

## DIFFEOMORPHISMS WITH PERSISTENCY

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**ABSTRACT.** The  $C^1$  interior of the set of all diffeomorphisms satisfying Lewowicz's persistency is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

In [5], Lewowicz introduced a notion of persistency for a homeomorphism of a compact metric space  $X$ , and it is remarked that persistence is a weaker property than topological stability when  $X$  is a manifold. It is also proved there that every pseudo-Anosov map (on a surface) is persistent. The purpose of this paper is to analyze the dynamics of diffeomorphisms having persistency. More precisely we shall prove the following theorem.

Let  $M$  be a  $C^\infty$  closed manifold and let  $\text{Diff}(M)$  be the space of  $C^1$  diffeomorphisms of  $M$  endowed with  $C^1$  topology. We denote by  $\mathcal{P}(M)$  the set of all  $f \in \text{Diff}(M)$  having persistency.

**Theorem.** *The  $C^1$  interior of  $\mathcal{P}(M)$  in  $\text{Diff}(M)$ ,  $\text{int}\mathcal{P}(M)$ , is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.*

It was proved in [7] and [9] respectively that the  $C^1$  interior of the set of all  $f \in \text{Diff}(M)$  having topological stability and the  $C^1$  interior of the set of all  $f \in \text{Diff}(M)$  having the pseudo-orbit tracing property were characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition. Therefore, if the theorem is established, then these two open sets and  $\text{int}\mathcal{P}(M)$  are equal.

Let  $\mathcal{E}(M)$  be the set of all expansive diffeomorphisms of  $M$ . Gerber and Katok [3] proved that if  $f$  is a pseudo-Anosov map on a surface  $M$  and if  $\mathcal{N}^0(f)$  is a  $C^0$  neighborhood of  $f$ , then there exists a smooth diffeomorphism  $g \in \mathcal{N}^0(f)$  conjugating to  $f$ . Thus it can be checked that  $g \in \mathcal{E}(M) \cap \mathcal{P}(M)$ , and more precisely, the following corollary implies that  $g$  belongs to  $\mathcal{E}(M) \cap \partial\mathcal{P}(M)$ . Here  $\partial\mathcal{P}(M)$  denotes the boundary of  $\mathcal{P}(M)$  in  $\text{Diff}(M)$ .

**Corollary.**  *$\mathcal{E}(M) \cap \text{int}\mathcal{P}(M)$  is characterized as the set of all Anosov diffeomorphisms.*

The corollary is an easy consequence of our theorem. Indeed, since every  $f \in \text{Diff}(M)$  satisfying Axiom A and the strong transversality condition is structurally

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stable, if  $f \in \mathcal{E}(M)$ , then it is Anosov (by [6]). Conversely, if  $f \in \text{Diff}(M)$  is Anosov, then  $f$  is persistent since  $f$  is topologically stable.

Let  $d$  be a metric on  $M$  which is induced from a Riemannian metric  $\|\cdot\|$  on  $TM$ , and let us denote by  $\mathcal{H}(M)$  the set of all homeomorphisms of  $M$ . We say that  $f \in \mathcal{H}(M)$  is *persistent* if for each  $\varepsilon > 0$ , there is  $\delta > 0$  such that for every  $x \in M$  and  $g \in \mathcal{H}(M)$  with  $d(f, g) < \delta$ , there is  $y \in M$  satisfying  $d(f^n(x), g^n(y)) < \varepsilon$  ( $\forall n \in \mathbf{Z}$ ). The notion is independent of a metric for  $M$  and is conjugacy invariant.

Let  $\Lambda(f)$  be a hyperbolic set of  $f \in \text{Diff}(M)$ . For any  $\varepsilon > 0$  and  $x \in \Lambda(f)$ , the *local stable manifold* and the *local unstable manifold* are denoted by  $W_\varepsilon^s(x, f)$  and  $W_\varepsilon^u(x, f)$  respectively. The *stable manifold*  $W^s(x, f)$  and the *unstable manifold*  $W^u(x, f)$  of  $x \in \Lambda(f)$  are defined by a usual way. Let  $f \in \text{Diff}(M)$  satisfy Axiom A. Then the *non-wandering set* of  $f$ ,  $\Omega(f)$ , is a disjoint union of basic sets  $\Lambda_1(f) \cup \cdots \cup \Lambda_\ell(f)$ . Recall that the periodic points of  $f|_{\Lambda_i(f)}$  are dense in  $\Lambda_i(f)$  ( $1 \leq i \leq \ell$ ) and that for every  $x \in M$  there are  $p \in \Lambda_i(f)$  and  $q \in \Lambda_j(f)$  ( $1 \leq i \neq j \leq \ell$ ) such that  $x \in W^s(p, f) \cap W^u(q, f)$ . We say that  $f$  satisfies the *strong transversality condition* if for every  $x \in M$ ,  $T_x W^s(p, f) + T_x W^u(q, f) = T_x M$  for some  $p, q \in \Omega(f)$ .

Let  $P(f)$  denote the set of all periodic points of  $f \in \text{Diff}(M)$ , and let  $\mathcal{F}(M)$  be the set of all  $f \in \text{Diff}(M)$  having a  $C^1$ -neighborhood  $\mathcal{U}(f) \subset \text{Diff}(M)$  such that every  $p \in P(g)$  ( $\forall g \in \mathcal{U}(f)$ ) is hyperbolic. Then such a set was characterized as the set of all diffeomorphisms satisfying Axiom A with no-cycles (see [1, 4]). It is well known that every  $f \in \text{Diff}(M)$  satisfying Axiom A and the strong transversality condition is persistent (because  $f$  is topologically stable (see [8])). Therefore our theorem follows from the following two propositions.

**Proposition A.** *The  $C^1$  interior of  $\mathcal{P}(M)$ ,  $\text{int}\mathcal{P}(M)$ , is a subset of  $\mathcal{F}(M)$ .*

**Proposition B.** *Let  $f \in \text{Diff}(M)$  satisfy Axiom A with no-cycles. If  $f \in \text{int}\mathcal{P}(M)$ , then  $f$  satisfies the strong transversality condition.*

## 1. PROOF OF PROPOSITION A

Let  $f \in \text{int}\mathcal{P}(M)$ . To get the conclusion, it is enough to show that every  $p \in P(f)$  is hyperbolic. Indeed, if this is established, then for every  $C^1$  neighborhood  $\mathcal{V}(f) \subset \text{int}\mathcal{P}(M)$  of  $f$ , every  $g \in P(g)$  ( $\forall g \in \mathcal{V}(f)$ ) is hyperbolic because  $g \in \text{int}\mathcal{P}(M)$ . Thus  $f \in \mathcal{F}(M)$  is obtained.

Fix a neighborhood  $\mathcal{U}(f) \subset \text{int}\mathcal{P}(M)$  of  $f$ , and by assuming that there is a non-hyperbolic periodic point  $p = f^n(p)$ , we shall derive a contradiction. Here  $n > 0$  is the prime period of  $p$ . The tangent space  $T_p M$  splits into the direct sum  $T_p M = E_p^u \oplus E_p^s \oplus E_p^c$  where  $E_p^u, E_p^s$  and  $E_p^c$  are  $D_p f^n$ -invariant subspaces corresponding to the absolute values of the eigenvalues of  $D_p f^n$  greater than one, less than one and equal to one, and suppose  $E_p^c \neq 0$ . Then, for every  $\varepsilon > 0$  there exists a linear automorphism  $\mathcal{O} : T_p M \rightarrow T_p M$  such that

$$\begin{cases} \|\mathcal{O} - I\| \leq \varepsilon, \\ \mathcal{O}(E_p^\sigma) = E_p^\sigma \text{ for } \sigma = s, u \text{ and } c, \\ \text{all eigenvalues of } \mathcal{O} \circ D_p f^n|_{E_p^c} \text{ are of a root of unity,} \end{cases}$$

where  $I : T_p M \rightarrow T_p M$  is an identity map. By making use of Franks's lemma (see [2, Lemma 1.1]), we can find  $\delta_0 > 0$  and  $g \in \mathcal{U}(f)$  such that

$$(i) \quad B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(f^j(p)) = \emptyset \text{ for } 0 \leq i \neq j \leq n-1,$$

- (ii)  $g(x) = f(x)$  for  $x \in \{p, f(p), \dots, f^{n-1}(p)\} \cup \{M \setminus \cup_{i=0}^{n-1} B_{4\delta_0}(f^i(p))\}$ ,
- (iii)  $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x)$  for  $x \in B_{\delta_0}(f^i(p))$  ( $0 \leq i \leq n-2$ ),
- (iv)  $g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{n-1}(p)} f \circ \exp_{f^{n-1}(p)}^{-1}(x)$  for  $x \in B_{\delta_0}(f^{n-1}(p))$ ,

where  $B_\varepsilon(x) = \{y \in M \mid d(x, y) \leq \varepsilon\}$  for  $\varepsilon > 0$ .

Define  $G = \mathcal{O} \circ D_p f^n$ . Then there exists  $m > 0$  such that  $G|_{E_p^c}$  is an identity map. For a sufficiently small  $0 < \delta_1 < \delta_0$ , we have

$$g|_{\exp_p T_p M(\delta_1)}^{mn} = \exp_p \circ G^m \circ \exp_p^{-1}$$

where  $T_p M(\delta_1) = \{v \in T_p M \mid \|v\| \leq \delta_1\}$ . Put  $E_p^c(\delta_1) = E_p^c \cap T_p M(\delta_1)$ . Then it is clear that

$$g|_{\exp_p E_p^c(\delta_1)}^{mn} = id|_{\exp_p E_p^c(\delta_1)}.$$

Let  $v = (v_1, v_2, \dots, v_r)$  ( $r = \dim E_p^c$ ) be the representation by components with respect to the fundamental vectors of  $\mathbf{R}^r = E_p^c$ . Put  $\hat{\varepsilon} = \delta_1/8$  and fix any  $0 < \delta \leq \hat{\varepsilon}$ . Let  $\varphi : \mathbf{R}^r \rightarrow \mathbf{R}^r$  be the time-one map given by the vector field

$$\dot{v}_i = \delta' \chi(v_1) \cdots \chi(v_r) v_i$$

for  $1 \leq i \leq r$ . Here  $\chi : \mathbf{R} \rightarrow \mathbf{R}$  is a  $C^\infty$  function ( $0 \leq \chi(t) \leq 1$ ) such that

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \leq \delta_1/2, \\ 0 & \text{if } |t| \geq 2\delta_1/3, \end{cases}$$

and  $\delta' > 0$  is a number chosen so that  $\|\varphi(v) - v\| < \delta$  for  $v \in \mathbf{R}^r$  and  $\|D_v \varphi - id|_{\mathbf{R}^r}\| \leq |e^{\delta'} - 1| < \delta$  for  $v \in T_p M(\delta_1) \cap \mathbf{R}^r$ . We shall denote by  $\tilde{\varphi} : T_p M(\delta_1) \rightarrow T_p M(\delta_1)$  the extension of  $\varphi$  such that  $\tilde{\varphi}(v) = \varphi(v)$  for  $v \in E_p^c(\delta_1)$  and  $\|\tilde{\varphi}(v) - v\| < \delta$  for  $v \in T_p M(\delta_1)$ . Put

$$\psi(x) = \begin{cases} \exp_p \circ \tilde{\varphi} \circ \exp_p^{-1}(x) & \text{if } x \in \exp_p(T_p M(\delta_1)), \\ x & \text{otherwise,} \end{cases}$$

and define  $\tilde{g} = \psi \circ g$ . Let  $\hat{\delta} = \hat{\delta}(g, \hat{\varepsilon}) > 0$  be a number as in the definition of persistency of  $g$ . Then  $d(\tilde{g}, g) < \hat{\delta}$  ( $\tilde{g} \in \mathcal{H}(M)$ ) for a sufficiently small  $\delta$ . Take and fix  $v \in E_p^c(\delta_1)$  such that  $d(\exp_p(v), p) = 2\hat{\varepsilon}$ . Clearly  $g^{mn}(\exp_p(v)) = \exp_p(v)$  for all  $m \in \mathbf{Z}$ . On the other hand, it is easy to see that for every  $y \in B_{\hat{\varepsilon}}(\exp_p(v))$ , there is  $m(y) \in \mathbf{Z}$  such that

$$d(\tilde{g}^{m(y)n}(y), g^{m(y)n}(\exp_p(v))) = d(\tilde{g}^{m(y)n}(y), \exp_p(v)) > \hat{\varepsilon}.$$

This is a contradiction.

## 2. PROOF OF PROPOSITION B

Before starting the proof of this proposition, we need some preparation. Throughout this section let  $f \in \text{Diff}(M)$  satisfy Axiom A with no-cycles. Take a basic set  $\Lambda(f)$  of  $f$  and fix  $\varepsilon_0 > 0$  sufficiently small. Since  $\dim W_{\varepsilon_0}^s(x, f) = \dim W_{\varepsilon_0}^s(y, f)$  for  $x, y \in \Lambda(f)$ , we denote by  $\text{Ind } \Lambda(f)$  the dimension of  $W_{\varepsilon_0}^s(x, f)$  for  $x \in \Lambda(f)$ . If  $g \in \text{Diff}(M)$  is  $C^1$  close to  $f$ , then the number of basic sets  $\{\Lambda_i(g)\}$  of  $g$  coincides with that of basic sets  $\{\Lambda_i(f)\}$  because of  $\Omega$ -stability of  $f$ .

The following lemma is induced by Franks's lemma (see [9, Lemma 3] for details).

**Lemma 1.** *Let  $\Lambda_1(f)$  and  $\Lambda_2(f)$  be basic sets for  $f$ . Suppose that there are  $p = f^n(p) \in \Lambda_1(f)$  ( $n > 0$ ),  $q \in \Lambda_2(f)$  and  $x \in M \setminus \Omega(f)$  satisfying  $x \in W^s(p, f) \cap W^u(q, f)$ . Then, for a neighborhood  $\mathcal{U}(f) \subset \text{Diff}(M)$  of  $f$ , there are  $0 < \varepsilon_1 < \varepsilon_0/2$ ,  $g \in \mathcal{U}(f)$  and two distinct basic sets  $\Lambda_1(g)$  and  $\Lambda_2(g)$  such that*

- (i)  $B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \emptyset$  for  $0 \leq i \neq j \leq n - 1$ ,
- (ii) 
$$g(x) = \begin{cases} \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x) & \text{if } x \in B_{\varepsilon_1}(f^i(p)) \text{ for } 0 \leq i \leq n-1, \\ f(x) & \text{if } x \notin \cup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$$
- (iii) 
$$\begin{cases} g^n(p) = p \in \Lambda_1(g) \text{ and } q \in \Lambda_2(g), \\ x \in W^s(p, g) \cap W^u(q, g), \\ T_x W^s(p, g) = T_x W^s(p, f) \text{ and } T_x W^u(q, g) = T_x W^u(q, f). \end{cases}$$

Since  $f$  satisfies Axiom A, there exist a  $Df$ -invariant continuous splitting  $T_{\Omega(f)}M = E^s \oplus E^u$  and a constant  $0 < \lambda < 1$  such that

$$\|Df|_{E^s}^m\| < \lambda^m, \|Df|_{E^u}^{-m}\| < \lambda^m$$

for all  $m \geq 0$ . We denote by  $E_x^\sigma$  a fiber of  $E^\sigma$  at  $x \in \Omega(f)$  and put  $E_x^\sigma(\varepsilon) = \{v \in E_x^\sigma \mid \|v\| \leq \varepsilon\}$  for  $\varepsilon > 0$  ( $\sigma = s, u$ ). Let  $g \in \text{Diff}(M)$ ,  $p = g^n(p) \in \Lambda_1(g)$  and  $\varepsilon_1 > 0$  be given as in Lemma 1. Then it is easily checked that for  $0 < \varepsilon \leq \varepsilon_1$ , we have

$$\exp_p(E_p^\sigma(\varepsilon)) \subset W_{\varepsilon_0}^\sigma(p, g) \text{ and } \dim \exp_p(E_p^\sigma(\varepsilon)) = \dim W_{\varepsilon_0}^\sigma(p, g)$$

for  $\sigma = s, u$ .

Now we shall prove Proposition B. Fix  $x \in M \setminus \Omega(f)$  and let  $\Lambda_i(f)$  and  $\Lambda_j(f)$  be basic sets of  $f$  such that  $x \in W^s(\Lambda_i(f), f) \cap W^u(\Lambda_j(f), f)$ . To simplify the proof we assume  $i = 1$  and  $j = 2$ . If  $\text{Ind } \Lambda_1(f) = \dim M$  or  $\text{Ind } \Lambda_2(f) = 0$ , then the conclusion of this proposition is clear. Thus we shall prove  $T_x M = T_x W^s(x, f) + T_x W^u(x, f)$  when  $\text{Ind } \Lambda_1(f) \leq \dim M - 1$  and  $\text{Ind } \Lambda_2(f) \geq 1$ .

Since  $f \in \text{int}\mathcal{P}(M)$  and  $\Omega(f) = \overline{P(f)}$ , there is  $f' \in \text{int}\mathcal{P}(M)$  arbitrarily near to  $f$  in the  $C^1$  topology such that

- (a)  $f(y) = f'(y)$  for all  $y$  outside of a small neighborhood of  $x$ ,
- (b) there are  $p = f'^n(p) \in \Lambda_1(f')$  for some  $n > 0$  and  $q \in \Lambda_2(f')$  such that  $x \in W^s(p, f') \cap W^u(q, f')$ ,  $T_x W^s(p, f') = T_x W^s(x, f)$  and  $W^u(q, f') = W^u(x, f)$ .

By (a), there are basic sets  $\Lambda_i(f')$  ( $i = 1, 2$ ) for  $f'$  such that  $\Lambda_i(f') = \Lambda_i(f)$  since  $f$  is  $\Omega$ -stable. Let us prove  $T_x M = T_x W^s(p, f') + T_x W^u(q, f')$ . We identify  $f'$  with  $f$  for simplicity, and let  $\mathcal{U}(f)$  be a small neighborhood of  $f$  such that  $\mathcal{U}(f) \subset \mathcal{P}(M)$ .

Then, by Lemma 1 there are  $g \in \mathcal{U}(f)$  and basic sets  $\Lambda_i(g)$  ( $i = 1, 2$ ) satisfying Lemma 1 (i), (ii) and (iii). Thus  $T_x W^s(p, g) = T_x W^s(x, f)$  and  $W^u(q, g) = W^u(x, f)$ . Pick  $\ell > 0$  such that  $g^\ell(x) \in W_{\varepsilon_1/2}^s(p, g)$  and  $g^{-\ell}(x) \in W_{\varepsilon_0/2}^u(g^{-\ell}(q), g)$ , and define

$$C^u(g^\ell(x)) = \text{the connected component of } g^\ell(x) \text{ in } W^u(g^\ell(q), g) \cap B_{\varepsilon_1}(p).$$

Clearly,  $\exp_p^{-1}(C^u(g^\ell(x))) \subset T_p M$ .

For a linear subspace  $E$  of  $T_pM$  and  $\nu > 0$ , we write

$$E_\nu(g^\ell(x)) = \{v + \exp_p^{-1}(g^\ell(x)) | v \in E \text{ with } \|v\| \leq \nu\}.$$

Then there are a linear subspace  $E' \subset T_pM$  and a number  $0 < \nu_0 \leq \varepsilon_1$  such that

$$T_{g^\ell(x)} \exp_p(E'_\nu(g^\ell(x))) = T_{g^\ell(x)} C^u(g^\ell(x))$$

and  $\exp_p(E'_\nu(g^\ell(x))) \subset B_{\varepsilon_1}(p)$  for  $0 < \nu \leq \nu_0$ . Since  $g^\ell(x) \notin \Omega(g)$ , there is  $0 < \nu_1 \leq \nu_0$  such that  $B_{\nu_1}(g^\ell(x)) \cap g^i(B_{\nu_1}(g^\ell(x))) = \emptyset$  for all  $i \in \mathbf{Z} \setminus \{0\}$ . If  $\mathcal{U}(g) \subset \mathcal{U}(f)$  is a neighborhood of  $g$ , then there are  $0 < \nu_2 < \nu_1/4$  and  $\varphi \in \text{Diff}(M)$  such that

$$\begin{cases} \varphi|_{(B_{4\nu_2}(g^\ell(x)))^c} = id, \\ \varphi(g^\ell(x)) = g^\ell(x), \\ \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) \subset C^u(g^\ell(x)), \\ \dim \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) = \dim C^u(g^\ell(x)), \\ g' \in \mathcal{U}(g) \text{ where } g' = \varphi^{-1} \circ g. \end{cases}$$

We denote  $\exp_p(E'_{\nu_2}(g^\ell(x)))$  by  $\exp_p(E'_{\nu_2}(g'^\ell(x)))$  because of  $g^\ell(x) = g^\ell(x)$ . It is clear that there are two distinct basic sets  $\Lambda_i(g')$  ( $i = 1, 2$ ) such that  $\Lambda_i(g') = \Lambda_i(g)$  since  $g$  is  $\Omega$ -stable, and such that

$$\begin{aligned} T_x W^\sigma(x, g') &= T_x W^\sigma(x, g) \quad (\sigma = s, u), \\ \exp_p(E'_{\nu_2}(g'^\ell(x))) &\subset W^u(g'^\ell(x), g') \cap B_{\varepsilon_1}(p), \\ \dim \exp_p(E'_{\nu_2}(g'^\ell(x))) &= \dim W^u(g, g') = \dim C^u(g^\ell(x)). \end{aligned}$$

**Lemma 2.** *Under the above notation,  $\exp_p(E'_{\nu_2}(g'^\ell(x)))$  meets transversely  $W_{\varepsilon_1}^s(p, g')$  at  $g'^\ell(x)$ .*

If this lemma is established, then we have  $T_x M = T_x W^s(x, f) + T_x W^u(x, f)$  since  $T_x W^\sigma(x, g') = T_x W^\sigma(x, g) = T_x W^\sigma(x, f)$  for  $\sigma = s, u$ .

*Proof of Lemma 2.* Put  $C_\varepsilon^u(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$  for  $\varepsilon > 0$ . Take  $0 < \tilde{\varepsilon} < \nu_2$  such that  $C_{\tilde{\varepsilon}}^u(g'^\ell(x))$  is the connected component of  $g'^\ell(x)$  in  $B_{\tilde{\varepsilon}}(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$  for  $0 < \varepsilon \leq \tilde{\varepsilon}$ , and such that  $B_{\tilde{\varepsilon}}(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g')) \subset \exp_p(E'_{\nu_2}(g'^\ell(x)))$ .

*Claim.* *For every  $0 < \varepsilon \leq \tilde{\varepsilon}$ , if  $d(g'^{-i}(g'^\ell(x)), g'^{-i}(w)) < \varepsilon$  for all  $i \geq 0$ , then  $w \in C_\varepsilon^u(g'^\ell(x))$ .*

Indeed, it is clear that  $d(g'^{-\ell-i}(x), g'^{-2\ell-i}(w)) < \varepsilon \leq \varepsilon_0/2$  for all  $i \geq 0$ . On the other hand, since  $d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0/2$  ( $\forall i \geq 0$ ),

$$d(g'^{-2\ell-i}(w), g'^{-\ell-i}(q)) \leq d(g'^{-2\ell-i}(w), g'^{-\ell-i}(x)) + d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0$$

for all  $i \geq 0$  and so  $g'^{-2\ell}(w) \in W_{\varepsilon_0}^u(g'^{-\ell}(q), g')$ . Thus  $w \in C_\varepsilon^u(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$  since  $d(g'^\ell(x), w) < \varepsilon$ . The claim is proved.

Suppose that  $\exp_p(E'_{\nu_2}(g'^\ell(x)))$  does not meet transversely  $W_{\varepsilon_1}^s(p, g')$  at  $g'^\ell(x)$ . Then there exist  $0 < \nu_3 < \min\{\tilde{\varepsilon}, \nu_2/2\}$  such that for every  $\delta > 0$  ( $\delta \ll \nu_3$ ) there is  $\psi_\delta \in \text{Diff}(M)$  satisfying

$$\begin{cases} \psi_\delta|_{(B_{\nu_3}(g'^\ell(x)))^c} = id, \\ d(\psi_\delta, id) < \delta, \\ \psi_\delta(\exp_p(E'_{\nu_3/2}(g'^\ell(x)))) \cap W_{\varepsilon_1}^s(p, g') = \emptyset. \end{cases}$$

Fix  $0 < \varepsilon' < \nu_3/2$  and let  $\delta' = \delta'(g', \varepsilon') > 0$  be a number as in the definition of persistency of  $g'$ . Take  $\delta > 0$  such that  $\tilde{g} = g' \circ \psi_\delta \in \mathcal{H}(M)$  and  $d(\tilde{g}, g') < \delta'$ . Then, for  $g'$ -orbit  $\{g'^i(x)\}_{i \in \mathbf{Z}}$  of  $x$ , there is  $y \in B_{\varepsilon'}(g'^\ell(x))$  such that  $d(\tilde{g}^i(y), g'^i(g'^\ell(x))) < \varepsilon'$  for all  $i \in \mathbf{Z}$ . By the claim  $y \in \exp_p(E'_{\nu_3/2}(g'^\ell(x)))$ , from which  $\tilde{g}(y) = g \circ \psi_\delta(y) \notin W_{\varepsilon_1}^s(p, g')$ . Thus, by the hyperbolicity,  $d(\tilde{g}^i(\tilde{g}(y)), g'^{i+1}(g'^\ell(x))) > \varepsilon'$  for some  $i \geq 0$ . This is a contradiction.

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