AD] it is proved that every sequence  $(x_n)$  in  $X$  satisfying  $\sum_n |\langle x_n, x^* \rangle|^2 < +\infty$  for all  $x^* \in X^*$  lies inside the range of an  $X$ -valued measure. Nevertheless, they show a sequence which does not lie in the range of an  $X$ -valued measure with bounded variation. In [PR] the authors proved that  $X$  is finite dimensional if and only if every nul sequence (equivalently, every compact set) in  $X$  lies inside the range of an  $X$ -valued measure having bounded variation. The purpose of this paper is to characterize, given a real number  $p \in (1, +\infty)$ , the Banach spaces in which every  $p$ -weakly summable sequence lies inside the range of an  $X^{**}$ -valued measure with bounded variation. We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [Ps] and [DU]. We only consider real Banach spaces. If  $X$  is a such space,  $B_X$  will denote its closed unit ball. The phrase "range of an  $X$ -valued measure" always means a set of the form  $rg(F) = \{F(A) : A \in \Sigma\}$ , where  $\Sigma$  is a  $\sigma$ -algebra of subsets of a set  $\Omega$  and  $F: \Sigma \rightarrow X$  is countably additive. Given  $p \geq 1$ ,  $l_w^p(X)$  will denote the vector space of all sequences  $(x_n)$  in  $X$  such that  $\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < +\infty$  for all  $x^* \in X^*$ . It is easy to see that if  $(x_n) \in l_w^p(X)$ , then

$$\varepsilon_p((x_n)) = \sup \left\{ \left( \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\} < +\infty$$

and  $(l_w^p(X), \varepsilon_p)$  is itself a Banach space.

---

Received by the editors September 12, 1994 and, in revised form, December 2, 1994.  
1991 *Mathematics Subject Classification*. Primary 46G10; Secondary 47B10.  
This research has been partially supported by the D.G.I.C.Y.T., PB 90-893.

If  $\hat{x} = (x_n) \in l_w^p(X)$  and  $P$  is a finite subset of  $\mathbb{N}$ ,  $\hat{x}(P) = (x_n(P))$  is the sequence defined by

$$x_n(P) = \begin{cases} x_n & \text{if } n \in P, \\ 0 & \text{if } n \notin P \end{cases}$$

for all  $n \in \mathbb{N}$ .  $l_u^p(X)$  will denote the subspace of  $l_w^p(X)$  consisting of the sequences  $\hat{x} = (x_n)$  such that the net  $(\hat{x}(P))_{P \in \mathcal{F}(\mathbb{N})}$  converges to  $(x_n)$  in  $l_w^p(X)$ , where  $\mathcal{F}(\mathbb{N})$  is the set of all finite subsets of  $\mathbb{N}$ . Recall that  $l_u^1(X)$  is formed by the unconditionally summable sequences in  $X$ . We need the following propositions that list some privileges that membership in  $l_w^p(X)$  or in  $l_u^p(X)$  entail.

**Proposition A.** *Let  $p > 1$  and  $X$  be a Banach space. The following statements are equivalent:*

- (i)  $(x_n) \in l_w^p(X)$ .
- (ii) The series  $\sum_{n=1}^{\infty} \alpha_n x_n$  converges unconditionally for every sequence  $(\alpha_n) \in l_q$ .
- (iii) The map  $(\alpha_n) \in l_q \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X$  defines a bounded operator.

**Proposition B.** *Let  $p \geq 1$ . If  $(x_n) \in l_u^p(X)$ , then the operator  $(\alpha_n) \in l_q \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X$  is compact.*

1. MAIN RESULT

Throughout this section  $X$  will be a Banach space and  $p \in (1, +\infty)$ .

**Theorem 1.** *The following statements are equivalent:*

- (i) For every unconditionally sequence  $(x_n)$  in  $X$  the operator  $x^* \in X^* \rightarrow \sum_{n=1}^{\infty} \langle x_n, x^* \rangle e_n \in l_q$  is nuclear.
- (ii) There exists a constant  $c > 0$  such that

$$(1) \quad \left| \sum_{k=1}^n \langle x_k, x_k^* \rangle \right| \leq c \sup \left\{ \sum_{k=1}^n |\langle x_k, x^* \rangle| : \|x^*\| \leq 1 \right\} \\ \cdot \sup \left\{ \left( \sum_{k=1}^n |\langle x, x_k^* \rangle|^p \right)^{1/p} : \|x\| \leq 1 \right\}$$

for all  $\{x_1, \dots, x_n\} \subset X$  and  $\{x_1^*, \dots, x_n^*\} \subset X^*$ .

*Proof.* (i) $\Rightarrow$ (ii) We consider the linear map

$$\hat{x} = (x_n) \in l_u^1(X) \rightarrow T_{\hat{x}} \in \mathcal{N}(X^*, l_q)$$

defined by  $T_{\hat{x}}(x^*) = \sum \langle x_n, x^* \rangle e_n$  for all  $x^* \in X^*$  ( $\{e_n : n \in \mathbb{N}\}$  is the unit basis of  $l_q$ ). It has closed graph, so there exists a positive constant  $c$  so that

$$\nu \left( \sum_{n=1}^{\infty} x_n \otimes e_n : X^* \rightarrow l_q \right) \leq c \sup \left\{ \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle| : \|x^*\| \leq 1 \right\}$$

for all  $(x_n) \in l_u^1(X)$ . By a standard argument we obtain

$$(2) \quad \nu \left( \sum_{n=1}^m x_n \otimes e_n : X^* \rightarrow l_q^m \right) \leq c \sup \left\{ \sum_{n=1}^m |\langle x_n, x^* \rangle| : \|x^*\| \leq 1 \right\}$$

for all  $m \in \mathbb{N}$  and  $\{x_1, \dots, x_m\} \subset X$ .

Now, given  $\{x_1, \dots, x_m\} \subset X$  and  $\{x_1^*, \dots, x_m^*\} \subset X^*$ , define two operators  $v: l_q^m \rightarrow X^*$  and  $u: X^* \rightarrow l_q^m$  by

$$v(\alpha_i) = \sum_{i=1}^m \alpha_i x_i^* \quad \text{and} \quad u(x^*) = \sum_{i=1}^m \langle x_i, x^* \rangle e_i.$$

Note that  $tr(u \circ v) = \sum_{i=1}^m \langle x_i, x_i^* \rangle$ , so we have

$$\begin{aligned} \left| \sum_{i=1}^m \langle x_i, x_i^* \rangle \right| &\leq \nu(u \circ v) \leq \nu(u) \|v\| \\ &= \nu(u) \sup \left\{ \left( \sum_{i=1}^m |\langle x, x_i^* \rangle|^p \right)^{1/p} : \|x\| \leq 1 \right\} \end{aligned}$$

and using (2) we obtain

$$\begin{aligned} \left| \sum_{k=1}^n \langle x_k, x_k^* \rangle \right| &\leq c \sup \left\{ \sum_{k=1}^n |\langle x_k, x^* \rangle| : \|x^*\| \leq 1 \right\} \\ &\cdot \sup \left\{ \left( \sum_{k=1}^n |\langle x, x_k^* \rangle|^p \right)^{1/p} : \|x\| \leq 1 \right\}. \end{aligned}$$

(ii) $\Rightarrow$ (i) Given  $(x_n^*) \in l_w^p(X^*)$ , we define a linear form  $\phi$  by

$$(x_n) \in l_u^1(X) \rightarrow \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle \in \mathbb{R}.$$

By (ii)  $\phi \in l_u^1(X)^*$  and  $\|\phi\| \leq c \|(x_n^*)\|_p$ . So, the linear map  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in l_\infty$  is integral (see [DU, p. 232]). Equivalently,  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in c_0$  is integral. Then, so is its adjoint  $(\alpha_n) \in l_1 \rightarrow \sum \alpha_n x_n^* \in X^*$ . Therefore, the linear map  $\psi$  defined by

$$(x_n^*) \in l_w^p(X^*) \rightarrow \sum_{n=1}^{\infty} e_n \otimes x_n^* \in I(l_1, X^*)$$

is well defined and  $\|\psi\| \leq c$ . Now denote the restriction map of  $\psi$  to  $l_u^p(X^*)$  by  $\psi_u$ . Since  $\hat{x}^* = \lim_P \hat{x}^*(P)$  for all  $\hat{x}^* \in l_u^p(X^*)$ , it follows that  $\psi_u$  takes all its values in  $\mathcal{N}(l_1, X^*)$  (note that  $\mathcal{N}(l_1, X^*)$  is a subspace of  $I(l_1, X^*)$  because  $(l_1)^*$  has the metric approximation property). If we also denote the operator

$$(x_n^*) \in l_u^p(X^*) \rightarrow \sum_{n=1}^{\infty} e_n \otimes x_n^* \in \mathcal{N}(l_1, X^*)$$

by  $\psi_u$ , then  $(\psi_u)^*$  maps  $B(X^*, l_1)$  into  $l_u^p(X^*)^*$ . In particular, for all  $(x_n) \in l_u^1(X)$ , the operator  $x^* \in X^* \rightarrow \sum \langle x_n, x^* \rangle e_n \in l_q$  is integral. This completes the proof because nuclear and integral operators into a reflexive space are the same.

Recall that an operator  $T: X \rightarrow Y$  is called  $(r, q, p)$ -summing if there is a constant  $c \geq 0$  such that

$$\left( \sum_{k=1}^n |\langle Tx_k, y_k^* \rangle|^r \right)^{1/r} \leq c \sup_{x^* \in B_{X^*}} \left( \sum_{k=1}^n |\langle x_k, x^* \rangle|^q \right)^{1/q} \cdot \sup_{y \in B_Y} \left( \sum_{k=1}^n |\langle y, y_k^* \rangle|^p \right)^{1/p}$$

for all finite families of elements  $x_1, \dots, x_n \in X$  and functionals  $y_1^*, \dots, y_n^* \in Y^*$ . So, Theorem 1 gives us a characterization of Banach spaces  $X$  for which  $I_X$  is  $(1, 1, p)$ -summing.

An operator  $T: X \rightarrow Y$  is  $(p, q)$ -summing if there is a constant  $c \geq 0$  such that

$$\left( \sum_{k=1}^n \|Tx_k\|^p \right)^{1/p} \leq c \sup \left\{ \left( \sum_{k=1}^n |\langle x_k, x^* \rangle|^q \right)^{1/q} : \|x^*\| \leq 1 \right\}$$

for all finite subset  $\{x_1, \dots, x_n\}$  of  $X$ .

Following [Ps] we will say that a Banach space  $X$  satisfies Grothendieck's Theorem (in short,  $X$  is a G.T. space) if  $B(X, l_2) = \Pi_1(X, l_2)$ . The next proposition shows the relationship between the Banach spaces  $X$  for which  $I_X$  is absolutely  $(1, 1, p)$ -summing and the above classes.  $\square$

**Proposition 2.** (i) *If  $X$  is a G.T. space, then  $I_X$  is  $(1, 1, p)$ -summing for  $1 < p \leq 2$ .*

(ii) *If  $1 < p < +\infty$  and  $T \in B(X, Y)$ , then*

$$T \text{ is } (1, 1, p)\text{-summing} \Rightarrow T \text{ is } (q, 1)\text{-summing.}$$

*Proof.* If  $(x_n) \in l_u^1(X)$ , then the operator  $T: x^* \in X^* \rightarrow \sum \langle x_n, x^* \rangle e_n \in l_q$  admits the following factorization

$$\begin{array}{ccc} X^* & \xrightarrow{T} & l_q \\ J \searrow & \nearrow I & \\ & l_1 & \end{array}$$

where  $I: l_1 \rightarrow l_q$  is the natural inclusion and  $J: X^* \rightarrow l_1$  is defined by  $Jx^* = (\langle x_n, x^* \rangle)$  for all  $x^* \in X^*$ .  $I$  is obviously 1-summing and  $J$  is 2-summing by [Ps, 6.6.2], so  $T$  is nuclear.

(ii) If  $T$  is  $(1, 1, p)$ -summing there is a constant  $c \geq 0$  such that

$$(3) \quad \left| \sum_{i=1}^n \langle Tx_i, y_i^* \rangle \right| \leq c \sup \left\{ \sum_{i=1}^n |\langle x_i, x^* \rangle| : \|x^*\| \leq 1 \right\} \cdot \varepsilon_p((y_i^*)_{i=1}^n)$$

for all  $\{x_1, \dots, x_n\} \subset X$  and  $\{y_1^*, \dots, y_n^*\} \subset Y^*$ . Given  $\{x_1, \dots, x_n\} \subset X$ , choose  $y_i^* \in B_{Y^*}$  so that  $|\langle Tx_i, y_i^* \rangle| = \|Tx_i\|$  for each  $i \leq n$ . By (3) we have

$$\begin{aligned} \sum_{i=1}^n |\alpha_i| \|Tx_i\| &= \sum_{i=1}^n |\alpha_i| |\langle Tx_i, y_i^* \rangle| \leq c\varepsilon_1((x_i)_{i=1}^n) \varepsilon_p((\alpha_i y_i^*)_{i=1}^n) \\ &\leq c\varepsilon_1((x_i)_{i=1}^n) \|(\alpha_i)\|_p \end{aligned}$$

for all  $(\alpha_i) \in l_p^n$ . Then

$$\left( \sum_{i=1}^n \|Tx_i\|^q \right)^{1/q} \leq c \sup \left\{ \sum_{i=1}^n |\langle x_i, x^* \rangle| : \|x^*\| \leq 1 \right\}$$

for all  $\{x_1, \dots, x_n\} \subset X$ .

In [P, 17.1.6], Pietsch formulated the following conjecture: for  $1/r > 1/q + 1/p - 1/2$ ,  $I_X$  is  $(r, q, p)$ -summing if and only if  $X$  is finite dimensional. The conjecture is true for  $q = r = 1$ . Certainly, let  $p > 2$ . If  $I_X$  is  $(1, 1, p)$ -summing, it follows from Proposition 2(ii) that  $I_x$  is  $(q, 1)$ -summing. By [P, Theorem 17.2.7.]  $X$  has to be finite dimensional since  $q < 2$ .  $\square$

## 2. SEQUENCES IN THE RANGE OF A VECTOR MEASURE WITH BOUNDED VARIATION

In this section we use Theorem 1 to obtain a characterization of Banach spaces  $X$  for which every  $p$ -weakly summable sequence  $(x_n)$  in  $X$  lies inside the range of an  $X^{**}$ -valued measure having bounded variation. The following lemma collects some elementary facts we need (see [Pi 2]).

**Lemma 3.** *Let  $X$  be a Banach space. If  $\hat{x} = (x_n)$  is a bounded sequence in  $X$ , we consider the linear operator  $T_{\hat{x}}: l_1 \rightarrow X$  defined by  $T_{\hat{x}}(\alpha_n) = \sum \alpha_n x_n$  for all  $(\alpha_n) \in l_1$ . Then the following assertions hold:*

- (i)  $(x_n)$  lies inside the range of an  $X^{**}$ -valued measure with bounded variation iff  $T_{\hat{x}}$  is integral.
- (ii)  $(x_n)$  lies inside the range of an  $X$ -valued measure with bounded variation iff  $T_{\hat{x}}$  is Pietsch-integral.

Now we are ready to face our problem.

**Theorem 4.** *Let  $X$  a Banach space and  $1 < p < +\infty$ . The following statements are equivalent:*

- (i) Every sequence  $(x_n) \in l_w^p(X)$  lies inside the range of an  $X^{**}$ -valued measure with bounded variation.
- (ii) Every sequence  $(x_n) \in l_w^p(X)$ , satisfying that the operator  $(\alpha_n) \in l_q \rightarrow \sum \alpha_n x_n \in X$  is compact, lies inside the range of an  $X$ -valued measure with bounded variation.
- (iii) Every sequence  $(x_n) \in l_u^p(X)$  lies inside the range of an  $X$ -valued measure with bounded variation.
- (iv)  $I_{X^*}$  is  $(1, 1, p)$ -summing.

*Proof.* (i) $\Rightarrow$ (ii) By Lemma 3, we can consider the linear map

$$\phi: \hat{x} \in l_w^p(X) \rightarrow T_{\hat{x}} \in I(l_1, X).$$

It is continuous because its graph is closed. Since  $(l_q)^*$  has the approximation property, for each sequence  $\hat{x} = (x_n) \in l_w^p(X)$  satisfying that the operator  $(\alpha_n) \in l_q \rightarrow \sum \alpha_n x_n \in X$  is compact, there exists a sequence  $(\hat{y}_k)$  in  $l_w^p(X)$  such that  $\hat{x} = \lim_{k \rightarrow +\infty} \hat{y}_k$  in  $l_w^p(X)$  and each sequence  $\hat{y}_k$  is finite dimensional. Then  $\phi(\hat{y}_k)$  belongs to  $\mathcal{N}(l_1, X)$  for all  $k \in \mathbb{N}$ . By continuity, so does  $\phi(\hat{x})$  (recall that  $\mathcal{N}(l_1, X)$  is a closed subspace of  $I(l_1, X)$ ). Hence, we have proved that such a sequence

$(x_n) \in l_w^p(X)$  actually lies inside a sum of segments

$$\sum [-z_n, z_n] = \left\{ \sum \alpha_n z_n : (\alpha_n) \in l_\infty, \|(\alpha_n)\|_\infty \leq 1 \right\}$$

where  $\sum \|z_n\| < +\infty$  (see [Pi 1]).

(ii) $\Rightarrow$ (iii) It is obvious because the operator

$$(\alpha_n) \in l_q \rightarrow \sum \alpha_n x_n \in X$$

is compact for each sequence  $(x_n) \in l_n^p(X)$ .

(iii) $\Rightarrow$ (iv) Now we consider the linear map

$$\psi: \hat{x} \in l_u^p(X) \rightarrow T_{\hat{x}} \in I(l_1, X).$$

Having a closed graph,  $\psi$  is continuous. Since  $\hat{x} = \lim_P \hat{x}(P)$  for every sequence  $\hat{x} \in l_u^p(X)$ , it follows that  $\psi$  takes its values into  $\mathcal{N}(l_1, X)$ . As mentioned earlier, using the trace duality it is easy to prove that  $\psi^*$  takes every  $(x_n^*) \in l_u^1(X^*)$  in  $\sum x_n^* \otimes e_n \in I(X, l_q)$ . Again the reflexivity of  $l_q$  yields (iv).

(iv) $\Rightarrow$ (i) In the same way as in the proof of Theorem 1 we can prove that the linear map

$$(x_n^{**}) \in l_w^p(X^{**}) \rightarrow \sum_{n=1}^{\infty} e_n \otimes x_n^{**} \in I(l_1, X^{**})$$

is well defined and continuous. In particular, it follows from the above lemma that every  $(x_n) \in l_w^p(X)$  lies inside the range of an  $X^{**}$ -valued measure of bounded variation.

In view of Theorem 4 and the notes at the end of section 1, for  $p > 2$ , only finite-dimensional Banach spaces  $X$  have the property that every sequence  $(x_n) \in l_w^p(X)$  lies inside the range of an  $X$ -valued measure having bounded variation. That is why from now on we only consider  $p \in [1, 2]$ . □

### 3. FINAL NOTES AND EXAMPLES

It is well known that every sequence  $(x_n) \in l_w^1(X)$  lies inside the range of an  $X$ -valued measure with bounded variation. In fact, the vector measure  $F$  defined by

$$F(A) = 2 \sum_{n=1}^{\infty} \left( \int_A r_n(t) dt \right) x_n,$$

for any Lebesgue measurable subset  $A$  of  $[0, 1]$ , has bounded variation whenever  $(x_n) \in l_w^1(X)$ . In [AD] it is proved that  $\{x_n : n \in \mathbb{N}\} \subset rg(F)$ . Then, given an infinite-dimensional Banach space  $X$ , we can consider the set  $r(X)$  formed by all real numbers  $r \in [1, 2]$  such that every sequence  $(x_n) \in l_w^r(X)$  lies inside the range of an  $X$ -valued measure having bounded variation. Then  $r(X)$  is an interval whose bounds are 1 and  $\sup(r(X))$ . In the following we will determine the set  $r(X)$  for some classical Banach spaces.

- (i)  $r(X) = [1, 2]$  for every Banach space  $X$  satisfying:
  - (a)  $X^*$  is a G.T. space,
  - (b)  $X$  is a dual space.

By Proposition 2(i),  $I_{X^*}$  is  $(1, 1, r)$ -summing for all  $r \in (1, 2]$ . Then Theorem 4 implies that  $r \in r(X)$  for all  $r \in [1, 2]$ .

In particular, if  $\mu$  is a  $\sigma$ -finite positive measure,  $r(L^\infty(\mu)) = [1, 2]$ .

(ii)  $r(l_p) = \{1\}$  for  $1 \leq p < +\infty$ .

$I_p$  will denote the identity map  $l_p \rightarrow l_p$ . First, we consider the case  $p = 1$ . If  $(e_n^*)$  denotes the unit basis of  $l_\infty = (l_1)^*$ , then  $(e_n^*) \in l_w^1(l_\infty)$ . Since  $\sum \|e_n^*\|^s = \infty$  for  $s \geq 1$  it follows that  $I_\infty$  cannot be  $(s, 1)$ -summing for  $s \geq 1$ . So, Proposition 2(ii) tells us that  $I_\infty$  is not  $(1, 1, r)$ -summing for  $r > 1$ . By Theorem 4,  $r(l_1) = \{1\}$ .

Now suppose  $1 < p < +\infty$ .

*Claim.*  $r(l_p) \cap (1, q) = \emptyset$ . Let  $r \in r(l_p) \cap (1, q)$ . Theorems 1 and 4 assure us that there is a constant  $c \geq 0$  such that

$$(4) \quad \sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c\varepsilon_1((x_i^*)_{i=1}^n)\varepsilon_r((x_i)_{i=1}^n)$$

for all  $\{x_1, \dots, x_n\} \subset l_p$  and  $\{x_1^*, \dots, x_n^*\} \subset l_q$ . Given  $(\alpha_n) \in l_q$  and  $(\beta_n) \in l_u$  with  $u = rq(q-r)^{-1}$ , define  $x_n^* = \alpha_n e_n^*$  and  $x_n = \beta_n e_n$  for all  $n \in \mathbb{N}$ . From (4) we get

$$\sum_{i=1}^m |\alpha_i| |\beta_i| \leq c\varepsilon_1((\alpha_i e_i^*)_{i=1}^m)\varepsilon_r((\beta_i e_i)_{i=1}^m)$$

for all  $m \in \mathbb{N}$ . Applying Holder's inequality we obtain

$$\varepsilon_1((\alpha_i e_i^*)_{i=1}^m) \leq \|(\alpha_n)\|_q \varepsilon_p((e_n^*)) = \|(\alpha_n)\|_q$$

and

$$\varepsilon_r((\beta_i e_i)_{i=1}^m) \leq \varepsilon_q((e_n)) \|(\beta_n)\|_u = \|(\beta_n)\|_u.$$

Then, for all  $m \in \mathbb{N}$  and  $(\alpha_n) \in l_q$ , we have

$$\sum_{i=1}^m |\alpha_i| |\beta_i| \leq c \|(\alpha_n)\|_q \|(\beta_n)\|_u.$$

This implies that  $(\beta_n) \in l_p = (l_q)^*$ . Choosing  $(\beta_n) \in l_u \setminus l_p$  we fall in a contradiction since  $rq(q-r)^{-1} > p$ .

With our claim established we already have proved that  $r(l_p) = \{1\}$  for  $p < 2$ . Finally, we are going to show that  $r(l_p) \cap [q, 2] = \emptyset$  for  $p \geq 2$ . This is the easy part. Certainly, the identity map  $l_1 \rightarrow l_p$  is not nuclear, hence Lemma 3(ii) allows us to conclude that the sequence  $(e_n)$  does not lie inside the range of an  $l_p$ -valued measure of bounded variation. Nevertheless,  $(e_n) \in l_w^r(l_p)$  for all  $r \geq q$ . Thus  $[q, 2] \cap r(l_p) = \emptyset$ .

(iii)  $r(X) = \{1\}$  for all infinite-dimensional  $\mathcal{L}_p$ -space  $X$  with  $1 \leq p < +\infty$ .

By [LP, Proposition 7.3],  $X$  has a complemented subspace  $H$  isomorphic to  $l_p$ . Then  $r(X) \subset r(H) = r(l_p) = \{1\}$ .

## REFERENCES

- [AD] R. Anantharaman and J. Diestel, *Sequences in the range of a vector measure*, Anna. Soc. Math. Polon. Ser. I Comment. Math. Prace Mat. **30** (1991), 221–235. MR **92g**:46049
- [DU] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, RI, 1977. MR **56**:12216
- [LP] J. Lindenstrauss and Pelczynski, *Absolutely summing operators in  $\mathcal{L}_p$ -spaces and their applications*, Studia Math. **29** (1968), 275–326. MR **37**:6743
- [P] A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1980. MR **81j**:47001

- [Pi 1] C. Piñeiro, *Operators on Banach spaces taking compact sets inside ranges of vector measures*, Proc. Amer. Math. Soc. **116** (1992), 1031–1040. MR **93b**:47076
- [Pi 2] ———, *Sequences in the range of a vector measure with bounded variations*, Proc. Amer. Math. Soc. **123** (1995), 3329–3334. CMP 95:16
- [PR] C. Piñeiro and L. Rodríguez-Piazza, *Banach spaces in which every compact lies inside the range of a vector measure*, Proc. Amer. Math. Soc. **114** (1992), 505–517. MR **92e**:46038
- [Ps] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conf. Ser. in Math., vol. 60, Amer. Math. Soc., Providence, RI, 1986. MR **88a**:47020
- [T] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs Surveys Pure Appl. Math., vol. 38, Longman Sci. Tech., Harlow, 1989. MR **90k**:46039

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, APTDO. 1160, SEVILLA, 41080, SPAIN

*Current address:* Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Huelva, 21810 La Rábida, Huelva, Spain