

## OBLIQUE PROJECTIONS IN ATOMIC SPACES

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ABSTRACT. Let  $\mathcal{H}$  be a Hilbert space,  $\mathbf{O}$  a unitary operator on  $\mathcal{H}$ , and  $\{\phi^i\}_{i=1,\dots,r}$ ,  $r$  vectors in  $\mathcal{H}$ . We construct an *atomic subspace*  $U \subset \mathcal{H}$ :

$$U = \left\{ \sum_{i=1,\dots,r} \sum_{k \in \mathbf{Z}} c^i(k) \mathbf{O}^k \phi^i : c^i \in l^2, \forall i = 1, \dots, r \right\}.$$

We give the necessary and sufficient conditions for  $U$  to be a well-defined, closed subspace of  $\mathcal{H}$  with  $\{\mathbf{O}^k \phi^i\}_{i=1,\dots,r, k \in \mathbf{Z}}$  as its Riesz basis. We then consider the oblique projection  $\mathbf{P}_{U \perp V}$  on the space  $U(\mathbf{O}, \{\phi_U^i\}_{i=1,\dots,r})$  in a direction orthogonal to  $V(\mathbf{O}, \{\phi_V^i\}_{i=1,\dots,r})$ . We give the necessary and sufficient conditions on  $\mathbf{O}$ ,  $\{\phi_U^i\}_{i=1,\dots,r}$ , and  $\{\phi_V^i\}_{i=1,\dots,r}$  for  $\mathbf{P}_{U \perp V}$  to be well defined. The results can be used to construct biorthogonal multiwavelets in various spaces. They can also be used to generalize the Shannon-Whittaker theory on uniform sampling.

### 1. INTRODUCTION

Shift-invariant spaces of  $L_2(\mathbf{R})$  of the form

$$(1.1) \quad U = \left\{ \sum_{i=1}^{i=r} \sum_{k \in \mathbf{Z}} c^i(k) \phi^i(x - k) : c^i \in l_2 \right\}$$

are the building blocks for the multiresolution and wavelet theory [13, 14, 15, 8, 1, 4]. Because of their richness and simple structure, these spaces have long been used in many other areas, such as numerical analysis [16, 6, 18], and sampling theory [5, 3, 19, 20], to name a few. If we replace  $L_2(\mathbf{R})$  by a complex Hilbert space  $\mathcal{H}$ , and if we choose  $r$  functions  $\{\phi^i\}_{i=1,\dots,r} \subset \mathcal{H}$ , we then obtain the abstract space

$$(1.2) \quad U(\mathbf{O}, \{\phi^i\}_{i \in S}) = \left\{ \sum_{k \in \mathbf{Z}} \mathbf{C}(k) \Phi^T(k) := \sum_{i \in S} \sum_{k \in \mathbf{Z}} c^i(k) \mathbf{O}^k \phi^i : c^i \in l^2, \forall i \in S \right\},$$

where  $\mathbf{O}$  is a unitary operator that replaces the shift operator  $\mathbf{O}\phi^i(x) = \phi^i(x-1)$  in (1.1), and where  $S = \{1, \dots, r\}$ ,  $\mathbf{C} = (c^1, \dots, c^r)$ , and  $\Phi^T(k)$  is the vector transpose of  $\Phi(k) := (\mathbf{O}^k \phi^1, \dots, \mathbf{O}^k \phi^r)$ . Since  $U$  is generated from a unitary operator  $\mathbf{O}$  and from a set of atoms  $\{\phi^i\}_{i \in S}$ , we will say that  $U$  is an *atomic* subspace of  $\mathcal{H}$ .

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By observing the relation between wavelets and wandering subspaces in operator theory, Goodman, Lee, and Tang [11] used atomic spaces to construct an abstract theory of multiresolutions and wavelets in Hilbert spaces. As a result, they developed the theory of multiwavelets in  $L_2(\mathbf{R})$  (see also [17, 9]).

In this paper, we completely characterize the atomic spaces. We give the necessary and sufficient conditions on  $\mathbf{O}$  and  $\{\phi^i\}_{i \in S}$  for  $U$  to be a well-defined, closed subspace of  $\mathcal{H}$ , with  $\{\mathbf{O}^k \phi^i\}_{i \in S, k \in \mathbf{Z}}$  as its Riesz basis. We then relate these results to the computation of the orthogonal projection of a vector  $g \in \mathcal{H}$  on the space  $U$ . This is done in section 2. In section 3, we give the necessary and sufficient conditions on  $\mathbf{O}$ ,  $\{\phi_U^i\}_{i \in S}$ , and  $\{\phi_V^i\}_{i \in S}$  for the existence of the oblique projection  $\mathbf{P}_{U \perp V}$  on the space  $U$  in a direction orthogonal to  $V$ . Our results can be used to extend the biorthogonal wavelet theory of Cohen-Daubechies-Feauveau to the case of biorthogonal multiwavelets of  $L_2$ . Our results can also be used to extend the biorthogonal theory to spaces other than  $L_2(\mathbf{R}^d)$  and to geometries other than the Euclidian geometry.

### 2. ATOMIC SPACES

The definition of the space  $U$  in (1.2) makes sense if there exists a positive number  $0 < M < \infty$  such that

$$(2.1) \quad \left\| \sum_{k \in \mathbf{Z}} \mathbf{C}(k) \Phi^T(k) \right\|_{\mathcal{H}}^2 \leq M \|\mathbf{C}\|_{l_2^r}^2 \quad \forall \mathbf{C} \in l_2^r$$

where  $l_2^r := l_2(\mathbf{Z}) \times \dots \times l_2(\mathbf{Z})$  and where  $\|\mathbf{C}\|_{l_2^r}^2 = \sum_{i \in S} \|c^i\|_{l_2}^2$ . If, in addition, we have the lower bound

$$(2.2) \quad m \|\mathbf{C}\|_{l_2^r}^2 \leq \left\| \sum_{k \in \mathbf{Z}} \mathbf{C}(k) \Phi^T(k) \right\|_{\mathcal{H}}^2,$$

where  $m > 0$ , then  $U$  is a closed subspace of  $\mathcal{H}$  and the set  $\{\mathbf{O}^k \phi^i\}_{i \in S, k \in \mathbf{Z}}$  is a Riesz basis for  $U \subset \mathcal{H}$ . Our goal is then to find conditions on  $\mathbf{O}$  and  $\{\phi^i\}_{i \in S}$  for  $U$  to be a well-defined subspace of  $\mathcal{H}$ , and for  $\{\mathbf{O}^k \phi^i\}_{i \in S, k \in \mathbf{Z}}$  to be its Riesz basis. To do this, we first observe that if  $U$  is well defined (i.e., condition (2.1) is satisfied), then we can write the norm of a vector  $u = \sum \mathbf{C}(k) \Phi^T(k)$  belonging to  $U$  as

$$(2.3) \quad \|u\|_{\mathcal{H}}^2 = \sum_{(k,l) \in \mathbf{Z}^2} \mathbf{C}(k) \mathbf{A}(l-k) \mathbf{C}^*(l),$$

where the sequence of matrices  $\mathbf{A}(k), k \in \mathbf{Z}$ , is defined to be

$$[\mathbf{A}]_{i,j}(k) := \langle \phi^i, \mathbf{O}^k \phi^j \rangle_{\mathcal{H}}.$$

Here,  $\langle \bullet, \bullet \rangle_{\mathcal{H}}$  denotes the inner product on  $\mathcal{H}$ .

Using Parseval's theorem and the fact that the convolution operator transforms into the product operator in Fourier domain, (2.3) becomes

$$(2.4) \quad \|u\|_{\mathcal{H}}^2 = \int_0^1 \widehat{\mathbf{C}}(f) \widehat{\mathbf{A}}(f) \widehat{\mathbf{C}}^*(f) df,$$

where the Fourier transform of a sequence  $\mathbf{b}(k)$  is defined to be

$$\widehat{\mathbf{b}}(f) = \sum \mathbf{b}(k)e^{-i2\pi kf},$$

and where  $\widehat{\mathbf{C}}^*$  denotes the Hermitian transpose of  $\widehat{\mathbf{C}}$ . From this definition, it follows that  $\widehat{\mathbf{A}}(f)$  is 1-periodic (i.e.,  $\widehat{\mathbf{A}}(f+1) = \widehat{\mathbf{A}}(f)$ ). Moreover, the  $r \times r$  matrix  $\widehat{\mathbf{A}}(f)$  in (2.4) is self-adjoint for almost all  $f$ . To see this, we use the fact that  $\mathbf{O}$  is unitary, and simply note that

(2.5)

$$\begin{aligned} [\widehat{\mathbf{A}}(f)]_{i,j} &= \langle \phi^i, \phi^j \rangle_{\mathcal{H}} + \sum_{k=1}^{k=\infty} \langle \phi^i, \mathbf{O}^k \phi^j \rangle_{\mathcal{H}} e^{-i2\pi fk} + \langle \phi^i, \mathbf{O}^{-k} \phi^j \rangle_{\mathcal{H}} e^{i2\pi fk} \\ &= \langle \phi^i, \phi^j \rangle_{\mathcal{H}} + \sum_{k=1}^{k=\infty} \langle \mathbf{O}^{-k} \phi^i, \phi^j \rangle_{\mathcal{H}} e^{-i2\pi fk} + \langle \mathbf{O}^k \phi^i, \phi^j \rangle_{\mathcal{H}} e^{i2\pi fk} \\ &= \overline{\langle \phi^j, \phi^i \rangle_{\mathcal{H}}} + \sum_{k=1}^{k=\infty} \overline{\langle \phi^j, \mathbf{O}^{-k} \phi^i \rangle_{\mathcal{H}} e^{i2\pi fk}} + \overline{\langle \phi^j, \mathbf{O}^k \phi^i \rangle_{\mathcal{H}} e^{-i2\pi fk}} \\ &= \overline{[\widehat{\mathbf{A}}(f)]_{j,i}} \end{aligned}$$

where we use  $\bar{z}$  to denote the complex conjugate of  $z$ . From (2.4), it also follows that  $\widehat{\mathbf{A}}(f)$  is positive. In fact, we have:

**Theorem 2.1.** *The space  $U$  is a well-defined, closed subspace of  $\mathcal{H}$  with Riesz basis  $\{\mathbf{O}^k \phi^i\}_{i \in S, k \in \mathbf{Z}}$  if and only if the  $r \times r$  matrix  $\widehat{\mathbf{A}}(f)$  is positive self-adjoint for almost all  $f \in \mathbf{R}$  and there exist two positive constants  $0 < m \leq M < \infty$  such that the smallest and largest eigenvalues  $\lambda_{\min}(\widehat{\mathbf{A}}(f))$  and  $\lambda_{\max}(\widehat{\mathbf{A}}(f))$  satisfy:*

(2.6)

$$m \leq \operatorname{ess\,inf}_{f \in [-\frac{1}{2}, \frac{1}{2}]} (\lambda_{\min}(\widehat{\mathbf{A}}(f))) \leq \operatorname{ess\,sup}_{f \in [-\frac{1}{2}, \frac{1}{2}]} (\lambda_{\max}(\widehat{\mathbf{A}}(f))) \leq M \quad a.e.$$

We should note that our definition of atomic spaces excludes finite-dimensional spaces. For related results in finite dimensions, we refer the reader to [10]. Other related results can also be found in [12].

*Proof. Necessity.* If conditions (2.1) and (2.2) hold, then—as we have shown above— $\widehat{\mathbf{A}}(f)$  is a 1-periodic and positive self-adjoint matrix for almost all  $f$ . Thus, it has  $r$  real eigenvalues. If we assume that the largest eigenvalue  $\lambda_{\max}(f)$  is not bounded above by  $M$ , then the set  $E_n := \{f \in [-\frac{1}{2}, \frac{1}{2}] : \lambda_{\max}(f) > n\}$  has a strictly positive measure:  $\operatorname{meas}(E_n) > 0$  (we only need to concentrate on the interval  $[-\frac{1}{2}, \frac{1}{2}]$  because  $\lambda_{\max}(f)$  is 1-periodic). We consider the periodic vector functions  $\widehat{\mathbf{C}}_n(f+1) = \widehat{\mathbf{C}}_n(f)$  defined by  $\widehat{\mathbf{C}}_n(f) := (\operatorname{meas}(E_n))^{-1/2} \chi_{E_n} \mathbf{V}_{\lambda_{\max}}(f)$  for all  $f \in [-\frac{1}{2}, \frac{1}{2}]$ , where  $\chi_{E_n}$  is the characteristic function of the set  $E_n$  and  $\mathbf{V}_{\lambda_{\max}}(f)$  is the eigenvector corresponding to  $\lambda_{\max}(f)$  with  $\|\mathbf{V}_{\lambda_{\max}}(f)\|_{\mathbf{R}^r} = 1$ . By construction,  $\widehat{\mathbf{C}}_n(f)$  are the Fourier transforms of vector sequences  $\mathbf{C}_n(k)$  in  $l_2^r$  with  $\|\mathbf{C}_n\|_{l_2^r} = 1$ . Using identity (2.4), and the fact that  $\|\mathbf{C}_n\|_{l_2^r}^2 = 1$ , we conclude that the norm of the vectors  $u_n = \sum_{k \in \mathbf{Z}} \mathbf{C}_n(k) \Phi^T(k)$  tends to infinity as  $n$  goes to infinity:  $\|u_n\|_{\mathcal{H}}^2 \geq n$ . This contradicts our hypothesis that  $\|u_n\|_{\mathcal{H}}^2 \leq M$ . Thus,

$\lambda_{\max}(f) \leq M$  almost everywhere. A similar argument using the smallest eigenvalue  $\lambda_{\min}(f)$  implies that  $\lambda_{\min}(f)$  is bounded below by a positive constant  $m$ .

**Sufficiency.** If  $\widehat{A}(f)$  is positive self-adjoint for almost all  $f$ , then from (2.4) we obtain

$$\int_0^1 \lambda_{\min}(f) \sum_{i \in S} |\widehat{c}^i(f)|^2 df \leq \int_0^1 \widehat{C}(f) \widehat{A}(f) \widehat{C}^*(f) df \leq \int_0^1 \lambda_{\max}(f) \sum_{i \in S} |\widehat{c}^i(f)|^2 df.$$

Condition (2.6) and the properties of the Fourier transform then imply that

$$m \|\mathbf{C}\|_{l_2^r}^2 \leq \left\| \sum_{k \in \mathbf{Z}} \mathbf{C}(k) \Phi^T(k) \right\|_{\mathcal{H}}^2 \leq M \|\mathbf{C}\|_{l_2^r}^2. \quad \square$$

Since  $\widehat{A}(f)$  is positive and self-adjoint, we have that  $\lambda_{\max}(f) = \|\widehat{A}(f)\|$  and that  $(\lambda_{\min}(f))^{-1} = \|\widehat{A}^{-1}(f)\|$  where  $\|\bullet\|$  denotes the norm of the operators on  $\mathbf{R}^r$ . Thus, condition (2.6) is equivalent to the conditions  $\|\widehat{A}(f)\| \leq M$  and  $\|\widehat{A}^{-1}(f)\| \leq m^{-1}$  almost everywhere. These two conditions imply that the convolution operator  $\bullet * \mathbf{A}(k)$  which takes a sequence  $\mathbf{C}(k)$  into the sequence  $\mathbf{C} * \mathbf{A}(k) = \sum_{l \in \mathbf{Z}} \mathbf{C}(l) \mathbf{A}(k-l)$ , is a bounded linear operator on  $l_2^r$  and has a bounded convolution inverse. This last assertion is an immediate consequence of the following theorem:

**Theorem 2.2.** *The convolution operator  $\bullet * \mathbf{X}(k)$ , generated from the  $r \times r$  matrix-sequence  $\mathbf{X}(k)$ , is a bounded linear operator on  $l_2^r$  if and only if there exists a positive constant  $M$  such that*

$$(2.7) \quad \left\| \widehat{\mathbf{X}}(f) \right\|^2 = \left\| \widehat{\mathbf{X}}(f) \widehat{\mathbf{X}}^*(f) \right\| \leq M < \infty \quad a.e.$$

Moreover, the norm of the convolution operator is given by

$$(2.8) \quad \|\bullet * \mathbf{X}\|_{B(l_2^r)} = \operatorname{ess\,sup}_{f \in [0,1]} \left\| \widehat{\mathbf{X}}(f) \right\|.$$

*Proof.* With an argument similar to the one that led to (2.4), we get

$$\|\mathbf{C} * \mathbf{X}\|_{l_2^r}^2 \leq \int_0^1 \lambda_{\max}(f) \widehat{C}(f) \widehat{C}^*(f) df \leq \Lambda \int_0^1 \widehat{C}(f) \widehat{C}^*(f) df,$$

where  $\lambda_{\max}(f)$  is the largest eigenvalue of the  $r \times r$  self-adjoint matrix  $\widehat{\mathbf{X}}(f) \widehat{\mathbf{X}}^*(f)$ , and where  $\Lambda := \operatorname{ess\,sup}_{f \in [0,1]} (\lambda_{\max}(f))$ . It follows that  $\|\bullet * \mathbf{X}\|_{B(l_2^r)} \leq \Lambda^{1/2}$ . Conversely,

using a proof by contradiction similar to the “necessity” proof of Theorem 2.1, we get that  $\Lambda^{1/2} \leq \|\bullet * \mathbf{X}\|_{B(l_2^r)}$ . The proof of the theorem then follows by simply

noting that  $\left\| \widehat{\mathbf{X}}(f) \right\|^2 = \left\| \widehat{\mathbf{X}}(f) \widehat{\mathbf{X}}^*(f) \right\| = \lambda_{\max}(f)$ . □

As a consequence of Theorem 2.2, we also conclude that the inverse Fourier transform of  $\widehat{\mathbf{A}}^{inv}(f) := \widehat{\mathbf{A}}^{-1}(f)$  makes sense, and that  $\mathbf{A}^{inv}(k)$  is the convolution inverse of  $\mathbf{A}(k)$ :

$$\sum_{k \in \mathbf{Z}} \mathbf{A}(k) \mathbf{A}^{inv}(l-k) = \delta(k) \mathbf{I},$$

where  $\mathbf{I}$  is the  $r \times r$  identity matrix and where  $\delta(k)$  is the Dirac impulse (i.e.,  $\delta(0) = 1$  and  $\delta(k) = 0$  for  $k \neq 0$ ).

If we assume that  $\|\phi^i\|_{\mathcal{H}} = 1$ , then the sequences  $[\mathbf{A}]_{i,j}(k) = \langle \phi^i, \mathbf{O}^k \phi^j \rangle_{\mathcal{H}}$  have the property that  $|\mathbf{A}_{i,i}(0)| = 1$  and  $|\mathbf{A}_{i,j}(k)| \leq 1$  for  $k \neq 0, i \neq j$ . For this reason, we will call  $\mathbf{A}(k)$  the *autocorrelation matrix-sequence*. In fact, if  $\mathcal{H} = L_2(\mathbf{R})$ , if  $r = 1$ , and if  $\mathbf{O}$  is the shift operator, then  $\mathbf{A}(k)$  is precisely the sampled autocorrelation function  $\mathbf{A}(k) = \int_{\mathbf{R}} \phi(x)\phi(x - k)dx$ .

**2.1. Orthogonal projection.** If the autocorrelation sequence  $\mathbf{A}(k)$  satisfies condition (2.6), then  $U$  is closed. Thus, the orthogonal projection is well defined. In fact, if  $\mathbf{P}g = \sum \mathbf{E}(k)\Phi^T(k)$  is the orthogonal projection of  $g \in \mathcal{H}$ , then from the orthogonality condition  $\langle \mathbf{P}g - g, \mathbf{O}^l \phi^j \rangle_{\mathcal{H}} = 0$ , we get the convolution equation

$$(2.9) \quad \sum_{k \in \mathbf{Z}} \mathbf{E}(k)\mathbf{A}(l - k) = \mathbf{G}(l) \quad \forall l \in \mathbf{Z},$$

where  $\mathbf{G}$  is the  $1 \times r$  vector-sequence  $[\mathbf{G}]_j(l) := \langle g, \mathbf{O}^l \phi^j \rangle_{\mathcal{H}}$ . Equation (2.9) allows us to write the orthogonal projection as

$$(2.10) \quad \mathbf{P}g = \sum_{k \in \mathbf{Z}} \mathbf{E}(k)\Phi^T(k) = \sum_{(k,l) \in \mathbf{Z}^2} \mathbf{G}(l)\mathbf{A}^{inv}(k - l)\Phi^T(k)$$

where, as before,  $\mathbf{A}^{inv}(k)$  is the inverse Fourier transform of  $\widehat{\mathbf{A}}^{-1}(f)$ , and  $\Phi^T(k)$  is the vector-transpose of  $\Phi(k) := (\mathbf{O}^k \phi^1, \dots, \mathbf{O}^k \phi^r)$ . Because of the simple atomic structure of  $U$ , all the operations involved in computing the vector-sequence  $\mathbf{E}(k)$  in (2.9) consist of simple convolutions and additions only. This feature is often useful for signal processing and numerical analysis because it allows for fast “filtering” implementations.

### 3. OBLIQUE PROJECTIONS

Let  $U(\mathbf{O}, \{\phi_U^i\}_{i \in S})$  and  $V(\mathbf{O}, \{\phi_V^i\}_{i \in S})$  be two atomic subspaces of  $\mathcal{H}$  that satisfy condition (2.6). Then, under appropriate conditions, the oblique projection  $\mathbf{P}_{U \perp V}g = \sum \mathbf{C}_g(k)\Phi_U^T(k)$  of a vector  $g \in \mathcal{H}$  on the space  $U$ , in a direction orthogonal to  $V$ , must satisfy  $\langle \mathbf{P}_{U \perp V}g - g, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}} = 0$  for all  $j \in S$  and for all  $l \in \mathbf{Z}$ . From this, we obtain the convolution equation

$$(3.1) \quad \sum_{k \in \mathbf{Z}} \mathbf{C}_g(k)\mathbf{X}_{UV}(l - k) = \mathbf{G}(l),$$

where  $\mathbf{X}_{UV}$  is the *cross-correlation* matrix-sequence  $[\mathbf{X}_{UV}]_{i,j}(k) = \langle \phi_U^i, \mathbf{O}^k \phi_V^j \rangle_{\mathcal{H}}$  and where  $\mathbf{G}$  is the  $1 \times r$  vector given by  $[\mathbf{G}]_j(l) = \langle g, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}}$ . In the Fourier domain, (3.1) becomes

$$(3.2) \quad \widehat{\mathbf{C}}_g(f)\widehat{\mathbf{X}}_{UV}(f) = \widehat{\mathbf{G}}(f).$$

Our first observation is that the convolution operator  $\bullet * \mathbf{X}_{UV}$  is a bounded operator on  $l_2^r$ , a fact solely based on our assumptions that  $U$  and  $V$  satisfy condition (2.6). To see why our last assertion is true, we pick a vector  $g = \sum \mathbf{D}(k)\Phi_U^T(k)$  in the space  $U$  and compute the orthogonal projection  $\mathbf{P}_V g$  of the vector  $g \in U$  on the

space  $V$ . A simple calculation shows that the sequence  $\mathbf{G}$  in (2.9) is given by  $\mathbf{G} = \mathbf{D} * \mathbf{X}_{UV}$ . Using eqs. (2.1), (2.2), (2.9), and Theorem 2.1, we get that

$$(3.3) \quad \begin{aligned} \|\mathbf{D} * \mathbf{X}_{UV}\|_{l^2_2}^2 &= \|\mathbf{E} * \mathbf{A}_V\|_{l^2_2}^2 \leq M_V \|\mathbf{E}\|_{l^2_2}^2 \leq m_V^{-1} M_V \|\mathbf{P}_V g\|_{\mathcal{H}}^2 \\ &\leq m_V^{-1} M_V \|g\|_{\mathcal{H}}^2 \leq m_U^{-1} m_V^{-1} M_V \|\mathbf{D}\|_{l^2_2}^2 \end{aligned}$$

where  $\mathbf{A}_V$  is the autocorrelation operator  $\mathbf{A}$  in Theorem 2.1 associated with the space  $V(\mathbf{O}, \{\phi_V^i\}_{i \in S})$ , and  $m_V, M_V$  are the corresponding constants in (2.6), while the constant  $m_U$  relates to the space  $U$ . From (3.3), we conclude that  $\bullet * \mathbf{X}_{UV}$  is a bounded convolution operator on  $l^2_2$ . Equivalently, as a consequence of Theorem 2.2, there exists a constant  $M$  such that, except possibly on a set of measure zero,  $\|\widehat{\mathbf{X}}_{UV}(f)\| \leq M < \infty$ . If, furthermore, we have that  $\|\widehat{\mathbf{X}}_{UV}^{-1}(f)\| \leq m < \infty$ , then the oblique projection  $\mathbf{P}_{U \perp V}$  is well defined.<sup>1</sup> The converse is also true, and we have:

**Theorem 3.1.** *Let  $U(\mathbf{O}, \{\phi_U^i\}_{i \in S})$  and  $V(\mathbf{O}, \{\phi_V^i\}_{i \in S})$  be two atomic subspaces of  $\mathcal{H}$  that satisfy condition (2.6). The oblique projection  $\mathbf{P}_{U \perp V}$  is well defined if and only if  $\widehat{\mathbf{X}}_{UV}^{-1}(f)$  exists for almost all  $f \in [0, 1]$ , and if there exists a constant  $m > 0$  such that*

$$(3.4) \quad \left\| \widehat{\mathbf{X}}_{UV}^{-1}(f) \right\| \leq m < \infty.$$

The proof of Theorem 3.1 is given below. This theorem can be restated in terms of an ‘‘angle’’ between the spaces  $U$  and  $V$ . In particular, we define the angle  $\theta(U, V)$  between the spaces  $U$  and  $V$  by

$$(3.5) \quad \cos(\theta(U, V)) := \inf_{\substack{u \in U \\ \|u\|_{\mathcal{H}}=1}} \|\mathbf{P}_V u\|_{\mathcal{H}}$$

Although  $\theta(U, V) = \theta(V, U)$  for the atomic spaces  $U$  and  $V$  (as we will see below), this is not true for arbitrary spaces. We have:

**Theorem 3.2.** *If the atomic spaces  $U(\mathbf{O}, \{\phi_U^i\}_{i \in S})$  and  $V(\mathbf{O}, \{\phi_V^i\}_{i \in S})$  satisfy condition (2.6), then*

- (i)  $\cos(\theta(U, V)) = \cos(\theta(V, U)) = \text{ess inf}_{f \in [0, 1]} \left( \lambda_{\min} \left( \widehat{\Xi}_{UV}(f) \right) \right)$ , where  $\widehat{\Xi}_{UV}(f)$  is defined by (3.10) below.
- (ii)  $\cos(\theta(U, V)) = \cos(\theta(V^\perp, U^\perp))$ .
- (iii) *The oblique projection  $\mathbf{P}_{U \perp V}$  is well defined if and only if  $\cos(\theta(U, V)) > 0$ .*

*Remark 3.1.* The identity (ii) in Theorem 3.2 holds even if  $U$  and  $V$  are not atomic spaces.

*Remark 3.2.* We would expect a condition similar to the one in (iii) for the existence of  $\mathbf{P}_{U \perp V}$  to be necessary, in general. In this case, the condition is also sufficient.

*Proof of Theorem 3.1. Necessity.* Our goal is to find a solution to the convolution equation

$$(3.6) \quad \mathbf{C} * \mathbf{X}_{UV} = \mathbf{B}$$

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<sup>1</sup>The oblique projection  $\mathbf{P}_{U \perp V}$  is well-defined if and only if, for each vector  $g \in \mathcal{H}$ , there exists a unique representation of the form  $g = u_1 + v_1$  with  $u_1 \in U$  and  $v_1 \in V^\perp$ .

where  $\mathbf{B}$  is an arbitrary vector-sequence of  $l_2^r$ . To find a solution  $\mathbf{C}$ , we construct the vector  $g \in V$ , given by

$$g = \sum_{k \in \mathbf{Z}} (\mathbf{B} * \mathbf{A}_V^{inv})(k) \Phi_V^T(k)$$

where  $\mathbf{A}_V^{inv}$  is the inverse of the autocorrelation matrix-sequence  $\mathbf{A}_V(k)$ . Since we are assuming that the oblique projection is well defined, we construct the vector  $\mathbf{P}_{U \perp V} g = \sum \mathbf{C}_g(k) \Phi_U^T(k)$ . A simple calculation using the fact that

$$\langle \mathbf{P}_{U \perp V} g - g, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}} = 0$$

shows that  $\mathbf{C}_g * \mathbf{X}_{UV} = \mathbf{B}$ . Moreover,  $\mathbf{C}_g$  is the only solution to (3.6). To see why this claim is true, we assume that there exists a nonzero sequence  $\mathbf{C} \in l_2^r$  such that  $\mathbf{C} * \mathbf{X}_{UV} = 0$ . A simple calculation then shows that the nonzero vector  $u = \sum \mathbf{C}(k) \Phi_U^T(k)$  is orthogonal to  $V$ . Using the fact that  $\mathbf{P}_{U \perp V}$  is well defined, we get that  $\mathbf{P}_{U \perp V} u = 0$ . But since  $u \in U$ , we also get that  $\mathbf{P}_{U \perp V} u = u$ . Hence, we deduce that  $u = 0$ , a fact that contradicts our assumption that  $\mathbf{C} \neq 0$ . Therefore the only solution to the homogeneous equation is  $\mathbf{C} = 0$ . It follows that our constructed solution  $\mathbf{C}_g$  is unique, and the operator  $\bullet * \mathbf{X}_{UV}$  has an inverse. From a calculation similar to (3.3) and from Theorem 2.2, we then conclude that (3.4) holds.

**Sufficiency.** If  $g \in \mathcal{H}$ , then the sequence  $[\mathbf{G}]_j(l) = \langle g, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}}$  is given by

$$(3.7) \quad \langle g, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}} = \langle g_V, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}} = \sum_{k \in \mathbf{Z}} \mathbf{D}(k) \mathbf{A}_V(l - k),$$

where  $g_V = \sum_{k \in \mathbf{Z}} \mathbf{D}(k) \Phi_V^T(k)$  is the orthogonal projection of  $g \in \mathcal{H}$  on  $V$ . Since by assumption  $\mathbf{A}_V$  satisfies condition (2.6), Theorem 2.2 then implies that  $\mathbf{G} \in l_2^r$ . If we assume that condition (3.4) holds, then from Theorem 2.2 it follows that  $\mathbf{C}_g = \mathbf{G} * \mathbf{X}_{UV}^{-1}$  belongs to  $l_2^r$ . It is not difficult to see that the vector  $u_g = \sum \mathbf{C}_g(k) \Phi_U^T(k)$  belongs to  $U$  and that the vector  $e_g = g - u_g$  is orthogonal to  $V$ . Moreover, the decomposition  $g = u_g + e_g$  is unique. This follows from the fact that if  $u = \sum \mathbf{C}(k) \Phi_U^T(k)$  is orthogonal to  $V$ , then (from  $\langle u, \mathbf{O}^l \phi_V^j \rangle_{\mathcal{H}} = 0$ ) we get  $\mathbf{C} * \mathbf{X}_{UV} = 0$ . The last equality implies that  $\mathbf{C} = 0$  by assumption. Thus,  $u = 0$ . It follows that the only vector  $u \in U$  which is orthogonal to  $V$  is the trivial vector (i.e.,  $U \cap V^\perp = \{0\}$ ). From this, we conclude that the decomposition of  $g$  into components in  $U$  and  $V^\perp$  is unique.  $\square$

*Proof of Theorem 3.2.* If  $\mathbf{P}_V u = \sum \mathbf{E}(k) \Phi_V^T(k)$  is the orthogonal projection of the vector  $u = \sum \mathbf{D}(k) \Phi_U^T(k) \in U$  on the space  $V$ , then by a simple calculation that uses the results of section 2.1, we obtain the relation

$$(3.8) \quad \widehat{\mathbf{E}}(f) = \widehat{\mathbf{D}}(f) \widehat{\mathbf{X}}_{UV}(f) \widehat{\mathbf{A}}_V^{-1}(f).$$

Using the above relation and (2.4), and using the superscript  $*$  to denote the Hermitian transpose, we obtain

$$(3.9) \quad \|\mathbf{P}_V u\|_{\mathcal{H}}^2 = \int_0^1 \widehat{\mathbf{Y}}(f) \widehat{\Xi}_{UV}(f) \widehat{\mathbf{Y}}^*(f) df,$$

where  $\widehat{\mathbf{Y}}(f) := \widehat{\mathbf{D}}(f)\widehat{\mathbf{A}}^{1/2}(f)$  and where  $\widehat{\Xi}_{UV}(f)$  is given by

$$(3.10) \quad \widehat{\Xi}_{UV}(f) = \widehat{\mathbf{R}}_{UV}(f)\widehat{\mathbf{R}}_{UV}^*(f),$$

$$(3.11) \quad \widehat{\mathbf{R}}_{UV}(f) = \widehat{\mathbf{A}}_U^{-1/2}(f)\widehat{\mathbf{X}}_{UV}(f)\widehat{\mathbf{A}}_V^{-1/2}(f).$$

From the definition of  $\widehat{\mathbf{Y}}(f)$  and from relation (2.4), we also get that

$$(3.12) \quad \|u\|_{\mathcal{H}}^2 = \int_0^1 \widehat{\mathbf{Y}}(f)\widehat{\mathbf{Y}}^*(f)df.$$

Combining equalities (3.9) and (3.12), and using the fact that  $\widehat{\Xi}_{UV}(f)$  is a positive self-adjoint matrix for almost all  $f$ , we get the lower bound

$$(3.13) \quad \rho^2 \|u\|_{\mathcal{H}}^2 \leq \int_0^1 \widehat{\mathbf{Y}}(f)\widehat{\Xi}_{UV}(f)\widehat{\mathbf{Y}}^*(f)df$$

where  $\rho = \cos(\theta(U, V)) = \operatorname{ess\,inf}_{f \in [-\frac{1}{2}, \frac{1}{2}]} (\lambda_{\min}(f))$  (here  $\lambda_{\min}(f)$  is the smallest eigenvalue of  $\widehat{\Xi}_{UV}(f)$ ). Since  $\widehat{\Xi}_{UV}(f)$  is a positive self-adjoint matrix, we conclude from (3.13) that  $\widehat{\Xi}_{UV}(f)$  has an inverse and  $\|\widehat{\Xi}_{UV}^{-1}(f)\|$  is bounded by a constant if and only if the lower bound  $\rho > 0$ . From this last assertion and from eqs. (3.10) and (3.11) and Theorem 3.1, we get part (iii) of Theorem 3.2.

The proof of (ii) can be found in [19]. We show it here for completeness. Let  $U^\perp$  denote the orthogonal complement of  $U$ . Using the fact that

$$\|\mathbf{P}_U v\|_{\mathcal{H}}^2 + \|\mathbf{P}_{U^\perp} v\|_{\mathcal{H}}^2 = \|v\|_{\mathcal{H}}^2,$$

we get that

$$(3.14) \quad \inf_{B_V^1} \|\mathbf{P}_U v\|_{\mathcal{H}}^2 = \sup_{B_V^1} \|\mathbf{P}_{U^\perp} v\|_{\mathcal{H}}^2$$

where  $B_V^1$  is the ball of radius 1 in  $V$ . Using (3.14) and the general identity

$$(3.15) \quad \|\mathbf{P}_W v\|_{\mathcal{H}} = \sup_{B_W^1} \langle w, v \rangle_{\mathcal{H}}$$

we obtain

$$(3.16) \quad \begin{aligned} \inf_{B_V^1} \|\mathbf{P}_U v\|_{\mathcal{H}} &= \sup_{B_V^1} \|\mathbf{P}_{U^\perp} v\|_{\mathcal{H}} = \sup_{B_V^1, B_{U^\perp}^1} |\langle u^\perp, v \rangle_{\mathcal{H}}| \\ &= \sup_{B_{U^\perp}^1} \|\mathbf{P}_V u^\perp\|_{\mathcal{H}} = \inf_{B_{U^\perp}^1} \|\mathbf{P}_{V^\perp} u^\perp\|_{\mathcal{H}} \end{aligned}$$

from which (ii) follows.

Finally, to prove part (i), we first observe that

$$(3.17) \quad [\mathbf{X}_{UV}]_{i,j}(k) = \langle \phi_V^i, \mathbf{O}^k \phi_U^j \rangle_{\mathcal{H}} = \overline{\langle \mathbf{O}^k \phi_U^j, \phi_V^i \rangle_{\mathcal{H}}} = \overline{\langle \phi_U^j, \mathbf{O}^{-k} \phi_V^i \rangle_{\mathcal{H}}}$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ . A simple computation in Fourier domain yields

$$(3.18) \quad \widehat{\mathbf{X}}_{UV}(f) = \widehat{\mathbf{X}}_{VU}^*(f),$$

where  $\widehat{\mathbf{X}}_{VU}(f)$  is the matrix associated with the oblique projection  $\mathbf{P}_{V\perp U}$ , and where the superscript  $*$  denotes the Hermitian transpose. It follows that the operators  $\widehat{\Xi}_{UV}$  and  $\widehat{\Xi}_{VU}$  (see eqs. (3.10) and (3.11)) can also be written as

$$(3.19) \quad \widehat{\Xi}_{UV}(f) = \widehat{\mathbf{R}}_{UV}(f)\widehat{\mathbf{R}}_{UV}^*(f) = \widehat{\mathbf{R}}_{VU}^*(f)\widehat{\mathbf{R}}_{VU}(f),$$

$$(3.20) \quad \widehat{\Xi}_{VU}(f) = \widehat{\mathbf{R}}_{VU}(f)\widehat{\mathbf{R}}_{VU}^*(f) = \widehat{\mathbf{R}}_{UV}^*(f)\widehat{\mathbf{R}}_{UV}(f).$$

Using the last two relations, we infer that if  $\lambda(f)$  is an eigenvalue for  $\widehat{\Xi}_{VU}(f)$  (i.e.,  $\widehat{\mathbf{R}}_{VU}\widehat{\mathbf{R}}_{VU}^*x = \lambda x$ ), then we have that  $\widehat{\mathbf{R}}_{VU}^*\widehat{\mathbf{R}}_{VU}(\widehat{\mathbf{R}}_{VU}^*x) = \lambda(\widehat{\mathbf{R}}_{VU}^*x)$ . Hence,  $\lambda$  is also an eigenvalue of  $\widehat{\Xi}_{UV}(f)$ . Since  $\widehat{\Xi}_{VU}(f)$  and  $\widehat{\Xi}_{UV}(f)$  have the same eigenvalues, we conclude from (3.9) and (3.13) that  $\cos\theta(U, V) = \cos\theta(V, U) = \rho$ .  $\square$

If the conditions of Theorems 3.1 or 3.2 are satisfied, then the oblique projection can be written as

$$(3.21) \quad \mathbf{P}_{U\perp V}g = \sum_{(k,l)\in\mathbf{Z}^2} \mathbf{G}(k)\mathbf{X}_{UV}^{inv}(l-k)\Phi_U^T(l)$$

where  $\mathbf{X}_{UV}^{inv}(k)$  is the inverse Fourier transform of  $\widehat{\mathbf{X}}_{UV}^{-1}(f)$ . The operations involved in computing the coefficients of the series (3.21) consist of simple convolutions and additions only. As previously mentioned, this feature is useful for implementation.

#### 4. EXAMPLES AND CONCLUDING REMARKS

The oblique projection in shift-invariant subspaces of  $L_2$  is implicit in the general theory of biorthogonal wavelets of A. Cohen, I. Daubechies, and J.C. Fauveau [7]. Our results can be used to extend the biorthogonal wavelet theory to spaces other than  $L_2$ . This has been done for the sequence space  $l_2$ , which is useful in digital signal processing [2]. Other extensions of interest would be to biorthogonal multiwavelets and to geometries other than the Euclidian space. In particular, mutiresolution on the sphere would be useful in geophysics.

Shift-invariant spaces have also been used to generalize the Shannon-Whittaker theory on uniform sampling [3], and the oblique projections have been used to take into account nonideal acquisition devices [19]. The present results may also be useful in multimodal signal and image processing.

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